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RESEARCH PAPER

Spatiotemporal Nonlocal Thermoelastic Model with Caputo-Tempered Fractional Derivatives for Infinite Thermoelastic Porous Half-Space with Voids

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Abstract

This study presents a novel generalized nonlocal thermoelastic model for porous materials with voids, addressing key limitations in traditional thermoelasticity frameworks. The proposed model builds on the two-phase lag (TPL) theory, incorporating spatial and temporal nonlocal effects to account for microscale and memory-dependent behaviors in porous structures. A significant innovation lies in integrating Caputo-tempered fractional derivatives, which introduce exponential tempering to mitigate the long-range memory effects associated with standard fractional derivatives. This refined mathematical framework provides an enhanced and accurate representation of the dynamic thermomechanical behavior of elastic materials with voids. To validate the model, the transient response of a semi-infinite porous medium subjected to a non-Gaussian laser-shaped heat flux on its free, stress-free surface is analyzed. This study fills a critical research gap by evaluating the combined influence of nonlocal spatial-temporal effects, phase delay, and tempered fractional parameters on the size-dependent thermomechanical responses of half-space porous nanostructures. Key findings reveal that incorporating tempered fractional derivatives significantly improves the predictability of thermal and mechanical responses while offering a more realistic depiction of energy dissipation and wave propagation. These contributions highlight the potential of the proposed model for advancing the understanding and optimization of porous nanostructures in engineering applications.

Keywords: Space-time nonlocal; thermoelastic model; porous materials; half-space; Caputo-tempered fractional

1. Introduction

Porous materials with voids represent a class of materials distinguished by interconnected pores or cavities embedded within their structure. These voids play a crucial role in shaping the physical, mechanical, and thermal

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characteristics of the material, rendering them indispensable in diverse engineering and scientific domains [1]. The exploration of porous materials holds paramount importance in disciplines like civil engineering, materials science, geotechnical engineering, and biomedical engineering, among others. Porosity stands out as a pivotal feature, denoting the ratio of void volume to total material volume. It profoundly impacts density, strength, and permeability, with higher porosity typically correlating with lower density and improved fluid or gas absorption capabilities [2]. The structural design and connectivity of voids within porous materials exhibit a wide spectrum, ranging from spherical to irregular shapes and uniform to heterogeneous distributions. This void geometry significantly influences the material's mechanical and thermal responses, shaping its behavior under various conditions [3].

The mechanical properties of porous materials, encompassing compressive strength, tensile strength, and elasticity, are notably affected by the presence of voids. Generally, increasing porosity leads to reduced mechanical strength due to the diminished load-bearing area. However, specific void configurations can enhance particular properties like energy absorption or impact resistance [4]. Thermal properties of porous materials are distinctive, owing to their capacity to trap air or gases within voids, resulting in lower thermal conductivity and effective thermal insulation. Factors such as pore size, shape, distribution, and overall porosity dictate the material's thermal behavior [5].

In classical elasticity, the stress and strain experienced at a specific point are determined solely by the local properties of the material [6]. However, in nonlocal elasticity, the response at a given point is influenced by the conditions of neighboring points within a defined range. This becomes crucial in materials with intricate microstructural characteristics, where local interactions might not fully capture the material's behavior.

Eringen's nonlocal theory [7, 8] is a foundational framework in continuum mechanics that accounts for small-scale effects in materials by incorporating nonlocal interactions. Unlike classical elasticity, which assumes that stress at a point depends only on strain at the same point, Eringen's theory [9] introduces the concept that the stress at a point is influenced by the strains in a surrounding region. This is particularly useful for capturing size-dependent behaviors observed in materials at micro- and nanoscale dimensions [10].

Temporal nonlocality, on the other hand, considers the effect of a material's past states on its current response. This is especially important in viscoelastic materials, where the material's behavior depends on its loading history. Temporal nonlocality is often represented mathematically using fractional derivatives or memory functions to account for the time-dependent characteristics of the material.

Spatial and temporal nonlocal elasticity theory is an advanced framework that expands classical elasticity to include nonlocal interactions in materials. This theory is especially pertinent for materials with microstructural features like porous materials, composites, and nanostructures, where neighboring points' conditions, even at a distance, can impact a point's behavior [11]. This nonlocal approach is essential for accurately simulating the mechanical response of materials under diverse loading scenarios, especially when addressing size-dependent effects and intricate microstructural interactions [12]. By accounting for nonlocal interactions in both space and time, this theory provides a more comprehensive framework for modeling the mechanical response of materials under various loading conditions. As research in this area continues to evolve, the potential for optimizing material design and performance in engineering applications will expand, leading to innovations in a wide range of fields [13, 14].

Ebrahimi et al. [15] introduced a novel fractional nonlocal time–space strain gradient viscoelasticity theory to analyze wave dispersion in functionally graded (FG) nanobeams. By integrating the Boltzmann superposition integral with nonlocal strain gradient elasticity, the authors developed a comprehensive model that captured both spatial and temporal nonlocal effects, providing insights into the behavior of nanostructures subjected to wave propagation. Hu and Oskay [16] presented a spatial–temporal nonlocal homogenization model to study transient anti-plane shear wave propagation in periodic viscoelastic composites. The model accounted for the inherent nonlocal interactions in both space and time, offering a more accurate representation of wave behavior in complex composite materials.

Agiasofitou and Lazar [17] explored a nonlocal elasticity model of Klein-Gordon type, incorporating internal length and time scales. The study focused on constitutive modeling and dispersion relations, providing a framework to understand wave propagation in materials with microstructural characteristics that influenced their mechanical response. Abouelregal et al. [18] introduced a size-dependent higher-order thermoelastic heat conduction model, incorporating spatial and temporal nonlocal effects to study the response of a viscoelastic micropolar medium subjected to short-pulse laser excitation. The model effectively captured the behavior of materials with instantaneous deformation and time-dependent effects.

Li et al. [19] proposed a spatiotemporally nonlocal homogenization method for viscoelastic porous metamaterial structures. The generalized Maxwell viscoelastic model with multiple branches was utilized to predict the viscoelastic behavior of materials more accurately, enhancing the understanding of complex metamaterial structures. Jiang et al. [20] developed a physically-based continuum theory that captured the microstructure-dependent and temporal effects of both permanent and transient polymer networks. The spatiotemporally nonlocal model provided insights into the mechanical behavior of polymer networks, considering their complex internal structures. Wang et al. [21] developed an analytical framework based on a spatiotemporal nonlocal model to investigate how microstructural characteristics influenced the overall dynamic response of composite materials. The authors addressed the limitations of conventional local theories that often neglected the effects of microstructure on macroscopic behavior by incorporating both spatial and temporal nonlocalities into their model.

Thermoelasticity is a specialized field within continuum mechanics that focuses on the interplay between thermal and elastic behaviors in solid materials. It investigates how variations in temperature affect the deformation and stress experienced by a material, as well as how these mechanical changes can, in turn, influence temperature distribution [22]. This reciprocal relationship is crucial for understanding the performance of materials under thermal loads, providing insights into phenomena such as thermal expansion, heat conduction, and the overall mechanical response of structures exposed to varying thermal conditions. The classical theory of thermoelasticity, often referred to as Fourier-based thermoelasticity theory, describes the coupled interactions between thermal and mechanical fields in elastic materials. A key characteristic of this theory is the coupling between heat conduction and elasticity, where changes in temperature induce strain in the material, and mechanical deformation can alter the thermal field. This relationship is governed by Fourier's Law for heat conduction, which states that the heat flux is directly proportional to the temperature gradient, leading to parabolic heat conduction equations [23].

However, one limitation of classical thermoelasticity is its assumption of instantaneous thermal wave propagation. Fourier's law implies infinite thermal wave speed, which is physically unrealistic in scenarios involving high-frequency or transient thermal loading [24]. While classical thermoelasticity provides a robust framework for many engineering applications, it fails to account for realistic time delays in heat propagation or material responses under dynamic loading conditions, particularly in microscale and nanoscale systems.

Generalized thermoelasticity was developed to overcome the limitations of classical theory by incorporating finite thermal wave speeds and time-dependent effects. Several formulations have been proposed to address these challenges [25]. The Lord-Shulman (LS) theory introduces a single relaxation time to account for finite thermal wave speed. It modifies Fourier's law by including a thermal inertia term, resulting in a hyperbolic heat conduction equation.

The Green-Lindsay (GL) theory [26] the classical theory of thermoelasticity theory by incorporating two relaxation times: one for heat flux and another for thermal strain. This formulation is particularly useful for analyzing complex thermomechanical interactions that involve multiple time-dependent phenomena.

The two-phase lag (TPL) theory [27, 28] incorporates two distinct phase lags: one for heat flux (τ_q) and another for the temperature gradient (τ_θ). This approach provides a comprehensive framework for capturing transient and memory-dependent behaviors in materials, making it particularly suitable for nanoscale materials and systems with delayed thermal and mechanical responses. Finally, fractional thermoelasticity [29, 30] integrates fractional calculus to model nonlocal and memory effects in both spatial and temporal domains. This formulation accounts for complex, long-range interactions and anomalous diffusion, which are often observed in porous and nanostructured materials.

Fractional calculus is a generalization of classical calculus that extends the concepts of derivatives and integrals to non-integer (fractional) orders. It provides powerful tools for modeling processes that exhibit memory, hereditary properties, or spatial and temporal nonlocality, often encountered in complex systems [31]. Fractional operators inherently account for historical and spatial interactions. For instance, fractional derivatives model time-dependent memory effects, while fractional integrals describe processes influenced by the state of the system over a given region or time interval. The advantages of using fractional operators include a better representation of complex systems that exhibit memory or fractal geometry, as well as the ability to unify models of classical and anomalous dynamics [32].

Applications of fractional operators are diverse. In viscoelasticity, they can describe materials where stress depends on the entire history of strain. In diffusion processes, they are used to model anomalous diffusion, where the mean squared displacement scales nonlinearly with time. In control theory, fractional-order controllers enhance system stability and response [33]. In signal processing, they analyze signals that exhibit fractal or power-law behaviors. Additionally, in electromagnetics and fluid dynamics, fractional operators effectively capture nonlocal or hereditary effects [34].

The most common definitions of fractional derivatives include the Riemann–Liouville fractional derivative, which generalizes the traditional integer-order derivative and is widely used in various applications. The Caputo fractional derivative, in contrast, is more suitable for initial value problems, as it allows for the inclusion of initial conditions in a more intuitive manner [34]. The Grünwald–Letnikov fractional derivative employs a limit process similar to the standard definition of derivatives, providing a discrete approximation to fractional order derivatives [35]. The Hadamard fractional derivative is defined using a specific integral form that emphasizes the role of limits in fractional order differentiation [36]. Finally, the conformable fractional derivative focuses on continuity and differentiability, offering a more intuitive approach to fractional derivatives that aligns closely with classical calculus [37].

Tempered fractional calculus is an extension of classical fractional calculus designed to address specific limitations associated with standard fractional derivatives and integrals, particularly their long-range memory effects. By introducing exponential tempering, this approach provides a more adaptable mathematical framework for modeling systems that exhibit localized or attenuated memory effects [38]. As a result, tempered fractional calculus enhances the ability to accurately describe the dynamic behavior of materials and processes, especially in scenarios where traditional fractional models may fall short.

Tempered fractional calculus offers several advantages. First, it effectively limits memory effects, unlike standard fractional derivatives that exhibit infinite memory. This makes tempered fractional operators suitable for physical systems with localized interactions [39]. Additionally, by varying the tempering parameter (λ), the behavior of the operator can transition smoothly between classical integer-order derivatives and fractional derivatives with long-range memory. Finally, tempered fractional models provide improved stability in modeling, making them more numerically stable and better suited for real-world applications where long-range memory is not dominant [40].

This study introduces a groundbreaking generalized nonlocal thermoelastic model specifically designed for porous materials containing voids. Traditional thermoelasticity frameworks often fall short in accurately capturing the complex behaviors exhibited by porous structures, particularly when considering microscale interactions and fractional effects. The proposed model addresses these limitations by integrating advanced theoretical concepts that enhance the understanding of the thermomechanical behavior of porous materials. The model builds upon the TPL theory, which accounts for the time lag in thermal and mechanical responses. In porous materials, the presence of voids leads to a delay in the propagation of thermal and mechanical waves. The TPL framework allows for a more accurate representation of these delays, capturing the transient behavior of the material under thermal loading.

A significant innovation of this model is the incorporation of spatial and temporal nonlocal effects. These effects consider the interactions between points in the material that are not in immediate proximity, which is particularly relevant in porous structures where microscale phenomena can influence macroscopic behavior. By accounting for these nonlocal interactions, the model provides a more comprehensive understanding of how thermal and mechanical responses are distributed throughout the material. The introduction of Caputo-tempered fractional derivatives is a pivotal aspect of the model. Standard fractional derivatives can exhibit long-range memory effects, which may not accurately reflect the behavior of real materials. The tempered fractional derivatives introduce an exponential tempering factor that mitigates these long-range effects, allowing for a more realistic representation of memory-dependent behaviors. This refinement enhances the model's ability to predict the dynamic responses of porous materials under various loading conditions.

2. Structure

The study of voided porous materials using fractional nonlocal thermoelastic models is essential for understanding their complex behavior under thermal and mechanical loading. These materials, characterized by their porous structure and the presence of voids, exhibit unique responses that classical elasticity theories cannot accurately capture [41]. By integrating nonlocal effects and fractional derivatives, a more comprehensive analysis of the interactions between thermal, mechanical, and void-related phenomena is achieved.

The governing equations for the linear homogeneous two-phase thermoelastic model with a vacuum, which considers displacement, volume fraction of voids, and temperature field without external forces, can be summarized as follows [42, 43]:

$$g = M\theta - \xi\phi - b_1 e_{kk}, \tag{1}$$

$$h_i = \alpha_1 \phi_{,i}, \tag{2}$$

$$\sigma_{ij} = 2\mu\varepsilon_{ij} - \beta\theta\delta_{ij} + b_1\phi\delta_{ij} + \lambda\varepsilon_{kk}\delta_{ij}, \tag{3}$$

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{4}$$

The parameters relevant to the model include M, which is a constitutive parameter related to thermal response; ξ, the coupling parameter between voids and thermal effects; and b_1, the mechanical response parameter due to voids. Additionally, $\theta = T - T_0$ represents the temperature variation from a reference temperature T_0. The parameter ϕ denotes the change in volume fraction due to voids, u_i are the displacement components, ε_{ij} are the strain tensor components, while ε_{kk} refers to cubic dilation. Also, δ_{ij} is the Kronecker delta, ϕ is the volume fraction of voids, α_1 serves as a constitutive parameter related to the volume fraction, h_i are the components of the equilibrated stress tensor, σ_{ij} are the stress tensor components. The parameter $\beta = (3\lambda + 2\mu)\alpha_t$ is a key term that connects the thermal and elastic properties of a material, α_t represents the linear thermal expansion coefficient, and λ and μ are the Lamé parameters.

In contrast to classical elasticity, which assumes localized stress and strain, nonlocal elasticity recognizes that the behavior of a material point is influenced by the states of surrounding points. This is particularly relevant for nanostructured materials, where size effects become significant [44]. The fundamental equations of nonlocal elasticity incorporate spatial integrals that account for the contributions of strain tensors from surrounding points, allowing for a more comprehensive understanding of material behavior at the nanoscale.

In this section, we shall first establish the constitutive relations and field equations for nonlocal thermoelastic materials with voids based on fractional derivative heat transfer.

The nonlocal force stress tensor $\tau_{kl}(r')$, equilibrated stress components $H_i(r)$, and equilibrated body force $G(r)$ are defined as follows [48, 49]:

$$\tau_{kl}(r) = \int \sigma_{kl}(r')\mathcal{K}(|r-r'|)\mathrm{d}V(r'), \tag{5}$$

$$H_i(r) = \int h_i(r')\mathcal{K}(|r-r'|)\mathrm{d}V(r'), \tag{6}$$

$$G(r) = \int g(r')\mathcal{K}(|r-r'|)\mathrm{d}V(r'), \tag{7}$$

$\mathcal{K}(|r-r'|,\xi)$ represents the attenuation function (kernel) that incorporates nonlocal influences, $|r-r'|$ is the Euclidean distance between points r and r'.

Inspired by the pioneering works of Eringen's nonlocal elastic model [7, 9] and the Boltzmann superposition integral model, both stress and strain are treated as convolution functions of time and space to account for nonlocality in both dimensions. Specifically, we assume that the stress at a reference point at a specified time depends on the historical time data and the stress at all points within the reference domain, as described by nonlocal kernel operators [11].

In the context of space-time nonlocal elasticity, stress and strain at a reference point are modeled as convolution functions of both time and space. This means that the stress and strain at a given point depend not only on the local state of the material but also on the historical state of the material at all points within a specified domain [16]. This approach allows for a more comprehensive understanding of how materials respond to external loads and thermal changes.

The fundamental assumption in this nonlocal framework is that the stress $\tau_{kl}(r,t)$ and strain $\varepsilon_{kl}(r,t)$ at a reference point r and time t can be expressed as integrals over the entire volume of the material, weighted by a kernel function that accounts for both spatial and temporal influences. The mathematical representation of this concept is as follows [14, 15]:

$$\tau_{kl}(r,t) = \int_{-\infty}^{t}\int_V \mathcal{K}(|r-r'|,t-t')\sigma_{kl}(r',t')\mathrm{d}V(r',t')\mathrm{d}\hat{t}. \tag{8}$$

Here, $\mathcal{K}(|r-r'|,t-t')$ is the kernel function that captures the influence of stress at point r' and time t' on the stress at point r and time t.

Similarly, the nonlocal equilibrated stress components $H_i(r)$, and equilibrated body force $G(r)$ are obtained as follow [14, 15]:

$$H_i(r,t) = \int \mathcal{K}(|r-r'|,t-t')h_i(r',t')\mathrm{d}V(r',t'), \tag{9}$$

$$G(r,t) = \int \mathcal{K}(|r-r'|,t-t')g(r',t')\mathrm{d}V(r',t'), \tag{10}$$

The choice of kernel function $\mathcal{K}$ is crucial in determining the nature and extent of nonlocal interactions. The kernel function typically exhibits properties that ensure the influence of distant points diminishes with increasing distance and time. To further advance the theory, we consider the nonlocal kernel as analogous to a Green's function for a linear differential operator. This relationship is expressed as [14, 17]:

$$\mathcal{L}\{\mathcal{K}(|x-\hat{x}|,t-\hat{t})\} = \delta(x-\hat{x})\delta(t-\hat{t}), \tag{11}$$

where $\delta(\cdot)$ represents the Dirac delta function and $\mathcal{L}$ is a differential operator encompassing both spatial and temporal derivatives.

We propose utilizing the Klein-Gordon (KG) operator to model nonlocal elasticity that encompasses both spatial and temporal nonlocalities. The differential operator $\mathcal{L}$ is defined as [17]:

$$\mathcal{L} = 1 - \ell^2 \nabla^2 + \tau^2 \frac{\partial^2}{\partial t^2}. \tag{12}$$

where ℓ is the internal length scale parameter associated with spatial nonlocality, τ is the characteristic time scale representing temporal nonlocality, and ∇^2 is the Laplacian operator accounting for spatial variations.

By applying the operator $\mathcal{L}$ to the nonlocal stress tensor$\tau_{kl}(r)$, equilibrated stress components $H_i(r)$, and equilibrated body force $G(r)$, we derive the equations for isotropic materials within the KG-type nonlocal elasticity framework:

$$\left(1 - \ell^2 \nabla^2 + \tau^2 \frac{\partial^2}{\partial t^2}\right)\tau_{kl} = \sigma_{kl} = 2\mu e_{ij} - \beta\theta\delta_{ij} + b_1\phi\delta_{ij} + \lambda e_{kk}\delta_{ij} \tag{13}$$

$$\left(1 - \ell^2 \nabla^2 + \tau^2 \frac{\partial^2}{\partial t^2}\right)G = g = M\theta - \xi\phi - b_1 e_{kk}, \tag{14}$$

$$\left(1 - \ell^2 \nabla^2 + \tau^2 \frac{\partial^2}{\partial t^2}\right)H_i = h_i = \alpha_1 \phi_{,i}, \tag{15}$$

The general equation of motion derived from Newton's second law is given by [22, 23]:

$$\tau_{ij,j} + F_i = \rho \frac{\partial^2 u_i}{\partial t^2}. \tag{16}$$

where χ refer to the equilibrated inertia.

By substituting equation (4) into equation (13), the equation of motion can be expressed as:

$$\mu u_{i,jj} + (\lambda + \mu)u_{j,ij} - \beta\theta_{,i} + b_1\phi_{,i} = \rho\left(1 - \ell^2 \nabla^2 + \tau^2 \frac{\partial^2}{\partial t^2}\right)\frac{\partial^2 u_i}{\partial t^2} \tag{17}$$

When no extrinsic equilibrated body force is present, the volume fraction field equation takes the form [45]:

$$G = \rho\chi\ddot{\phi} - H_{i,i}. \tag{18}$$

Substituting G from Equation (14) and H_i from Equation (15), the volume fraction field equation (18) can be expressed as follows:

$$\alpha_1\phi_{,ii} - \xi\varphi + M\theta - b_1 e_{kk} = \chi\rho\left(1 - \ell^2 \nabla^2 + \tau^2 \frac{\partial^2}{\partial t^2}\right)\ddot{\varphi} \tag{19}$$

By incorporating both spatial and temporal nonlocality, the model offers a more accurate prediction of material behavior under complex loading conditions, such as dynamic or transient loads. This approach is critical for nanostructured materials, porous media, and viscoelastic systems where nonlocal effects are pronounced.

The governing equation encapsulates the interplay between thermal and mechanical responses of materials with voids by linking entropy (η), temperature variation (θ), void fraction (ϕ), and strain components (e_{kk}) in the following equation [46]:

$$\rho\eta = M\phi + \frac{\rho C_E}{T_0}\theta + \beta e_{kk}. \tag{20}$$

where C_E represents the specific heat of the material.

The energy equation governs the heat balance, incorporating internal heat generation (Q) and heat flux components (q_i) is given by [47]:

$$\rho T_0 \dot{\eta} = -q_{i,i} + Q, \tag{21}$$

The TPL model advances Fourier's law by introducing phase lags for heat flux (τ_q) and temperature gradient (τ_θ), accounting for finite thermal propagation speeds [29, 30]:

$$\left(1 + \tau_q \frac{\partial}{\partial t}\right)q_i = -K\left(1 + \tau_\theta \frac{\partial}{\partial t}\right)\theta_{,i}, \tag{22}$$

where K is the thermal conductivity.

To further enhance the precision of heat conduction models, fractional derivatives are integrated into the TPL framework. This approach accounts for memory effects and non-local interactions, which are especially significant in materials exhibiting hereditary properties. The modified DPL equation incorporating fractional derivatives is given by [48, 49]:

$$\left(1 + \tau_q^\alpha D_t^\alpha\right)q_i = -K(1 + \tau_\theta^\alpha D_t^\alpha)\theta_{,i}. \tag{23}$$

In this modification, D_t^α denotes the fractional derivative operator of order $\alpha \in (0,1)$, which generalizes the concept of differentiation to account for non-local effects and memory.

Various formulations of fractional derivatives exist, each with distinct characteristics and applications. Some notable examples include:

The Caputo Fractional Derivative [50]:

$$ {}_0^C D_0^{(\alpha)} Y(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-s)^{\alpha}} \frac{d}{ds} Y(s)\, ds. \tag{24} $$

Caputo tempered (CT) fractional derivative [57, 58]:

$$ {}_0^{CT} D_t^{\alpha,\sigma} Y(t) = \frac{e^{-\sigma t}}{\Gamma(1-\alpha)} \int_0^t \frac{d}{ds}\left(e^{\sigma s} Y(s)\right) (t-s)^{-\alpha}\, ds. \tag{25} $$

The tempering parameter σ modulates the extent of memory effects, allowing for a more flexible modeling framework. The tempering parameter (σ) modulates long-range memory, balancing short- and long-term dynamics. When σ=0, it reduces to the Caputo derivative. Conversely, when σ>0, it introduces exponential tempering, which is relevant for systems exhibiting diminishing memory effects [51]. The Caputo-tempered fractional derivative is particularly useful in applications where long-term memory effects gradually diminish, providing a balance between short-term and long-term dynamics. See also [52-54].

Integrating the fractional TPL model with the coupled thermal-mechanical framework yields the fractional heat conduction equation:

$$ \left(1+\tau_q^{\alpha} D_t^{\alpha}\right)\left[\rho C_E \frac{\partial \theta}{\partial t} + \beta T_0 \frac{\partial e_{kk}}{\partial t} + M T_0 \frac{\partial \phi}{\partial t} - \varrho \frac{\partial Q}{\partial t}\right] = K\left(1+\tau_\theta^{\alpha} D_t^{\alpha}\right)\theta_{,ii}. \tag{26} $$

3. Problem Formulation

This section explores the behavior of a homogeneous, isotropic, thermoelastic porous half-space medium subjected to combined thermal and mechanical interactions, with an emphasis on understanding its response in scenarios where these effects are highly significant. The formulation will be based on the governing equations derived from the previous section, and we will establish the necessary boundary and initial conditions to facilitate the analysis.

The porous medium for $x \geq 0$ subjected to a time-dependent heat flux applied to its free surface is depicted in Fig. 1. The surface is assumed to be stress-free (drag-free), with no variation in the voids' volume fraction field. The problem's symmetry is considered, meaning all relevant physical variables—displacement, stress, and temperature—depend solely on the variables x and t. This symmetry simplifies the analysis to a one-dimensional framework.

The displacement components in this one-dimensional scenario are defined as:

$$ u_x = u(x,t), \;\; u_y(x,t) = 0 = u_z(x,t). \tag{27} $$

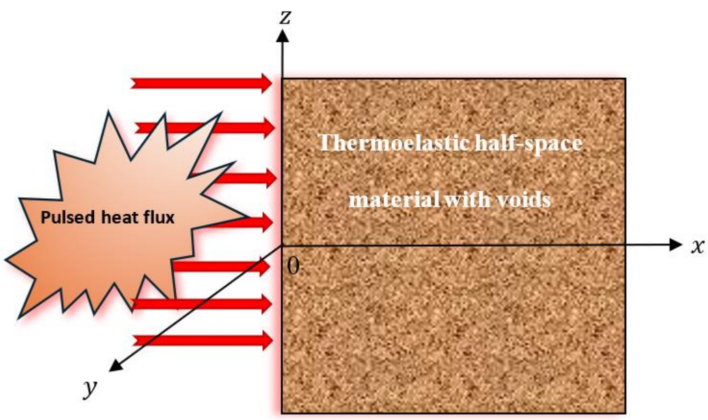

Figure1: The schematic representation of a thermoelastic half-space material containing voids

Then, the governing equations for the system can be reformulated as follows:

$$ \left(1 - \ell^2 \frac{\partial^2}{\partial x^2} + \tau^2 \frac{\partial^2}{\partial t^2}\right)\tau_{xx} = (\lambda + 2\mu)\frac{\partial u}{\partial x} - \beta\theta + b_1\phi, \tag{28} $$

$$\rho\left(1-\ell^2\frac{\partial^2}{\partial x^2}+\tau^2\frac{\partial^2}{\partial t^2}\right)\frac{\partial^2 u}{\partial t^2}=(\lambda+2\mu)\frac{\partial^2 u}{\partial x^2}-\beta\frac{\partial\theta}{\partial x}+b_1\frac{\partial\phi}{\partial x},\tag{29}$$

$$\chi\rho\left(1-\ell^2\frac{\partial^2}{\partial x^2}+\tau^2\frac{\partial^2}{\partial t^2}\right)\frac{\partial^2\phi}{\partial t^2}=\alpha_1\frac{\partial^2\phi}{\partial x^2}-\xi\phi+M\theta-b_1\frac{\partial u}{\partial x},\tag{30}$$

$$K(1+\tau_\theta^\alpha D_t^\alpha)\frac{\partial^2\theta}{\partial x^2}=\left(1+\tau_q^\alpha D_t^\alpha\right)\left[\rho C_E\frac{\partial\theta}{\partial t}+MT_0\frac{\partial\phi}{\partial t}+\beta T_0\frac{\partial^2 u}{\partial t\partial x}\right].\tag{31}$$

Assuming the medium is initially at rest, the following initial conditions apply:

$$\begin{gathered}\phi(x,0)=0,\frac{\partial\phi(x,0)}{\partial t}=0,\theta(x,0)=0,\frac{\partial\theta(x,0)}{\partial t}=0,\\ \tau_{xx}(x,0)=0,u(x,0)=0,\frac{\partial u(x,0)}{\partial t}=0.\end{gathered}\tag{32}$$

The surface of the medium is assumed to be traction-free, with no flux of voids across the boundary. The following mechanical boundary conditions are then assumed:

$$\tau_{xx}(x,t)=0,\ \frac{\partial\phi(x,t)}{\partial x}=0\quad\text{at}\quad x=0.\tag{33}$$

The surface at $x=0$ is assumed to be subjected to a pulsed heat flux. Consequently, the thermal boundary condition can be expressed as follows [63]:

$$-K\frac{\partial\theta(x,t)}{\partial x}=q_0\frac{t^2}{16t_p^2}\exp\left(-\frac{t}{t_p}\right)\quad\text{at}\quad x=0,\tag{34}$$

where q_0 represents the maximum intensity of the applied heat flux, while t_p is a time parameter chosen to control the rise and decay rates of the pulse.

Using the modified Fourier's law (23), the thermal condition can be expressed as:

$$-K(1+\tau_\theta^\alpha D_t^\alpha)\frac{\partial\theta(x,t)}{\partial x}=q_0\left(1+\tau_q^\alpha D_t^\alpha\right)\left[\frac{t^2}{16t_p^2}\mathrm{e}^{\left(-\frac{t}{t_p}\right)}\right]\quad\text{at}\quad x=0,\tag{35}$$

4. Non-Dimensional Formulation

Making dimensionless the governing equations of the thermoelastic porous half-space simplifies their mathematical handling and highlights key parameters governing system behavior. This process enhances understanding of thermal and mechanical interactions. In this section, the nondimensional variables are defined, and the equations are reformulated accordingly.

To achieve a non-dimensional formulation, we introduce the following non-dimensional variables

$$\begin{gathered}(x',u',\ell')=\upsilon_1 z_1(x,u,\ell),\ \left(t',\tau_q',\tau_\theta',\tau'\right)=\upsilon_1^2 z_1\left(t,\tau_q,\tau_\theta,\tau\right),\ \varphi'=\chi\upsilon_1^2 z_1^2\varphi,\\ \tau_{ij}'=\frac{\tau_{ij}}{\upsilon_1^2\rho},\ \theta'=\frac{\beta\theta}{\upsilon_1^2\rho},\ \eta_1=\frac{\rho C_E}{K},\ \alpha_1'=\frac{\alpha_1}{\upsilon_1^2\chi\rho},\ \upsilon_1^2=\frac{\lambda+2\mu}{\rho}.\end{gathered}\tag{36}$$

After applying the non-dimensional quantities, the governing equations for displacement (u), temperature (θ), stress (σ_{xx}), and void fraction (ϕ) can be rewritten as follows:

$$\left(1-\ell^2\frac{\partial^2}{\partial x^2}+\tau^2\frac{\partial^2}{\partial t^2}\right)\tau_{xx}=\frac{\partial u}{\partial x}-\theta+\delta_1\phi,\tag{37}$$

$$\left(1-\ell^2\frac{\partial^2}{\partial x^2}+\tau^2\frac{\partial^2}{\partial t^2}\right)\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}-\frac{\partial\theta}{\partial x}+\delta_1\frac{\partial\phi}{\partial x},\tag{38}$$

$$\left(1-\ell^2\frac{\partial^2}{\partial x^2}+\tau^2\frac{\partial^2}{\partial t^2}\right)\frac{\partial^2\phi}{\partial t^2}=\alpha_1\frac{\partial^2\phi}{\partial x^2}-\delta_4\phi-\delta_5\frac{\partial u}{\partial x}+\delta_6\theta,\tag{39}$$

$$(1+\tau_\theta^\alpha D_t^\alpha)\frac{\partial^2\theta}{\partial x^2}=\left(1+\tau_q^\alpha D_t^\alpha\right)\left[\delta_2\frac{\partial^2 u}{\partial t\partial x}+\delta_3\frac{\partial\phi}{\partial t}+\frac{\partial\theta}{\partial t}\right],\tag{40}$$

where

$$\delta_1=\frac{b_1}{\upsilon_1^4\eta_1^2\chi\rho},\ \delta_2=\frac{T_0\beta^2}{\upsilon_1^2\eta_1},\ \delta_3=\frac{MT_0\beta}{\upsilon_1^4\eta_1^3\chi\rho K},\ \delta_4=\frac{\xi}{\upsilon_1^4\eta_1^2\chi\rho},\ \delta_5=\frac{b_1}{\upsilon_1^2\rho},\ \delta_6=\frac{M}{\beta}.\tag{41}$$

5. Methodology for solving the thermoelastic half-space problem

The solution approach focuses on transforming the coupled partial differential equations (PDEs) into ordinary differential equations (ODEs) using the Laplace transform, which simplifies the problem in the Laplace domain. This methodology enables the derivation of analytical solutions for displacement (u), temperature (θ), and void fraction (ϕ) fields under given boundary and initial conditions. The Laplace transform of a time-dependent function $g(x,t)$ is defined as:

$$\bar{g}(x,s) = \mathcal{L}[g(x,t)] = \int_0^\infty e^{-st}\, g(x,t)\,dt. \tag{42}$$

Applying the Laplace transform to the governing equations (27) to (30), along with the initial conditions specified in (22), yields the transformed equations:

$$\left(1 - \mathcal{R}\frac{d^2}{dx^2}\right)\bar{\tau}_{xx} = g_1\frac{d\bar{u}}{dx} - g_2\bar{\theta} + g_3\bar{\phi}, \tag{43}$$

$$\left(\frac{d^2}{dx^2} - g_4\right)\bar{u} = g_5\frac{d\bar{\theta}}{dx} - g_6\frac{d\bar{\phi}}{dx}, \tag{44}$$

$$\left(\frac{d^2}{dx^2} - g_0\right)\bar{\theta} = g_7\frac{d\bar{u}}{dx} + g_0\delta_3\bar{\phi}, \tag{45}$$

$$\left(\frac{d^2}{dx^2} - g_9\right)\bar{\phi} = g_8\frac{d\bar{u}}{dx} - g_{10}\bar{\theta}, \tag{46}$$

where

$$\begin{gathered} g_1 = \frac{1}{1+\tau^2 s^2}, g_2 = \frac{1}{1+\tau^2 s^2}, g_3 = \frac{\delta_1}{1+\tau^2 s^2}, g_4 = \frac{s^2(1+\tau^2 s^2)}{1+\ell^2 s^2}, \\ g_5 = \frac{1}{1+\ell^2 s^2}, g_6 = \frac{\delta_1}{1+\ell^2 s^2}, g_7 = \mathcal{M}_0\delta_2, \mathcal{M}_0 = \frac{s(1+\tau_q\mathcal{F}(s))}{(1+\tau_\theta\mathcal{F}(s))}, \\ g_8 = \frac{\delta_5}{\alpha_1+\ell^2 s^2}, g_9 = \frac{s^2(1+\tau^2 s^2)+\delta_4}{\alpha_1+\ell^2 s^2}, g_{10} = \frac{\delta_6}{\alpha_1+\ell^2 s^2}, \mathcal{R} = \frac{\ell^2}{1+\tau^2 s^2} \end{gathered} \tag{47}$$

The transformed fractional operator $\mathcal{F}(s)$ is defined based on the type of fractional derivative used:

$$\mathcal{M}(s) = \begin{cases} s^\alpha & \text{for C fractional operotor,} \\ (s+\chi)^\alpha & \text{for CT fractional operotor.} \end{cases} \tag{48}$$

Combining equations (44)-(46) results in a single sixth-order differential equation in the Laplace domain:

$$\frac{d^6\bar{\mathcal{G}}}{dx^6} - L_1\frac{d^4\bar{\mathcal{G}}}{dx^4} + L_2\frac{d^2\bar{\mathcal{G}}}{dx^2} - L_3\bar{\mathcal{G}} = 0, \tag{48}$$

where $\bar{\mathcal{G}} = \{\bar{u}, \bar{\theta}, \bar{\phi}\}$ and coefficients L_1, L_2, and L_3 are defined as:

$$\begin{gathered} L_1 = \frac{1}{g_7}[p_5 + g_7 p_1 + g_8 p_2], \\ L_2 = \frac{1}{g_7}[p_1 p_5 + g_7 g_4 g_0 + p_2 p_4 + g_8 p_3], \\ L_3 = \frac{1}{g_7}[g_4 g_0 p_5 + p_3 p_4], \end{gathered} \tag{49}$$

Here,

$$\begin{gathered} p_1 = g_4 + \mathcal{M}_0 + g_5 g_7, p_2 = \mathcal{M}_0\delta_3 - g_6 g_7, p_3 = g_4\mathcal{M}_0\delta_3, \\ p_4 = g_8\mathcal{M}_0 + g_7 g_{10}, p_5 = g_9 g_7 - g_8\mathcal{M}_0\delta_3. \end{gathered} \tag{50}$$

he sixth-order differential equation can be factorized into three second-order factors as:

$$\left(\frac{d^2}{dx^2} - m_1^2\right)\left(\frac{d^2}{dx^2} - m_2^2\right)\left(\frac{d^2}{dx^2} - m_3^2\right)\{\bar{u}, \bar{\theta}, \bar{\phi}\} = 0, \tag{51}$$

where m_1^2, m_2^2, and m_3^2 are roots of the characteristic equation:

$$m^6 - L_1 m^4 + L_2 m^2 - L_3 = 0. \tag{52}$$

The roots m_1, m_2, and m_3 are calculated as:

$$\begin{gathered} m_1 = \sqrt{\frac{1}{3}(2\mathbb{R}\sin\varpi + L_1)}, m_2 = \sqrt{-\frac{\mathbb{R}}{3}\left(\sqrt{3}\cos\varpi + \sin\varpi\right) + \frac{L_1}{3}} \\ m_3 = \sqrt{\frac{\mathbb{R}}{3}\left(\sqrt{3}\cos\varpi - \sin\varpi\right) + \frac{L_1}{3}}, \end{gathered} \tag{53}$$

where

$$\mathbb{R} = \sqrt{L_1^2 - 3L_2},\ \varpi = \frac{1}{3}\sin^{-1}\mathbb{g},\ \mathbb{g} = \frac{1}{2\mathbb{R}^3}\ (9L_1L_2 - 2L_1^3 - 27L_3). \tag{54}$$

The general solution of the sixth-order differential equation, assuming regularity at infinity, is given by:

$$\{\bar{u}, \bar{\theta}, \bar{\phi}\} = \sum_{i=1}^{3} C_i e^{-m_i x}\{1, \Omega_i, \Psi_i\}, \tag{55}$$

where C_i are constants determined by boundary conditions, and:

$$\Psi_i = \frac{m_i^4 - p_1 m_i^2 + g_4 \mathcal{M}_0}{p_2 m_i^2 - p_3}, \Gamma_i = \frac{m_i(g_4\Psi_i - g_5)}{m_i^2 - g_4},\ \ i = 1,2,3. \tag{56}$$

Substituting the general solution into the stress equation (33), we get:

$$\bar{\tau}_{xx} = \sum_{i=1}^{3}\left(\frac{\Omega_i}{1 - \mathcal{R} m_i^2}\right) C_i e^{-m_i x}, \tag{57}$$

where $\Omega_i = g_3\Psi_i - m_i\Gamma_i g_1 - g_2$.

The Laplace-transformed boundary conditions (23) and (25) are:

$$\bar{\tau}_{xx}(x,s) = 0,\ \frac{d\bar{\phi}(x,s)}{dx} = 0 \quad \text{at} \quad x = 0, \tag{58}$$

$$\frac{d\bar{\theta}(x,t)}{dx} = -\frac{\delta_0 q_0 t_p}{8(st_p+1)^3} = -\bar{G}(s) \quad \text{at} \quad x = 0. \tag{59}$$

To solve for C_1, C_2, and C_3, we substitute the expressions for $\bar{\tau}_{xx}$, $\bar{\phi}$, and $\bar{\theta}$ into the boundary conditions.

6. Numerical Laplace Transform Inversion Method

Obtaining time-domain solutions for functions initially defined in the Laplace domain is often infeasible through direct analytical inversion. For complex systems, numerical inversion methods provide an efficient and practical approach to computing the inverse Laplace transform. Below is a description of one such robust method based on Fourier series. The Fourier series method, leveraging the Fast Fourier Transform (FFT), is particularly advantageous for its efficiency and accuracy when applied to complex systems without closed-form time-domain solutions.

The Fourier series method approximates the inverse Laplace transform $\mathcal{g}(x,t)$ by using properties of exponential functions and the FFT. Mathematically, the transformation is given by:

$$\mathcal{g}(x,t) = \frac{e^{\mathbb{c}t}}{2\mathbb{t}_1}\left[\bar{\mathcal{g}}(x,\mathbb{c}) + 2\mathrm{Re}\sum_{\mathbb{k}=1}^{\mathbb{m}_0} e^{i\mathbb{k}\pi t/\mathbb{t}_1}\bar{\mathcal{g}}\left(x, \mathbb{c} + \frac{i\mathbb{k}\pi}{\tau_1}\right)\right], 0 \le t \le 2\mathbb{t}_1. \tag{60}$$

In this algorithm, $\mathbb{c}$ is a positive coefficient that must be greater than or equal to the real parts of all singularities of $\bar{\mathcal{g}}(x,s)$. $\mathbb{t}_1$ is a positive constant that helps define the temporal range of the Fourier series. $\mathbb{m}_0$ is the minimum number of terms required to achieve the desired accuracy.

When implementing the numerical inversion method, it is essential to consider the level of precision required for the inversion. The choice of $\mathbb{m}_0$ directly impacts the accuracy of the result. The condition for determining $\mathbb{m}_0$ is given by:

$$e^{\mathbb{c}t}\mathrm{Re}\left(e^{i\mathbb{m}_0\pi t/\mathbb{t}_1}\bar{\mathcal{g}}\left(x, \mathbb{c} + \frac{i\mathbb{m}_0\pi}{\tau_1}\right)\right) \le \epsilon_0. \tag{61}$$

where ϵ_0 is a small positive number that defines the acceptable error margin for the inversion. This condition ensures that the contributions from higher-order terms in the series do not exceed the specified error threshold.

7. Numerical Results of the Thermoelastic Semi-Infinite Porous Problem

This section presents numerical simulations of the thermoelastic behavior of a semi-infinite porous elastic magnesium medium, focusing on field variables including horizontal displacement (u), volume fraction (ϕ), horizontal nonlocal thermal stress (τ_{xx}), and temperature change (θ). The results offer insights into the material's coupled thermo-mechanical responses under a space-time nonlocal thermoelastic model incorporating fractional derivatives and phase lags. The choice of porous elastic magnesium for this investigation is motivated by its significant mechanical and thermal engineering applications. The material properties used in the simulations are as follows ($T_0 = 298$ K) [42, 55]:

$$\lambda = 2.17 \times 10^{10}\mathrm{Nm}^{-2},\ \ \mu = 3.278 \times 10^{10}\mathrm{Nm}^{-2},\ \ K = 1.7 \times 10^{2}\ \mathrm{Wm}^{-2}\mathrm{K}^{-1},$$
$$\rho = 1740\mathrm{kgm}^{-3},\ \ \beta = 2.68 \times 10^{6}\mathrm{Nm}^{-2}\mathrm{K}^{-1},\ \ C_E = 1.04 \times 10^{10}\mathrm{J\ kg\ K}^{-1},$$
$$M = 2 \times 10^{6}\mathrm{Nm}^{-2}\mathrm{K}^{-1},\ \ \alpha_t = 3.688 \times 10^{-5}\mathrm{K}^{-1},\ \ b_1 = 1.13849 \times 10^{10}\mathrm{Nm}^{-2},$$
$$\alpha_1 = 2.1 \times 10^{6}\mathrm{Nm}^{-2}\mathrm{K}^{-1}, \chi = 1.753 \times 10^{-15}\mathrm{m}^{2},\ \ \xi = 1.475 \times 10^{10}\mathrm{Nm}^{-2}.$$

In the study of thermoelastic materials with voids, particularly those exhibiting porous characteristics, it is essential to understand how various parameters influence their behavior under different loading conditions. This section delves into the critical parameters that govern the dynamic and transient responses of these materials, focusing on phase lags (τ_q and τ_θ), fractional operators (D_t^α) and orders (α), and space-time nonlocality parameters (τ and ℓ). By analyzing these factors, we can gain valuable insights into the coupled thermal and mechanical behavior of porous materials, which is crucial for their application in engineering and technology.

7.1. Comparison of Fractional Derivatives in Thermoelastic Response

The application of fractional calculus to the analysis of thermoelastic materials with voids offers a deeper understanding of their behavior. Fractional derivatives and integrals model memory effects and non-local interactions, which are particularly relevant in porous materials. The voids within these materials create complex stress and thermal distributions that traditional integer-order models often fail to capture.

Fractional operators enable a more precise depiction of material responses to thermal and mechanical loads. They capture gradual stress buildup due to thermal loads, reflecting the viscoelastic nature of the material. This approach is invaluable in biomechanics, materials science, and structural engineering, where materials often exhibit complex, time-dependent behaviors. Fractional calculus enhances the ability to analyze and predict these behaviors accurately.

Fractional Caputo derivatives are a key concept in fractional calculus, enabling the incorporation of memory effects into material responses. These derivatives are especially useful for modeling viscoelastic behavior, capturing the time-dependent nature of stress and deformation in materials. Tempered Caputo fractional derivatives further refine this approach by introducing a tempering mechanism that addresses transient behaviors. By modifying the standard fractional derivative to include time-dependent changes, they enhance the model's ability to represent complex loading conditions more accurately.

In this subsection, we present numerical results comparing the behavior of the studied physical fields under two types of fractional derivatives: fractional-Cabuto derivatives (C) and tempered-Cabuto fractional derivatives (CT) and fractional orders (α). This analysis aims to provide insights into how these memory effects influence the thermoelastic response of porous materials. The discussion analyzes the numerical results shown in Figures 2–5, highlighting the behavior of temperature (θ), displacement (u), nonlocal thermal stress (τ_{xx}), and volume fraction field (ϕ) under the influence of fractional operators.

Figure 2 highlights the dependence of the temperature field (θ) on the type of fractional derivative and the fractional order (α). For the classical fractional derivative (C), a power-law memory effect is evident, causing the temperature (θ) to decay more slowly over time. This suggests that the material retains heat longer, making it well-suited for applications like thermal insulation where sustained thermal effects are desired. The temperature curves indicate that past thermal states significantly influence the present behavior, prolonging elevated temperatures.

In contrast, the tempered fractional derivative (CT) incorporates an exponential tempering parameter (σ), which diminishes long-term memory effects. As a result, the temperature change (θ) stabilizes more quickly, making this approach ideal for applications requiring rapid thermal recovery, such as dynamic thermal management systems. The exponential factor $\mathrm{e}^{-\sigma x}$ ensures faster decay of memory effects, yielding a more responsive thermal behavior. The fractional order (α) also plays a critical role. Lower orders ($\alpha < 1$) correspond to slower heat dissipation, reflecting stronger memory effects and longer thermal histories. Conversely, as α approaches 1, the temperature decay accelerates (θ), aligning more closely with classical behavior and exhibiting reduced memory effects. This behavior is essential for designing materials optimized for either heat retention or rapid dissipation.

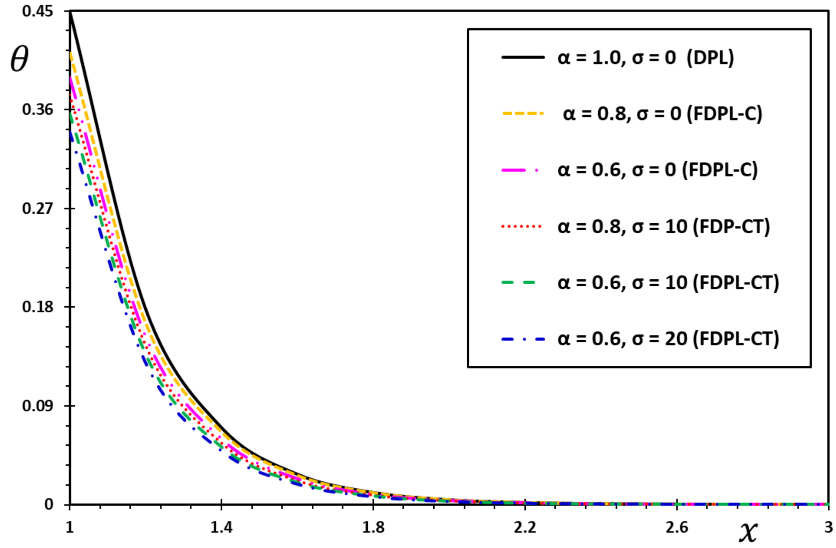

Figure 2: Temperature distribution θ under fractional derivative operators.

Figure 3 depicts the displacement distribution (u), illustrating the deformation behavior of the porous material under combined thermal and mechanical loads. The figure clearly indicates that in the case of fractional derivative C, the displacement develops gradually, indicating a delayed mechanical response to the applied forces. This behavior reflects the material's retention of past deformations, characteristic of viscoelastic or energy-dissipative systems. Such a gradual response is advantageous in applications requiring controlled deformation over time.

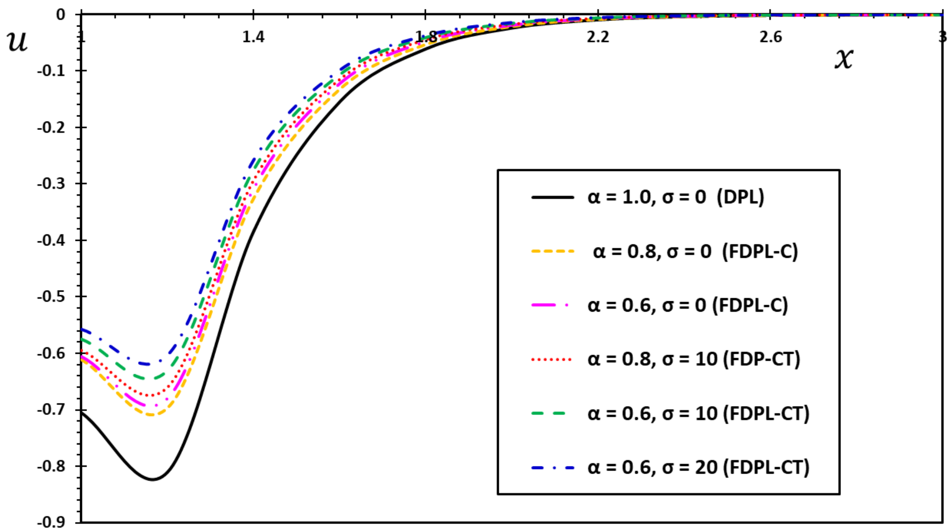

Figure 3: Displacement distribution u under fractional derivative operators.

On the other hand, the CT fractional derivative, with its exponential tempering, diminishes the effect of past deformations (u), leading to faster mechanical stabilization. This feature is particularly beneficial for dynamic systems requiring rapid adjustments, such as high-speed mechanical applications. The results also show that lower fractional orders produce slower displacement responses, emphasizing long-term deformation effects, while higher fractional orders enable quicker responses, minimizing lag in displacement evolution.

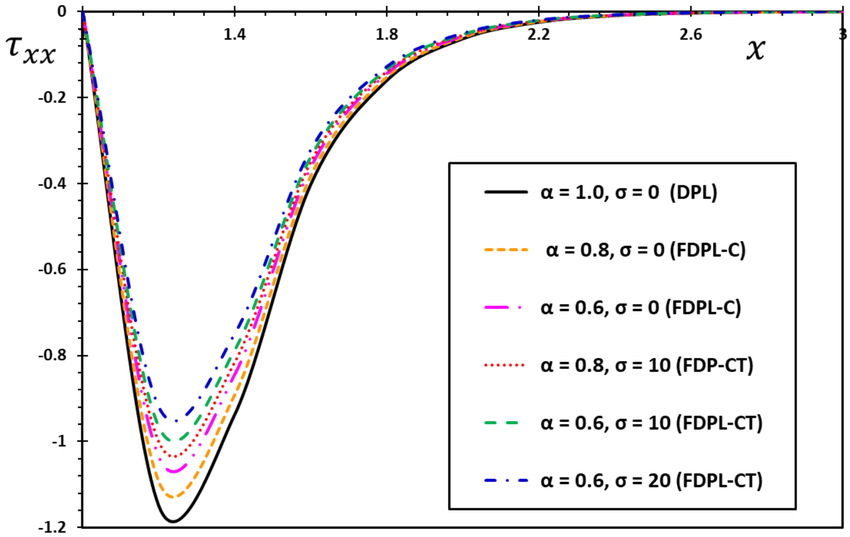

Figure 4: Nonlocal thermal stress τ_{xx} under fractional derivative operators.

Figure 4 focuses on the nonlocal thermal stress field (τ_{xx}), which is critical for understanding material responses under thermal and mechanical coupling. The C derivative results in a gradual buildup and relaxation of thermal stress τ_{xx}, reflecting the material's retention of past loading conditions. This behavior is suitable for applications that require controlled stress dissipation, such as structural components under sustained loading. The gradual accumulation of stress τ_{xx} indicates that the material can withstand prolonged thermal and mechanical loads without immediate failure.

The figure further demonstrates that when the fractional CT operator is applied, long-term stress accumulation is mitigated by moderate memory effects. This results in a faster relaxation of non-local thermal stress τ_{xx}, enhancing the material's ability to adapt to dynamic loading conditions. This feature is particularly advantageous for high-performance materials where minimizing residual stress is essential, especially in dynamic loading environments. The fractional order (α) plays a crucial role: lower fractional orders cause slower stress relaxation, highlighting the impact of long-term stress accumulation, while higher fractional orders promote faster stress decay, diminishing the influence of historical loading.

Figure 5 illustrates the volume fraction field (ϕ), depicting the distribution of pores and voids within the material and their response to thermal and mechanical interactions. The C derivative shows a delayed response to changes in the volume fraction (ϕ) due to its strong memory effect, resulting in a gradual evolution of the pore structure. This behavior makes the C derivative ideal for systems where slow, controlled changes in the pore structure are important for maintaining material integrity. As shown in Figure 5, the CT derivative diminishes the persistence of historical effects on the volume fraction (ϕ), enabling the field to stabilize more rapidly. This characteristic is particularly beneficial for adaptive porous systems that need to respond rapidly to varying loading conditions. The results show that lower fractional orders lead to slower pore evolution, highlighting the importance of historical effects, while higher fractional orders promote faster stabilization and lessen the impact of past states.

The comparative analysis of temperature, displacement, thermal stress, and volume fraction fields under the influence of fractional operators and fractional orders reveals significant insights into the behavior of porous elastic materials. The C and CT fractional derivatives demonstrate distinct effects on the dynamics of these materials, with the C operator exhibiting a smooth, traditional decay, while the CT operator shows faster decay rates and stronger damping in both thermal and mechanical responses.

These findings underscore the importance of selecting appropriate fractional derivatives and orders when modeling the behavior of porous materials under thermal and mechanical loading. Understanding these dynamics is crucial for optimizing material performance in various applications, including thermal insulation, structural components, and dynamic systems. Future research should continue to explore the implications of these fractional effects, particularly in the context of advanced materials and complex loading scenarios, to enhance our understanding of their behavior under real-world conditions.

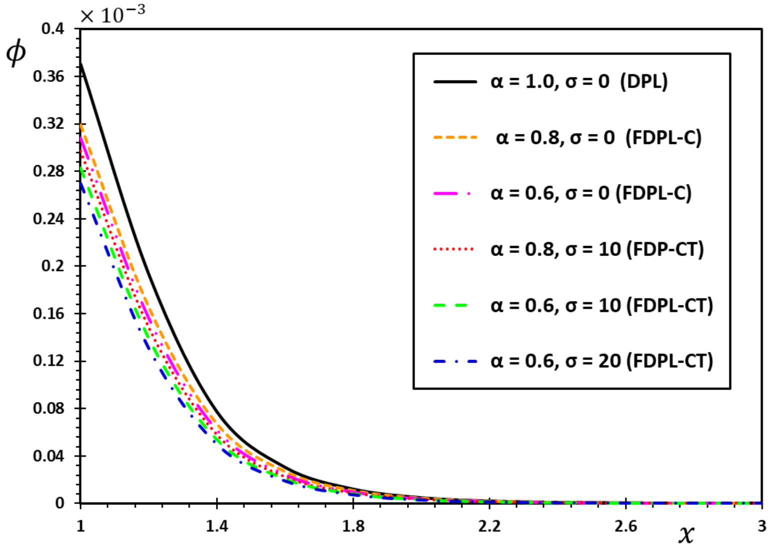

Figure 5: Volume fraction field distribution ϕ under fractional derivative operators.

7.2. Influence of Internal Length Scale and Time Scale Parameters on Porous Materials

The behavior of nanostructures, particularly nanoporous elastic materials, is significantly influenced by intrinsic length scale (ℓ) and characteristic time scale (τ) parameters. These parameters are crucial for accurately modeling size-dependent dynamic phenomena that emerge at the nanoscale, where classical continuum theories often fall short. This section explores the effects of these parameters on the dynamic responses of a thermoelastic porous medium subjected to variable heat flux, specifically in the form of laser pulses. The analysis focuses on key responses, including non-dimensional volume fraction field (ϕ), displacement (u), temperature distribution (θ), and non-local thermal stress (τ_{xx}). The comparison between classical and non-local cases in the context of elastic porous materials reveals significant differences, especially when incorporating non-local and time-delay effects. Figures 6–9 illustrate how varying values of ℓ and τ impact these mechanical and thermal fields.

The transition from classical to nonlocal cases highlights the importance of incorporating these parameters for a more accurate representation of nanostructured materials.

In the classical case, where both $\ell = 0$ and $\tau = 0$, the model reverts to the classical local scenario utilizing the two-phase lag (FTPL) thermoelastic model with fractional derivatives. In this case, non-local effects or time scale influences are not considered, resulting in a typical and well-understood response where stress and strain are directly related to local properties. This approach does not account for any size-dependent or time-dependent effects.

When $\ell \neq 0$ but $\tau = 0$, the results align with the fractional nonlocal dual-phase lag (N-FTPL) theory. This theory incorporates length scale effects while disregarding time scale impacts, allowing for the consideration of the material's microstructure and its influence on macroscopic behavior. This approach illustrates how the size of the material or structural element, particularly in porous materials, alters its behavior. In this context, nonlocal effects significantly influence the dynamic response of the material, resulting in size-dependent behavior, especially in nanoscale systems.

In the case where both $\ell \neq 0$ and $\tau \neq 0$, the analysis corresponds to the nonlocal thermoelastic theory of the Klein-Gordon type (N-KG-FTPL). This model accounts for both length scale and time scale effects, offering a more comprehensive understanding of the system's behavior. In this model, the material's response is influenced by both the size of the system (length scale) and the delayed interactions between thermal and mechanical fields (time scale). The time delay introduces a memory effect, meaning the system's response is not immediate, but depends on past states. This leads to a more nuanced dynamic behavior that better captures real-world phenomena such as wave propagation, damping, and stability under thermal and mechanical stresses.

This section examines the effects of these parameters on the non-dimensional temperature distribution (θ), displacement (u), non-local thermal stress (τ_{xx}), and volume fraction field (ϕ), in porous media. Figures 6–9 illustrate how varying values of ℓ and τ impact these mechanical and thermal fields.

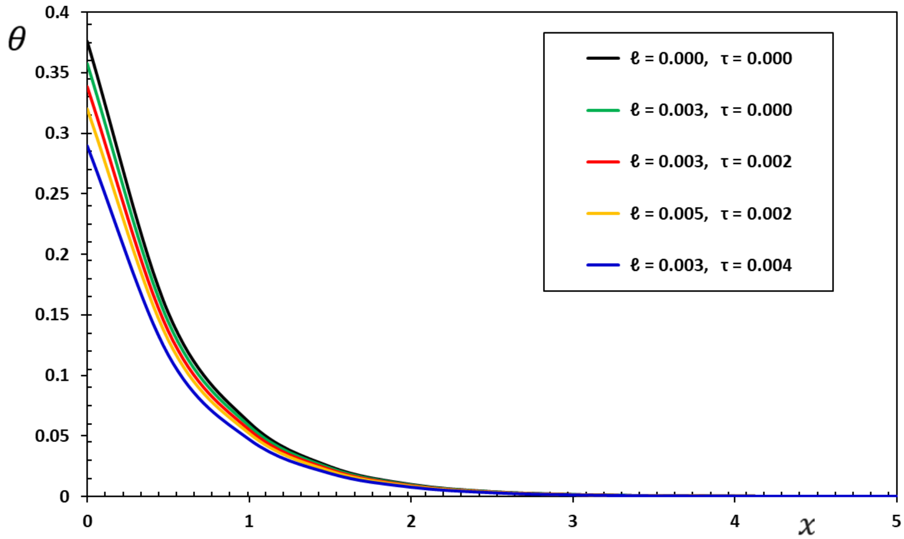

Figure 6: Temperature distribution θ via length and time scale parameters.

Figure 8 illustrates the temperature distribution (θ) in a porous medium, highlighting how the behavior changes under different conditions. In the classical case ($\ell = 0$ and $\tau = 0$), the material reaches thermal equilibrium quickly, with a larger temperature magnitude, and heat propagates instantaneously through the material. There are no significant gradients, and the system behaves in a typical manner based on classical heat conduction principles, where thermal diffusion occurs uniformly.

When non-local effects are introduced by incorporating the length scale parameter ($\ell > 0$), the temperature distribution changes dramatically. The heat no longer propagates purely locally; instead, distant regions of the material influence the thermal state. This results in a more gradual temperature decrease across the material, with the temperature spreading more slowly compared to the classical case. The non-local interactions cause the heat to propagate with a delayed response, reflecting the influence of interactions that extend over a larger scale within the material.

Finally, when both non-local and time scale effects are considered by introducing both ℓ and τ, the temperature distribution becomes even more sluggish. The time delays in heat flux and the temperature gradient lead to a thermal response that evolves over time, rather than reaching an instantaneous equilibrium. This dynamic change in the temperature gradients shows that the thermal state is not immediately stabilized but instead varies over time, reflecting more complex heat dissipation behavior. This is particularly important for applications where precise thermal management is required, such as in temperature-sensitive environments or advanced engineering systems. Understanding the time-dependent nature of heat propagation is crucial for optimizing performance, preventing overheating, and maintaining stability in such systems.

Figure 7 shows the displacement field (u) under the influence of the length scale parameter (ℓ) and the time scale parameter (τ). In the classical case (FTPL model), the absence of non-local effects results in a linear displacement profile that responds strongly to the applied loads. In contrast, when the non-local TPL theory (N-FTPL) is applied with only the ℓ effect, the displacement magnitude decreases, and the response becomes softer. This indicates that the material's behavior is influenced by the states of surrounding points, causing deviations within the porous medium. Finally, with both Klein-Gordon type effects (N-KG-FTPL), the displacement field becomes more dynamic and smaller, likely due to time delays that slow the response, leading to a lower displacement profile that reflects both immediate and historical effects.

Figure 9 investigates the behavior of non-local thermal stress (τ_{xx}) in a porous medium subjected to a variable heat flux. The figure shows that in the classical case; the thermal stress distribution is larger and directly correlates with the applied thermal loads. When the non-local DPL theory, incorporating the length scale effect (ℓ), is applied, the thermal stress distribution becomes more gradual and eventually diminishes within the medium. This behavior may be significant for applications where controlled stress dissipation over time is required. Additionally, the figure reveals that when both Klein-Gordon type effects (ℓ and τ) are present, the thermal stress distribution decreases considerably compared to the classical case. The time scale effects contribute to a more dynamic response, reflecting both current and historical thermal states. This reduction in thermal stress is crucial for understanding the performance of porous materials under dynamic thermal loads.

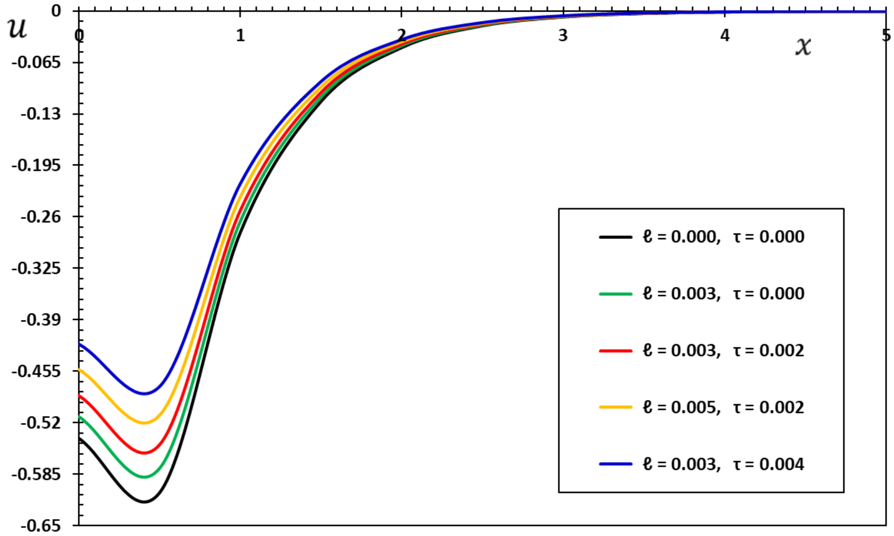

Figure 7: Displacement distribution u via length and time scale parameters.

Figure 6 shows the dimensionless volume fraction field (ϕ) under different values of the length scale factor and the time scale factor (ℓ and τ). From the numerical results, it is clear that in the classical case ($\ell = 0, \tau = 0$), the material response is direct, and the volume fraction field is more widely distributed. This reflects the assumption of local behavior without any size-dependent effects.

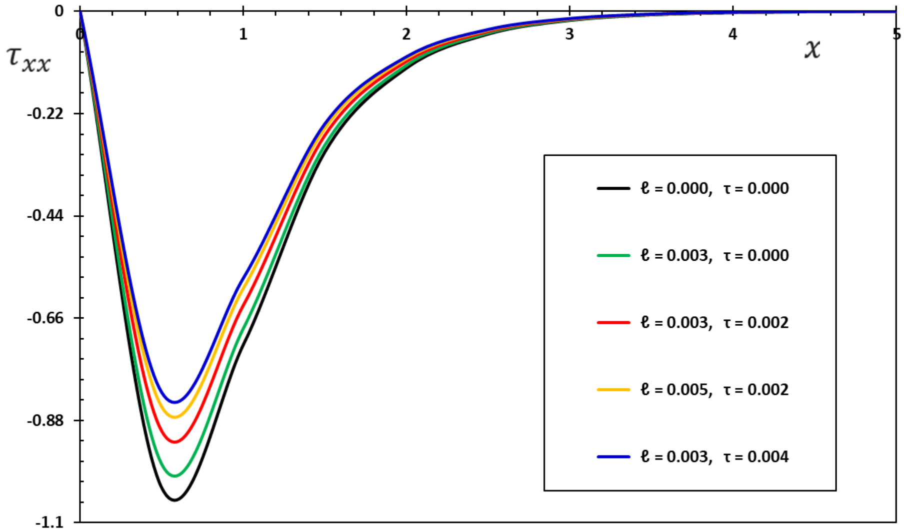

Figure 8: Nonlocal thermal stress τ_{xx} via length and time scale parameters.

When non-local effects are introduced ($\ell \neq 0, \tau = 0$), the volume fraction field begins to show variations that reflect the non-local interactions. The internal length scale introduces size-dependent behavior, resulting in a more heterogeneous and reduced distribution of pores and voids. This indicates that the influence of distant points is now accounted for, leading to a more varied field. With the addition of time-scale effects ($\ell \neq 0$, $\tau \neq 0$), the figure shows a further reduction in the volume fraction field. Time-scale effects introduce delays in the response to external loading, causing the pore structure to evolve dynamically over time. This results in a more complex and reduced distribution of pores and voids.

Figures 6–9 provide a visual representation of these effects, illustrating how varying ℓ and τ can significantly alter the mechanical and thermal fields in porous media. Understanding these influences is critical for the design and application of advanced materials in various engineering contexts, including thermal management, structural integrity, and dynamic loading scenarios. Future studies should continue to explore the implications of these parameters to enhance the predictive capabilities of models used in the analysis of porous materials.

The influence of internal length scale and time scale parameters on the non-dimensional thermo-physical fields in porous media is profound. The analysis demonstrates that incorporating these parameters leads to a more accurate representation of the behavior of porous materials under mechanical and thermal loading. The transition from classical to nonlocal theories highlights the significance of size-dependent and time-dependent effects, paving the way for future research and optimization in the field of porous media and advanced materials.

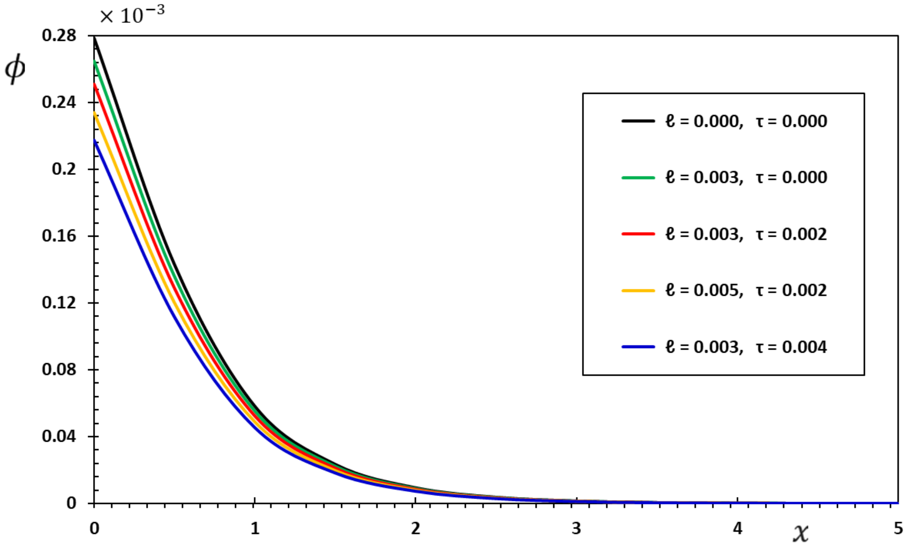

Figure 9: Volume fraction field distribution ϕ via length and time scale parameters.

8. Conclusions

This research introduces an innovative generalized nonlocal thermoelastic model specifically designed for porous materials containing voids. Conventional models often struggle to accurately capture the complex interactions and behaviors exhibited by porous structures, especially under dynamic thermal and mechanical loading conditions. This novel approach integrates several advanced theoretical concepts, including the two-phase lag (TPL) theory, spatial and temporal nonlocal effects, and Caputo-tempered fractional derivatives, to provide a more comprehensive and precise representation of the thermomechanical characteristics of porous structures. To validate the effectiveness of the proposed model, the study investigates the transient responses of porous materials subjected to non-Gaussian laser-shaped heat flux.

The study reveals several important findings regarding the influence of nonlocal spatial-temporal effects, phase delay, and tempered fractional parameters on the thermomechanical responses of half-space porous nanostructures:

1. The incorporation of tempered fractional derivatives significantly enhances the predictability of thermal and mechanical responses. The model captures the transient behavior more accurately, reflecting the complex interactions within the porous medium.
2. The refined mathematical framework provides a more realistic depiction of energy dissipation and wave propagation. This is crucial for understanding how porous materials behave under dynamic loading conditions, particularly in applications where energy management is essential.
3. The model highlights the size-dependent nature of thermomechanical responses in porous nanostructures. As the scale of the material decreases, the effects of nonlocal interactions and memory-dependent behaviors become more pronounced, necessitating the use of advanced modeling techniques.
4. The findings underscore the potential of the proposed model for advancing the understanding and optimization of porous nanostructures in various engineering applications, including thermal insulation, energy storage, and structural materials.

The analysis of nanoporous elastic materials under variable heat flux emphasizes the important roles of intrinsic length scale and time scale parameters in shaping dynamic responses. Understanding these effects is key for designing advanced materials in applications like thermal management, energy storage, and structural integrity in nanoscale systems. Further research and simulations will enhance our understanding, enabling the development of materials tailored for specific engineering needs.

Based on the current findings, future research could explore several avenues. One area is the extended applications of the model to investigate its relevance to other materials, such as composites and biomaterials. This expansion would broaden its scope across different fields. Another priority is the development of numerical simulations, where robust methods can be developed to solve the governing equations of the model. This advancement would allow for the analysis of more complex geometries and loading conditions. The final crucial aspect is experimental validation. Conducting experimental studies would help confirm the model's predictions and refine its parameters, ensuring its accuracy and reliability in practical scenarios.

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