نمایش فرایندهای گاوسی با خاصیت تقابل بوسیله مارتینگل‌ها

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چکیده

در این مقاله مفهوم خاصیت تقابل فرایندهای گاوسی در فضای هیلبرت معرفی و به کمک تکنیک‌های موجود در این فضا نمایشی بوسیله مارتینگل‌ها برای فرایندهای گاوسی با خاصیت تقابل ارائه میگردد. ثابت می‌شود که این نمایش منحصر بفرد و مشخص‌کننده فرایندهای گاوسی با خاصیت تقابل است.

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A Martingale Reperesentation of Gaussian Reciprocal Processes.

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Abstract

In This paper the notion of reciprocality in the Hilbert space is considered, and using techniques in a Hilbert space we introduce a martingale representation for the Gaussian mean zero reciprocal processes. It is shown that the representation is unique and completely characterizes the Gaussian reciprocal processes.

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Introduction

Let $\{X_t\ ; t\in[a,b]\}$ be a mean zero Gaussian process on a complete probability space (S,F,P). It is well known (see for example A.N. Shiryayev, 1984) that for each t , X_t can be considered as an element of the Hilbert space

$$H=\overline{Sp}\{X_t\ ; t\in[a\ ,\ b]\}$$

where "Sp" stands for the span closure under the inner product given by

$$(X_t, X_s)=E(X_t.X_s)=\int_S X_t\ .X_s dP\ .$$

Thus, the probabilistic independence of two random variables is equivalent to their orthogonality as elements of H.Because of this property the counterpart of the conditional expectation would be the orthogonal projections on the subspaces of H. (J.Neveu, 1975). Hence the definition of Markov, martingale and reciprocal mean zero Gaussian processes, which are based on the conditional expectation, can be stated as orthogonal projections on the subspaces of H. In Section1 we give these definitions and provide some of their elementary property.

In Section 2 we give the martingale representation of reciprocal processes. In this section we give a modified result of Dang-Ngoc and Royer (1978)which provides a condition under which a reciprocal process will be a Markov one. The converse is alwys true, i,e. a Markov process is always reciprocal.

The notion of reciprocal processes was introduced first by Jamison (1970). A reciprocal process is a process that have Markov property on finite intervals. The Markov property on sets (in particular on finite intervals) is discussed by V.Mandrekar (1976). Jamison classified the covariance function of stationary mean zero Gaussian reciprocal processes.

The technique of Hilbert space that we have considered in this paper is an alternative approach to the study of reciprocal procsses.

The representation that we give in this paper will provide the use of stochastic calculus theory as well as other properties of martingales in the area of reciprocal processes.

1. Preliminaries and notations

Throughout this paper we consider only mean zero real valued Gaussian processes on a complete pobability space (S,F,P).

Let $\{X_t\}$ be a process and consider the Hilbert space generated by $\left\{\sum_{i=1}^{n} a_i\ X_{t_i} : n\in N\ , a_i\in R\right\}$ with the inner product given by

$$(X_t\ ,X_s\ ) = E\ (X_t\ \ .X_s) = \int_S X_t\ \ .X_s\ dP.$$

We denote this space by $H=H(X)=Sp\{X_t\}$.

For a real t we define the following subspaces of H:

$H_t^-(X) = \overline{Sp}\{X_s\ : s\le t\}$ "Past"

$H_t^+(X) = \overline{Sp}\{X_s\ : s\ge t\}$ "Future"

$L_t\ (X) = \overline{Sp}\{X_t\}$ "Present"

and for the reals u,v; u<v let:

$$H^-_{u,v}\ (X) = \overline{Sp}\{X_s\ : s\le u \quad or\ s\ge v\}$$

$$H^+_{u,v}\ (X) = \overline{Sp}\{X_s\ : s\in [u\ ,\ v]\}$$

$$L_{u,v}\ (X) = \overline{Sp}\{X_u, X_v\}.$$

If there is no ambiguity we will write $H_t^-, H_t, L_t^+, H^-_{u,v}, H^+_{u,v}$ and $L_{u,v}$ instead of the above notations.

1.1. Definitions. The process $\{X\}$ is a

(i) Markov process if for each t, the orthogonal projection of H_t^+ on H_t^- is exactly the same as the orthogonal projection of H_t^+ on L_t. . In symbols:

$P_{H_t^-}^{H_t^+}=P_{L_t}^{H_t^+}$,(P stands for orthogonal projection).

For convinience we write$(H_t^+ \mid H_t^-)$ for $P_{H_t^-}^{H_t^+}$, and use similar notations in similar cases.

(ii) Reciprocal process, if for each u≤v we have:

$$(H^+_{u,v} \mid H^-_{u,v})=(H^+_{u,v} \mid L_{u,v})$$

(iii) Martingale, if for each t≥s we have:

$$(X_t \mid H_s^-)=X_s.$$

We have the following symmetry property for these notions.

1.2. Theorem. If {X } is a

(i) Markov-process, then for each t;

$$(H_t^- \mid H_t^+)=(H_t^- \mid L_t)$$

(ii) reciprocal process, then for each u<v;

$$(H^-_{u,v} \mid H^+_{u,v})=(H^-_{u,v} \mid L_{u,v})$$

(iii) martingale, then for each t<s;

$$(X_t \mid H_s^+)=X_s .$$

Proof. We give a proof for (ii). The other parts follow by a similar technique.

Let $t\notin(u,v)$ and $Y_t=(X_t \mid H^+_{u,v})$ .In order to show the equality in (ii) it suffices to show that $Y_t\in L_{u,v}$, or equivalently $(Y_t \mid L_{u,v})=Y_t$ To show this, we observe that

$$\begin{aligned}(Y_t \mid L_{u,v}) &= \big((X_t \mid H^+_{u,v}) \mid L_{u,v}\big)\\ &=(X_t \mid L_{u,v}) \qquad (L_{u,v}\subset H^+_{u,v})\\ &=(X_t \mid H^+_{u,v})=Y_t \quad (\{X_t\} \text{ is reciprocal}),\end{aligned}$$

which completes the proof.

The following theorem states that reciprocal property is weaker than Markov property.

1.3.Theorem. *A Markov process is reciprocal.*

Proof. Let $\{X_t\}$ be a Markov proces and $t\in(u,v)$, we have:

$$\begin{aligned}H^-_{u,v}&=H^-_u \oplus (H^-_{u,v} \ominus H^-_u)\\ &=H^+_u \oplus (H^-_{u,v} \ominus H^-_u).\end{aligned}$$

Thus

$$\begin{aligned}(X_t \mid H^-_{u,v})&=(X_t \mid H^-_u)+(X_t \mid H^-_{u,v} \ominus H^-_u)\\ &=(X_t \mid L_u)+(X_t \mid H^-_{u,v} \ominus H^-_u).\end{aligned}$$

Similarly

$$(X_t \mid H^-_{u,v})=(X_t \mid L_v)+(X_t \mid H^-_{u,v} \ominus H_v^+).$$

Subtracting the two sides of the above equality we get:

$$(X_t \mid L_u)-(X_t \mid L_v)=(X_t \mid H^-_{u,v} \ominus H_v^+)-(X_t \mid H^+_{u,v} \ominus H^-_u)$$

Since $(X_t \mid L_u)$ is in the space L_u it is of the form AX_u, where A is a real which depends on t and u, similarly $(X_t \mid L_v)$ is of the form BX_v; thus

$$A X_u-BX_v=(X_t \mid H^-_{u,v} \ominus H_v^+)-(X_t \mid H^+_{u,v} \ominus H^-_u).$$

We project the two sides of the above equation on H_v^+ and get

$$A (X_u \mid H_v^+)-BX_v=(X_t \mid H^+_{u,v} \ominus H^-_u);$$

the above equality is based on the facts that $X_v\in H_v^+$and that $(X_t \mid H^-_{u,v} \ominus H_v^+)$is orthogonal to H_v^+ .Again $(X_u \mid H_v^+)$ is of the form CX_v for some constant C. We have thus

$$(X_t \mid H^+_{u,v} \ominus H^-_u)= \lambda X_v \qquad \text{(for some real } \lambda).$$

Similarly

$$(X_t \mid H^+_{u,v} \ominus H_v^-)= BX_u$$

Therefore

$$(X_t \mid H^-_{u,v})= \lambda X_v+BX_u$$

which states that the projection of X_t on $H^-_{u,v}$ is in $L_{u,v}$, thus the proof is complete.

In Section 2 we refer to the following theorem. The proof is omitted.

1.4. Theorem (V. Mandrekar (1974)).

Let $\{X_t\}$ be a Markov process, then there exists a unique martingale M adopted to the filter $F_t= \sigma\{X_s : s \le t\}$ and a unique deterministic never *vanishing function f such that for each t*

$$X_t = f(t)M_t.$$

The follwing theorem is a result which provides a condition that a reciprocal process will be a Markov one.

1.5. Theorem. *Let $\{X_t : t \in [a,b]\}$ be a mean zero Gaussian reciprocal process. If either Xa or Xb is constant then the process is a Markov one.*

Proof. Assume that x is constant (since $EX_b = 0$, we will have $X_b \equiv 0$). Let a<s<t<b, we have

$$\begin{aligned}(X_t \mid H_s^-) &= (X_t \mid H_s^- \oplus \overline{Sp}\{X_b\}) \\ &= (X_t \mid H_{s,b}^-) \qquad (X_t \text{ is reciprocal}) \\ &= (X_t \mid \overline{Sp}\{X_s, X_b\}) \\ &= (X_t \mid \overline{Sp}\{X_s\}) \\ &= (X_t \mid L_s)\end{aligned}$$

i.e. $\{X_t\}$ is a Markov process.

2. A representation of reciprocal processes

Let $\{X_t : t \in [a, b]\}$ be a mean zero Gaussian reciprocal proess. Let

$$Z_t = (X_t \mid \overline{Sp}\{X_a, X_b\}) = \alpha(t)X_a + B(t)X_b, \quad Y_t = X_t - Z_t$$

It is clear that $\{Y_t\}$ is orthogonal to $\{Z_t\}$. Now we have the following theorem:

2.1.Theorem. *The above introduced $\{Y_t\}$ is a Markov process.*

Proof. First we observe that for each u<v

$$H_{u,v}^+(X_t) = H_{u,v}^+(Y_t + Z_t) + H_{u,v}^+(Y_t) + H_{u,v}^+(Z_t)$$

Now we show that $\{Y_t\}$ is a reciprocal process. Let a<u<v<b and t∈(u,v), we have:

$$\begin{aligned}&(Y_t \mid H_{u,v}^-(Y)) \\ &= (Y_t \mid H_{u,v}^-(X)) \qquad (\{Z_t\} \text{and} \{Y_t\} \text{are orthogonal}), \\ &= (X_t - Z_t \mid H_{u,v}^-(X)) \\ &= (X_t \mid H_{u,v}^-(X)) - Z_t \qquad (Z_t \in H_{u,v}^-(X)) \\ &= (X_t \mid L_{u,v}(X)) - Z_t \qquad (X_t \text{ is reciprocal}) \\ &= AX_u + BX_v - Z_t \qquad (\text{for same reals } A,B)\end{aligned}$$

Now substituting for X_u and X_v in terms of Y and Z we get:

$$\begin{aligned}A X_u + BX_v &= A(Y_u + Z_u) + B(Y_v + Z_v) \\ &= AY_u + BY_v + AZ_u + BZ_v \\ &= AY_u + BY_v + (AX_u + BX_v \mid \overline{Sp}\{X_a, X_b\}) \\ &= AY_u + BY_v + ((X_t \mid L_{u,v}(X)) \mid \overline{Sp}\{X_a, X_b\}) \\ &= AY_u + BY_v + ((X_t \mid H_{u,v}^-(X)) \mid \overline{Sp}\{X_a, X_b\}) \\ &= AY_u + BY_v + (X_t \mid \overline{Sp}\{X_a, X_b\}) \\ &= AY_u + BY_v + Z_t.\end{aligned}$$

Thus

$$\begin{aligned}(Y_t \mid H_{u,v}^-(Y)) &= AY_u + BY_v + Z_t - Z_t \\ &= AY_u + BY_v.\end{aligned}$$

Hence

$$\begin{aligned}(Y_t \mid \overline{Sp}\{Y_u, Y_v\}) &= ((Y_t \mid H_{u,v}^-) \mid \overline{Sp}\{Y_u, Y_v\}) \\ &= (AY_u + BY_v \mid \overline{Sp}\{Y_u, Y_v\}) \\ &= AY_u + BY_v.\end{aligned}$$

i.e. $(Y_t \mid H_{u,v}^-(Y)) = (Y_t \mid \overline{Sp}\{Y_u, Y_v\})$

which completes the proof of reciprocality of $\{Y_t\}$.

In order to show that $\{Y_t\}$ is a Markov process, by Theorem 1.5 it suffices to show that the "remote past" of the process is trivial. In other words, it suffices to show that

$$\bigcap_{u<v} \overline{Sp}\{Y_t : t \notin (u,v)\} = \{O\}$$

To show this we observe that

$$\begin{aligned}\overline{Sp}\{Y_t : t \notin (u,v)\} &= \overline{Sp}\{X_t : t \notin (u,v)\} \ominus \overline{Sp}\{Z_t : t \notin (u,v)\} \\ &= \overline{Sp}\{X_t : t \notin (u,v)\} \ominus \overline{Sp}\{X_a, X_b\}\end{aligned}$$

Therefore

$$\begin{aligned}\bigcap_{u<v} \overline{Sp}\{Y_t : t \notin (u,v)\} &= \{\bigcap_{u<v} \overline{Sp}\{X_t : t \notin (u,v)\} \ominus \overline{Sp}\{X_a, X_b\}\} \\ &= \overline{Sp}\{X_a, X_b\} - \overline{Sp}\{X_a, X_b\}\end{aligned}$$

which completes the proof.

Now we have the follwing main result which provides a representation for the mean zero Gaussian reciprocal processes.

2.2.Theorem (main result).

Let $\{X_t\ ,\ t\in [a,b]\}$ be a mean zero Gaussian process. $\{X_t\}$ is reciprocal if and only if it has the following representation

$$X_t = Y_t + Z_t$$

where $\{Y_t\}$ is a Markov process, $\{Z_t\}$ is orthogonal to $\{Y_t\}$ and lies in the "remote past" of $\{X_t\}$.

Moreover this representation is unique, in the sense that if

$$X_t = Y_t' + Z_t' ,$$

where $\{Y_t'\}$ and $\{Z_t'\}$ have the same property as $\{Y_t\}$ and $\{Z_t\}$ respectively, then

$$Y_t' = Y_t \quad and \quad Z_t' = Z_t$$

Proof. If $\{X_t\}$ is reciprocal then by Theorem 2.1 X_t has the required form.

Conversely let $X_t = Y_t + Z_t$, $\{Y_t\}$ and $\{Z_t\}$ satisfy the condition mentioned in the theorem. Let $t\in (u,v)$, $u<v$; then

$$\begin{aligned}&\left(X_t \mid H^-_{u,v}(X)\right)\\ &=\left(Y_t + \left(X_t \mid \overline{Sp}\{X_a ,X_b\}\right) \mid H^-_{u,v}(X)\right)\\ &=\left(Y_t \mid H^-_{u,v}(X)\right) + \left(X_t \mid \overline{Sp}\{X_a ,X_b\}\right)\\ &=\left(Y_t \mid H^-_{u,v}(Y) \oplus H^-_{u,v}(Z)\right) + \left(X_t \mid \overline{Sp}\{X_a ,X_b\}\right)\\ &=\left(Y_t \mid H^-_{u,v}(Y)\right) + \left(X_t \mid \overline{Sp}\{X_a ,X_b\}\right)\end{aligned}$$

the last equality is due to the orthogonality of $\{Y_t\}$ and $\{Z_t\}$. By Markov property of $\{Y_t\}$ we have:

$$\begin{aligned}&\left(X_t \mid H^-_{u,v}(X)\right)\\ &=\left(Y_t \mid \overline{Sp}\{Y\ ,Y\}\right) + \left(\left(X_t \mid \overline{Sp}\{X_a ,X_b\}\right)\right.\\ &= AY_u + BY_v + \left(\left(X_t \mid \overline{Sp}\{X_a ,X_b\}\right)\right)\end{aligned}$$

for some conslants A and B.

On the other hand:

$$\begin{aligned}&\left(X_t \mid H^-_{u,v}(X)\right)\\ &=\left(X_t \mid \left(H^-_{u,v}(X)\right) \ominus \overline{Sp}\{X_a ,X_b\} \oplus \overline{Sp}\{X_a ,X_b\}\right.\\ &=\left(X_t \mid H^-_{u,v}(X) \ominus \overline{Sp}\{X_a ,X_b\}\right) + \left(X_t \mid \overline{Sp}\{X_a ,X_b\}\right)\end{aligned}$$

Comparing the two values of $\left(X_t \mid H^-_{u,v}(X)\right)$ we conclude that:

$$\begin{aligned}&\left(X_t \mid H^-_{u,v}(X) \ominus \overline{Sp}\{X_a ,X_b\}\right)\\ &= AX_u+BX_v - \left(AX_u+BX_v \mid \overline{Sp}\{X_a ,X_b\}\right)\end{aligned}$$

which implies that

$$\left(X_t \mid H^-_{u,v}(X)\right) = AX_u + BX_v .$$

By (ii) of Theorem 1.2. we conclude that (Y_t) is reciprocal.

Uniqueness: Since $\overline{Sp}\{Y_a\ ,Y_b\} = (O)$ and Y is orthogonal to $\overline{Sp}\{Z_a\ ,Z_b\}$ we get

$$\overline{Sp}\{X_a ,X_b\} = \overline{Sp}\{Z_a ,Z_b\} = \overline{Sp}\{Z_a' ,Z_b'\}.$$

So

$$\begin{aligned}\left(X_t \mid \overline{Sp}\{X_a ,X_b\}\right) &= \left(Y_t + Z_t \mid \overline{Sp}\{X_a ,X_b\}\right) = Z_t\\ \left(X_t \mid \overline{Sp}\{X_a ,X_b\}\right) &= \left(Y_t' + Z_t' \mid \overline{Sp}\{X_a ,X_b\}\right) = Z_t'\end{aligned}$$

Therefore $Z_t = Z_t'$ and $Y_t = Y_t'$ and this proves the uniqueness of the represention.

Combining Theorems 2.2 and 1.4 we have the following theorem.

2.3. Theorem. *$\{X_t\}$ is a reciprocal process if and only if it has the following representation:*

$$X_t = f(t)\, M_t + Z_t$$

where

f(.) is a deterministic function.

$\{M_t\}$ is a $\{X_t\}$ adapted martingale

$\{Z_t\}$ is orthogonal to $\{M_t\}$ and lies in the remote past of $\{X_t\}$.

Moreover this representation is unique.

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