Economical Design of Double Variables Acceptance Sampling With Inspection Errors

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Abstract

The paper presents an economical model for double variable acceptance sampling with inspection errors. Taguchi cost function is used as acceptance cost while quality specification functions are normal with known variance. An optimization model is developed for double variables acceptance sampling scheme at the presence of inspection errors with either constant or monotone value functions. The monotone value functions could be descending or ascending exponentially. In the case that inspection errors have exponentially functions, we can find the best value for inspection errors regarding to the sample number and other economical parameters. Finally sensitivity analysis has done on model parameters and some numerical examples are given to demonstrate how the developed model is applied.

Keywords: Acceptance Sampling - Double Variable - Inspection Errors - Optimization - Taguchi Cost function

Introduction

The literature in variable acceptance sampling to control the ratio of non-conformity is very little, that Jackson [1] remarked this area undeveloped. The variable acceptance sampling international standard (ISO 3951:1989), in paragraph 9g of section 1.2.b. notes that in the case of more than one variable acceptance sampling, sampling method must be applied for all factors, and the lot will be accepted if and only if all factors are accepted. It is clear that in this case OC curve is different from single variable type and consumer's risk is smaller than the one in single variable type and producer risk is greater than the one in single variable type, in the case that factors increase in multi variable acceptance sampling , the efficiency of this method will decrease.

Other authors like Montgomery [2], Ryan [3], proposed methods for multi variable acceptance models using m factors with single variable methods. Shakun [4] presented a model for alternative variable sampling when the covariance matrix is known and the specification limits are approximately elliptical. Dantziger and Papp [5] developed a single variable method for alternative variable where the specifications are independent. Wesolowsky [6] developed the graph for double variable acceptance sampling, in his method he set the control limits with paying attention to economical specifications, assuming the variance and the covariance are known.

In recent decade applying acceptance sampling methods brought many questions in quality control and now the main target was production specification and reducing manufacturing tolerances, but in many cases because of human and manufacturing system errors, acceptance sampling is a desired method.

Vanderman [7] and Schilling [8] kept working on how much accuracy of the acceptance sampling is used in qualified environments. Hamilton and Lesperance [9] developed a method for single and multi variable acceptance sampling assuming that the process quality can be found out with estimation from lot defects while the variance and mean are known. Tagares [10] proposed an economical model for single variable acceptance sampling plan by using Taguchi cost model when inspections are free of errors. Arshadi [11] presents a new model for single variable acceptance sampling plan considering inspection errors.
In this paper we use Taguchi cost function by considering inspection errors for economical design of double variable acceptance sampling problems. The main reason for using Taguchi cost function is its view on cost of deviation as below:

When there is a deviation from target value, in traditional method the cost of this deviation was a constant value regardless of the measure of this deviation, but in Taguchi model the cost of this deviation is related to the square distance from the target ($\sigma^2$). It seems that this view is more effective to decrease the deviations.

**Notations and assumptions**

**Notations**

- $y_1 =$ Measured variable for specification type 1
- $y_2 =$ Measured variable for specification type 2
- $\mu_{01} =$ Target value for lot in specification type 1
- $\mu_{02} =$ Target value for lot in specification type 2
- $\mu_1 =$ Deviation of the mean of the quality characteristic type 1 in a given inspection lot from target value ($\mu_{01}$).
- $\mu_2 =$ Deviation of the mean of the quality characteristic type 2 in a given inspection lot from target value ($\mu_{02}$).
- $\sigma_1^2 =$ Variance of $y_1$ in a given inspection lot
- $\sigma_2^2 =$ Variance of $y_2$ in a given inspection lot
- $N =$ Lot size
- $n_1 =$ Sample size for type 1
- $n_2 =$ Sample size for type 2
- $x_i = y_i - \mu_{01} =$ Deviation from target value in each inspection for type 1
- $x_i = y_i - \mu_{02} =$ Deviation from target value in each inspection for type 2
- $L_1 =$ Lower acceptance limit for type 1
- $L_2 =$ Lower acceptance limit for type 2
- $U_1 =$ Upper acceptance limit for type 1
- $U_2 =$ Upper acceptance limit for type 2
- $c_i =$ Variable sampling and inspection cost per unit
- $c_r =$ Rejected cost per unit
- $k =$ Constant of the quality cost $kx^2$
- $\alpha =$ Type 1 inspection error
- $\beta =$ Type 2 inspection errors

**Assumptions**

1) The variance of $x_i$, $\sigma^2$ is known and constant
2) The variance of $\mu_i$, $\sigma^2_{\mu_i}$ is known and constant, $\sigma^2_{\mu_i} = \sigma^2_{\mu}/D_i$, $D_i$ is positive constant, expected to be larger than 1.
3) Measurement are not free of errors
4) The distribution of $x_i$, $f(x_i | \mu_i)$ is normal with mean $\mu_i$.
5) The distribution of $\mu_i$, $h(\mu_i)$ is normal with mean 0
6) $L_1 + U_1 = 2\mu_{01}$ and $L_i = \mu_{0i} - z_i, U_i = \mu_{0i} + z_i$
7) Inspections are destructive
8) Variables are independent

**Description of cost model**

Two samples are taken randomly with sizes $n_1, n_2$, after measuring $y_1, y_2$ then $x_1, x_2, \bar{x}_1, \bar{x}_2$ will be calculated and if $y_1$ lies between $L_1, U_1$ or $\bar{x}_1$ lies between $z_1, z_1$ and $y_2$ lies between $L_2, U_2$ or $\bar{x}_2$ lies between $z_2, z_2$ the lot will be accepted otherwise the lot is rejected and rejected lots will be returned to the suppliers with $c_r$ cost.

In this model screening of rejected lots is not considered because in some cases screening is not a practically feasible solution.

Regarding to the above notations and assumptions, the following three types of cost are recognized:

1) Inspection cost (CI)
2) Acceptance cost (CA)
3) Rejection cost (CR)

In this model these three costs are compared with each other and the solution will be gained through minimizing the expected total cost:

a) Expected total cost per inspection (ETCI)
b) Expected total cost without sampling/accept the lot (ETCA)
c) Expected total cost in rejection (ETCR)

And ETC=Expected totals cost of model=min (ETCI, ETCA, ETCR)

When there is no inspection error as mentioned by Tagares [10], single variable model, $Pa (\mu)$, the probability of acceptance of a lot with given $\mu$ is:

$$Pa (\mu) = \int_{-\infty}^{\infty} g(x | \mu) d\bar{x}$$

But when we have inspection error this probability will be written as (single variable):
\[ P_a(\mu) = P(\text{accept the lot} \mid \text{lot is ok}) \times P(\text{lot is ok}) + P(\text{accept the lot} \mid \text{lot is not ok}) \times P(\text{lot is not ok}) \]

So

\[ P_a(\mu) = (1 - \alpha) \times \int \int g(x \mid \mu) d\mu d\alpha + \beta \times (1 - \alpha) \times \int \int g(x \mid \mu) d\mu d\alpha \]

(1)

So we have:

\[ P_a(\mu) = (1 - \alpha) \times \int \int g(x \mid \mu) h(\mu) d\mu d\alpha + \beta \times (1 - \alpha) \times \int \int g(x \mid \mu) h(\mu) d\mu d\alpha \]

(2)

When inspection errors are predetermined and fixed, we have:

\[ P_a(\mu) = (1 - \alpha - \beta) \times \int \int g(x \mid \mu) h(\mu) d\mu d\alpha + \beta \]

(3)

When we have two independent variables, we can write:

\[ P_a(\mu_1, \mu_2) = (1 - \alpha) \times \int \int g(x_1 \mid \mu_1) d\mu_1 \times (1 - \alpha) \times \int \int g(x_2 \mid \mu_2) d\mu_2 \]

(4)

\[ P_a(\mu_1, \mu_2) = (1 - \alpha - \beta) \times \int \int g(x_1 \mid \mu_1) h(\mu_1) d\mu_1 d\mu_2 + \beta \times (1 - \alpha - \beta) \times \int \int g(x_1 \mid \mu_1) h(\mu_1) d\mu_1 d\mu_2 \]

(5)

then

\[ P_a(\mu_1, \mu_2) = (1 - \alpha - \beta) \times \int \int g(x_1 \mid \mu_1) \times h(\mu_1) d\mu_1 d\mu_2 + \beta \times (1 - \alpha - \beta) \times \int \int g(x_1 \mid \mu_1) \times h(\mu_1) d\mu_1 d\mu_2 \]

(6)

and

\[ P_a(\mu_1, \mu_2) = P_a(\mu_1) \times P_a(\mu_2) = (1 - \alpha - \beta)^2 \times \]

\[ \int \int g(x_1 \mid \mu_1) h(\mu_1) \times d\mu_1 d\mu_2 \]

(7)

\[ \int \int g(x_1 \mid \mu_1) h(\mu_1) \times d\mu_1 d\mu_2 + \beta \times (1 - \alpha - \beta) \times \int \int g(x_1 \mid \mu_1) h(\mu_1) \times d\mu_1 d\mu_2 \]

(8)

for double independent variable we have:

\[ q(x_1, x_2) = k_1 x_1^2 + k_2 x_2^2 \]

(9)

in the case that we have double variable model:

\[ CA = \int CA(\mu_1, \mu_2) d(\mu_1, \mu_2), \]

\[ CA(\mu_1, \mu_2) = (N - n_1 - n_2) \times \]

\[ \int \int q(x_1, x_2) f(x_1, x_2 \mid \mu_1, \mu_2) \times P_a(\mu_1, \mu_2) \]

(10)

\[ CR = (N - n_1 - n_2) \times \]

(11)

\[ CI = C + (n_1 + n_2) c_i \]

(12)

\[ ETC = \min(ETCA, ETCI, ETCR) \]

(13)

From Tagares [10] we have:

\[ \psi \sim N(0, \sigma^2(n+D)/nD) \]

(14)

and

\[ \psi \sim N(0, \sigma^2(n+D)/nD) \]

(15)

by using above relations for \( \mu_1, \mu_2 \) and relations in Appendix to find the optimal solution for this problem we must have the first order condition for \( z_i \) \( \frac{\partial ETCI}{\partial z_i} \) as
In this section we consider that both of them follow:

\[ q_1 = n_1 - t_1^2 - 2(n_1 + d_1)^2 + \sigma_1^2 (n_1 + d_1) \]

&

\[ q_2 = n_2 - t_2^2 - 2(n_2 + d_2)^2 + \sigma_2^2 (n_2 + d_2) \]

We must have both of \( \delta^2 \text{ETCI} / \partial z_1^2 \) and \( \delta^2 \text{ETCI} / \partial z_2^2 \) positive, because \( \delta^2 \text{ETC}_1 \text{I} / \partial z_1^2 < 0 \) and \( \delta^2 \text{ETC}_1 \text{I} / \partial z_2^2 < 0 \), so the absolute minimum value for this model because of: \( (-)(-)(-)(+>-0) \) or \( (-)(-)(+)(+)>0 \) for the absolute minimum value for this model.

In the case of \( \delta^2 \text{ETCI} / \partial z_1^2 > 0 \) and \( \delta^2 \text{ETCI} / \partial z_2^2 > 0 \), we consider that both of them positive:

Example

Then by modeling and programming with Maple 9.5 and comparing this problem with determining the sign of second order statement in the following example could explain it clearly.

Let

\[ \begin{align*}
C & = 10, C_1 = C_2 = 5, A = 2, \sigma = 2, \alpha = 0, \\
\sigma & = 0.5, N = 100,000, d_1 = 5, d_2 = 5
\end{align*} \]

and calculating the \( \psi(z) \) are normal distributions so \( \psi(z) \) are normal distributions.

By calculating \( \psi(z) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(z-\mu)^2}{2\sigma^2}} \) and also:

\[ \begin{align*}
\frac{\delta \psi}{\delta z_1} & = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(z-\mu)^2}{2\sigma^2}} \left( -\frac{z-\mu}{\sigma^2} \right) \\
\frac{\delta^2 \psi}{\delta z_1^2} & = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(z-\mu)^2}{2\sigma^2}} \left( \frac{1}{\sigma^4} \right) \\
\end{align*} \]

by equation (14) we will found the optimal method for this problem. In this model we can not say the optimality condition of the problem. The following example could explain it clearly.

\[ \frac{\delta \psi}{\delta z} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(z-\mu)^2}{2\sigma^2}} \left( -\frac{z-\mu}{\sigma^2} \right) \]

and for evaluation the above statement we should compare \( \delta^2 \text{ETCI} / \partial z_1^2 \) and these values must be found after the problem solved.

In this section we consider that both of them positive:

\[ \delta^2 \text{ETCI} / \partial z_1^2 > 0 \]

and \( \delta^2 \text{ETCI} / \partial z_2^2 > 0 \), by the same sign to have the absolute minimum value for this model.
the case that we have only one variable ((K=1, \(\sigma=.8\)),(K=2.2, \(\sigma=.65\))) we have the following table (Table 1 in index):

For solving this problem (double variable model) we must check \(k_1\sigma_1^2 + k_2\sigma_2^2 - cr < 0\):

\[
2 \times .8^2 + 2.2 \times .65^2 - 2.5 < 0
\]

Using above equations and programming the model by maple software, we have:

\(z_1=0.33, \ z_2=0.3, \ n_1=109, \ n_2=109,\)

Pae=43.69%,ETCI=245689.3, ETCA=265140, ETCR=250000 So the optimal decision is acceptance with inspection (Figure 4).

Figure 1: Double variables model for \(\alpha=\beta=0\).

Figure 2: Single variable for \(\alpha=\beta=0, k=2, \sigma=0.8\).

Figure 3: Single variable for \(S=.65, K=2.2, \alpha=\beta=0\).

Let have inspection errors with deterministic values: in above example we only change inspection error values to \(\alpha=5\%, \ \beta=10\%\).

Let \(k_1=1.2, k_2=0.8, \ \sigma_1=1, \ \sigma_2=1, \alpha=5\%, \ \beta=10\%, cr=2.5, d_1=d_2=5, N=100000\)

First of all, we should check the condition: \(k_1\sigma_1^2 + k_2\sigma_2^2 - cr < 0\), this condition holds true, so modeling the problem by above equations and programming by Maple 9.5, we have:

\(n_1=79, \ n_2=109, \ z_1=0.71, z_2=0.59,\)

ETCI=232762, ETCA=2400000, ETCR=250000 so the best decision is acceptance with inspection corresponding to these inspection error values.

Note: If we want to have two discrete sampling plan by variables No.1 and No.2 We must have:

\(n_1=50, \ z_1=1.1, \ \text{ETCI}=148980\)

ETCA=144000, ETCR=250000 and \(n_2=50, \ z_2=1.7, \ \text{ETCI}=103895.8, \ \text{ETCA}=960000, \ \text{ETCR}=250000\)

in these two problems, the best solution is lot acceptance without sampling with the total cost 240000(96000+144000), but when we have a double variable acceptance sampling problem with two independent variable, in the same time the best solution is acceptance
sampling by inspection with lower cost (232762 < 240000).

**Exponentially inspection errors**

In this section we propose two types of increasing and decreasing exponential functions for error types:

**a) Increasing type**

In this model inspection error will be increased by the number of sample size. We consider inspection error as below:

\[ e(n) = e^{-\frac{(n_1 + n_2)}{1000}} - 1 \]

and

\[ \alpha = e(n)/5, \beta = 4e(n)/5 \]

by replacing above statement in double variable inspection model and modeling with Maple we have following results:

\[ n_1 = 70, n_2 = 100, Z_1 = .73, Z_2 = .6, P_{ae} = 230203, P_{ae} = 71.53\% \]

**Note**

In this model the best value for inspection errors will be calculated by model and by knowing this information about the best value for inspection error parameter we can have a good sight to calibrate inspection instrument, considering cost values.

**b) Decreasing type**

In this model inspection error will be decreased by the number of sample size:

We consider inspection error as follows:

\[ e(n) = e^{-\frac{(n_1 + n_2)}{7000}} - 0.36 \]

and

\[ \alpha = e(n)/5, \beta = 4e(n)/5 \]

Figure 5: double variable with exponentially increasing errors.

Figure 6: double variable with exponentially decreasing errors.

By replacing above statement in double variable inspection model and modeling with Maple we have following results:

\[ n_1 = 70, Z_1 = .74, n_2 = 100, Z_2 = .58, P_{ae} = 66.6\%, ETC_{CA} = 240000, ETC_{CI} = 230216.7, ETC_{CR} = 250000 \]

so the best decision is acceptance sampling with best values for inspection errors as:

\[ \alpha = 12.3\%, \beta = 49.2\% \]

**Concluding remarks**

An new economical model for the selection of cost minimizing acceptance sampling plans for double variable model with two independent variables has been developed when inspection errors are present. A cost model is proposed for situations of fixed and variable inspection errors and also using quadratic cost in Taguchi method.

**Acknowledgments**

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Table 1: Results for single and double variables modeling.

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<th>B (%)</th>
<th>K</th>
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<th>n₁</th>
<th>z₁</th>
<th>n₂</th>
<th>z₂</th>
<th>ETCI</th>
<th>Pae(%)</th>
<th>ETCA</th>
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References

Appendix

a) \[ \frac{\partial}{\partial z} \left\{ \iiint_{\mu_1, \mu_2, x_1, x_2} \{ g(\bar{x}_1 | \mu_1) \times g(\bar{x}_2 | \mu_2) \} h(\mu_1) h(\mu_2) d\bar{x}_1 d\bar{x}_2 d\mu_1 d\mu_2 \right\} = \]
\[ \iiint_{\mu_1, x_1, \mu_2, x_2} \{ g(-z_1 | \mu_1) + g(z_1 | \mu_1) \} h(\mu_1) g(\bar{x}_2 | \mu_2) h(\mu_2) d\bar{x}_2 d\mu_2 d\mu_1 = \]

\((x_1, x_2 \text{ are independent})\) so:
\[ \iiint_{\mu_2, x_2} 2\psi(z) g(\bar{x}_2 | \mu_2) h(\mu_2) d\bar{x}_2 d\mu_2 = 2\psi(z_1) x \int_{-z_2}^{z_2} \psi(z_2) d\mu_2 \]

b) \[ \frac{\partial}{\partial z} \left\{ \iiint_{\mu_1, \mu_2, x_1, x_2} \{ g(\bar{x}_1 | \mu_1) \times g(\bar{x}_2 | \mu_2) \} h(\mu_1) h(\mu_2) d\bar{x}_1 d\bar{x}_2 d\mu_1 d\mu_2 \right\} = \]
\[ \iiint_{\mu_1, x_1, \mu_2, x_2} \{ g(-z_2 | \mu_2) + g(z_2 | \mu_2) \} h(\mu_2) g(\bar{x}_1 | \mu_1) h(\mu_1) d\bar{x}_1 d\mu_1 d\mu_2 = \]

\((x_1, x_2 \text{ are independent})\) so:
\[ \iiint_{\mu_1, x_1} 2\psi(z) g(\bar{x}_1 | \mu_1) h(\mu_1) d\bar{x}_1 d\mu_1 = 2\psi(z_2) x \int_{-z_1}^{z_1} \psi(z_1) d\mu_1 \]

\[ \frac{\partial}{\partial z} \left\{ \iiint_{\mu_1, \mu_2, x_1, x_2} \{ k1 \mu_1^2 g(\bar{x}_1 | \mu_1) \times g(\bar{x}_2 | \mu_2) \} h(\mu_1) h(\mu_2) d\bar{x}_1 d\bar{x}_2 d\mu_1 d\mu_2 \right\} \]
\[ = k1 \iiint_{\mu_1, \mu_2, x_1, x_2} \mu_1^2 \{ g(z_1 | \mu_1) + g(-z_1 | \mu_1) \} h(\mu_1) g(\bar{x}_2 | \mu_2) d\bar{x}_2 d\mu_2 d\mu_1 \]

\((x_1, x_2 \text{ are independent})\) so:
\[ k1 \iiint_{\mu_1, x_1} 2\psi(z) \{ n_1^2 z_1^2 / (n_1 + D_1)^2 + \sigma_1^2 / (n_1 + D_1) \} g(\bar{x}_1 | \mu_2) d\bar{x}_2 d\mu_2 \]

d) \[ \frac{\partial}{\partial z_2} \left\{ \iiint_{\mu_1, \mu_2, x_1, x_2} \{ k2 \mu_2^2 g(\bar{x}_2 | \mu_2) \times g(\bar{x}_1 | \mu_1) \} h(\mu_2) h(\mu_1) d\bar{x}_1 d\bar{x}_2 d\mu_1 d\mu_2 \right\} \]
\[ = k2 \iiint_{\mu_1, \mu_2, x_1, x_2} \mu_2^2 \{ g(z_2 | \mu_2) + g(-z_2 | \mu_2) \} h(\mu_2) d\mu_2 g(\bar{x}_1 | \mu_1) d\bar{x}_1 d\mu_1 \]

\((x_1, x_2 \text{ are independent})\) so:
\[ k2 \iiint_{\mu_1, x_1} 2\psi(z) \{ n_2^2 z_2^2 / (n_2 + D_2)^2 + \sigma_2^2 / (n_2 + D_2) \} g(\bar{x}_1 | \mu_1) d\bar{x}_1 d\mu_1 \]

e) \[ \frac{\partial}{\partial z} \left\{ \iiint_{\mu_2, \mu_1, x_1} \{ g(\bar{x}_1 | \mu_1) \} h(\mu_1) h(\mu_2) d\mu_1 d\mu_2 d\bar{x}_1 \right\} = \]
\[ \iiint_{\mu_2, \mu_1, x_1} \{ g(-z_1 | \mu_1) + g(z_1 | \mu_1) \} h(\mu_1) h(\mu_2) d\mu_1 d\mu_2 d\bar{x}_1 = 2\psi(z_1) \]

Similarly:
f) \( \partial / \partial z_2 \{ \int \int g(\tilde{x}_2 | \mu_2) h(\mu_2) h(\mu_1) d\mu_1 d\mu_2 d\tilde{x}_2 \} = 2\psi(z_2) \)

g) \( \partial / \partial z_1 \{ \int \int \mu_1^2 g(\tilde{x}_1 | \mu_1) h(\mu_1) h(\mu_2) d\tilde{x}_1 d\mu_1, d\mu_2 \} = 2\psi(z_1) \times \{(n_i^2 z_i^2)/(n_i + D_i)^2 + \sigma_i^2/(n_i + D_i)\} \)

h) \( \partial / \partial z_2 \{ \int \int \mu_2^2 g(\tilde{x}_2 | \mu_2) h(\mu_1) h(\mu_2) d\tilde{x}_2 d\mu_1, d\mu_2 \} = 2\psi(z_2) \times \{(n_i^2 z_i^2)/(n_i + D_i)^2 + \sigma_i^2/(n_i + D_i)\} \)

i) \( \partial / \partial z_1 \)

\[ \int \int \mu_1^2 g(\tilde{x}_1 | \mu_1) h(\mu_1) h(\mu_2) d\tilde{x}_1 d\mu_1 d\mu_2 = 2\psi(z_1) \{(n_i^2 z_i^2)/(n_i + D_i)^2 + \sigma_i^2/(n_i + D_i)\} \]

j) \( \partial / \partial z_2 \)

\[ \int \int \mu_2^2 g(\tilde{x}_2 | \mu_2) h(\mu_1) h(\mu_2) d\tilde{x}_2 d\mu_1 d\mu_2 = 2\psi(z_2) \{(n_i^2 z_i^2)/(n_i + D_i)^2 + \sigma_i^2/(n_i + D_i)\} \]

k) \( \partial / \partial z_1 \)

\[ \int \int \mu_1^2 g(\tilde{x}_1 | \mu_1) g(\tilde{x}_2 | \mu_2) h(\mu_1) h(\mu_2) d\tilde{x}_1 d\mu_1 d\mu_2 = \]

\[ \int \mu_2^2 g(\tilde{x}_2 | \mu_2) h(\mu_2) d\tilde{x}_2 d\mu_2 \times \partial / \partial z_1 \int g(\tilde{x}_1 | \mu_1) h(\mu_1) d\tilde{x}_1 d\mu_1 = \]

\[ 2\psi(z_1) \int \mu_2^2 g(\tilde{x}_2 | \mu_2) h(\mu_2) d\tilde{x}_2 d\mu_2 = 2\psi(z_2) \times \int \psi(z_2) n_i^2 z_i^2/(n_i + D_i)^2 + \sigma_i^2/(n_i + D_i) d\mu_2 \]

l) \( \partial / \partial z_2 \)

\[ \int \int \mu_1^2 g(\tilde{x}_1 | \mu_1) g(\tilde{x}_2 | \mu_2) h(\mu_1) h(\mu_2) d\tilde{x}_1 d\mu_1 d\mu_2 = \]

\[ \partial / \partial z_2 \int \mu_2^2 g(\tilde{x}_2 | \mu_2) h(\mu_2) d\tilde{x}_2 d\mu_2 \times \partial / \partial z_1 \int g(\tilde{x}_1 | \mu_1) h(\mu_1) d\tilde{x}_1 d\mu_1 = \]

\[ 2\psi(z_2) \int \mu_2^2 g(\tilde{x}_2 | \mu_2) h(\mu_2) d\tilde{x}_2 d\mu_2 = 2\psi(z_2) \times \int \psi(z_1) n_i^2 z_i^2/(n_i + D_i)^2 + \sigma_i^2/(n_i + D_i) d\mu_1 \]

\[ \partial / \partial z_2 \int \mu_2^2 g(\tilde{x}_2 | \mu_2) h(\mu_2) d\tilde{x}_2 d\mu_2 = \int \left\{ \int \mu_2^2 h(\mu_2) d\mu_2 \right\} g(\tilde{x}_1 | \mu_1) h(\mu_1) d\tilde{x}_1 d\mu_1 = \]

\[ (\sigma_i^2 / D_i) \times \int g(\tilde{x}_1 | \mu_1) h(\mu_1) d\tilde{x}_1 d\mu_1 = (\sigma_i^2 / D_i) \psi(z_1) \]