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Two Beautiful Geometrical Theorems by Abū Sahl Kūhī in a 17th Century Dutch Translation

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Abstract (received: 21/09/2008 - accepted: 15/11/2008)
This article is devoted to two theorems on tangent circles, which were discovered by the Iranian geometer Abū Sahl Kūhī (4th century A.H.). The two theorems were inspired by the Book of *Lemmas* (ma'khūdhāt) attributed to Archimedes. Kūhī's original treatise is lost, but the two theorems are found in Naṣīr al-Dīn Ṭūsī's edition of the *Lemmas* of Archimedes. They then appeared in Latin translations in 1659 in London, and again in 1661 in Florence, and in 1695 in a revised Dutch version in Amsterdam. The present article compares the original Arabic version of Kūhī's theorems (in the presentation of Ṭūsī) with the revised Dutch version.

Keywords: Kūhī, Ṭūsī, Archimedes, geometry, circles, 17th century Dutch mathematics

Introduction

Waijan ibn Rustam Abū Sahl Kūhī was an Iranian geometer and astronomer, who flourished in the second half of the 4th century A.H./ 10th century A.D. (for biographical data and a list of works of him, see sezgin, V, 314-321, VI, 218-219; Rosenfeld and Ihsanoğlu, 102-105; for a general analysis of his works, see Berggrenn). Kūhī had an outstanding reputation among his contemporaries: he was even called the "Master of his Age in the Art of Geometry" (the Arabic term is *shaykhʿasrihi fī ṣināʿat al-handasa*; see Berggren, 178). No works by Kūhī were known in medieval and Renaissance Europe. In the seventeenth century A.D., however, fragments of his work were translated into Latin. This paper is devoted to two beautiful geometrical theorems in Kūhī's *Ornamentation of the Lemmas of Archimedes*. The theorems were twice translated into Latin, in 1659 and 1661, and they also appeared in an edited form in the Latin edition of the works of Archimedes by Isaac Barrow (1630-1677) (see *Dictionary of Scientific Biography*, I, 473-476), which appeared in 1675 in London (see Barrow in references). In 1695 they were published in Amsterdam in an appendix to a Dutch version of the *Elements* of Euclid.[1]

In the seventeenth century, there was a certain interest in Islamic science in Holland. Between 1629 and 1667, Jacobus Golius held a joint professorship in mathematics and Arabic at the University of Leiden, and he translated a few scientific texts from Arabic into Latin. But no 17th-century Dutch paraphrase of an Islamic mathematical text was hitherto known to exist, and the document in this paper is probably unique. Thus it deserves to be published and compared to the original.

Section 2 of this paper contains an English translation of the medieval Arabic text of Kūhī's two geometrical theorems and some additional material. In Section 3 of this paper, the 17th-century Dutch paraphrase of Kūhī's theorems is presented, together with an English translation. In the brief mathematical analysis in Section 4, I will compare the Dutch paraphrase in Section 3 with the original in Section 2. Barrow's Latin edition will turn out to be an intermediary chain in the transmission from Iran to the Netherlands.

The rest of this introduction is about Kūhī's two geometrical theorems, their complicated transmission, and the way in which they were judged by the translators and by the mathematicians Barrow and Voogt.

Kūhī's theorems were inspired by proposition 5 of the *Lemmas* of Archimedes, a text on elementary Euclidean geometry consisting of 15 propositions on circles. It is unlikely that the *Lemmas* were written by Archimedes himself; the work is probably a Greek compilation

1. For an introduction to 17th century mathematics in Holland, see Dijksterhuis, Fokko J., "The Golden Age of Mathematics: Stevin, Huygens and the Dutch republic", *Nieuw Archief voor Wiskunde*, fifth series, 9 no. 2, (2008), 100-107, soon to be available on the internet at http://www.math.leidenuniv.nl/ ADDTILDE naw/serie5.

made in late antiquity. From now on, we will call its author "Archimedes."

The two theorems of Kūhī's concern variations of a figure which "Archimedes" calls *arbēlos*, or shoemaker's knife; the Latin term is sicila, "sickle". This shoemaker's knife consists of three semicircles with the same diameter, which are mutually tangent at their endpoints, as shown in Figure 1. In proposition 5 of the *Lemmas*, "Archimedes" draws a perpendicular at the point of tangency of the two small semicircles, and he describes two complete circles which are tangent to the perpendicular and to two boundary semicircles of the shoemaker's knife. "Archimedes" shows that the two complete circles are of equal size.

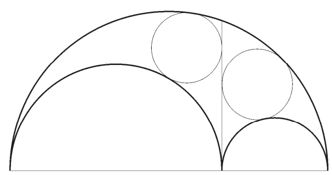


Figure 1

Kūhī generalized the shoemaker's knife to a figure with three semicircles with the same diameter, such that the largest semicircle is tangent to the two smaller circles, but the two smaller semicircles are no longer mutually tangent.

If the two smaller semicircles intersect, as in Figure 2, Kūhī drops the perpendicular through the point of intersection to the diameter and he defines the two additional complete circles as before. He proves that the complete circles are also of equal size.

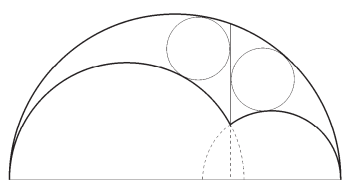


Figure 2

Finally, if the two small semicircles do not meet, as in Figure 3, Kūhī considers on the common diameter the point from which the tangents to the two small circles are equal. He draws the perpendicular through that point, constructs the complete circles as before, and proves that the complete circles are again of equal size.

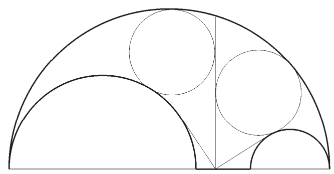


Figure 3

The mathematically interested reader is invited to give the proofs of the theorems of "Archimedes" and Kūhī, before reading the rest of this paper. "Archimedes" and Kūhī do not explain, at least not explicitly, the problem how to draw the complete circles by ruler and compass in such a way that they are tangent to two semicircles and the perpendicular. This is another interesting problem for the reader.

In his proof, "Archimedes" determines the diameter of one of the complete circles. In modern terms, the diameter turns out to be $ab/(a + b)$, where a and b are the diameters of the smaller semicircles and $a + b$ the diameter of the larger semicircle. Since this expression is symmetric in a and b, the diameters of the circles on both sides of the perpendicular must be equal. Kūhī's proof is more complicated but based on the same symmetry principle. If the smaller circles do not intersect, the radius of the complete circle in terms of the diameters a and b of the smaller semicircle and the closest distance between them is $(a + c)(b + c)/(a + b + 2c)$. Since the expression is symmetric in a and b, again the two small circles on both sides of the perpendicular are equal. We should note, however, that Kūhī does not determine the radius in this way.

Between the 17th and 19th centuries, similar problems about circles were very popular in Japan as a form of art, called sangaku. The figures were displayed in Japanese temples and visitors were invited to discover the 'nice' property in the figure and then to prove the property.[1] [2] Figures 1-3 have not been found in Japan, but they can be considered sangaku figures if all explanations are omitted. The 'nice' property to be discovered and proved is the equality of the complete circles. Each of Kūhī's figures 2 and 3 could be used as the logo of an institution or organization dedicated to the Islamic-Persian heritage in mathematics.

We now turn to the transmission of the *Lemmas* of "Archimedes" and of Kūhī's theorems. The Greek text of the *Lemmas* is lost. The *Lemmas* were translated into Arabic by Thābit ibn Qurra (836-901 AD) (on the mathematical works and translations by Thābit ibn Qurra see, e.g., Sezgin, V, 264-272.). The Arabic title of the work is *ma'kh̲ūdh̲āt*, literally: *Assumptions*, but scholars believe that the Arabic title is a translation of the Greek word *lēmmata* (compare Heiberg, II, 511 note), which is the reason why the work is called *Lemmas* in the modern literature. Thābit ibn Qurra's translation inspired Kūhī to write his *Ornamentation of the Lemmas of Archimedes*. The complete version of this *Ornamentation* is also lost; only the two geometrical theorems were preserved in the commentary to the *Lemmas* by Abu'l-Ḥasan ʿAlī ibn Aḥmad Nasawī (ca. 400/1010) (see Sezgin V, 345-348). By the time of Nasawī, the *Lemmas* of "Archimedes" had been included in the *Middle Books* (mutawassiṭāt) that is the collection of texts which had to be read by students of mathematics and astronomy between the *Elements* of Euclid and the *Almagest* of Ptolemy. When Naṣīr al-Dīn Ṭūsī (d. 672/1274) produced a new edition of the *Middle Books*, he included the *Lemmas* of "Archimedes" with the commentary by Nasawī and Kūhī's two theorems.

Ṭūsī's edition of the *Middle Books* is extant today in numerous Arabic manuscripts (see Sezgin 5/133), and it is the source of all (Arabic and Latin) versions of the *Lemmas* which have been published hitherto. In the 17th century, some manucripts of Ṭūsī's edition were available to European orientalists and mathematicians who were

1. See Fukagawa, H.; Rothman, A., *Sacred Mathematics: Japanese Temple Geometry*, Princeton University Press, 2008.
2. As far as I know, the question whether the Japanese sangaku figures were influenced by Greek and possibly Islamic mathematics has not been investigated.

interested in recovering lost works by Archimedes from Arabic texts. The first Latin translation of the *Lemmas* was made by John Greaves (1602-1652) (see Toomer 126-179) and published posthumously (London, 1659); two years later, in 1661, a much superior translation appeared in Florence.[1] This translation was a joint product of the Christian philosopher Ibrāhīm al-Ḥāḳilānī (1605-1664), from Ḥāḳil in Northern Lebanon, whose name was Latinized as Abraham Ecchellensis, and the mathematician Giovanni Alfonso Borelli (1608-1679), who did not know Arabic (on Borelli see *Dictionary of Scientific Biography*, II, 306-314). Borelli added his own introduction as well as commentaries to some of the propositions. The two Latin translations include the two theorems by Kūhī with references to him. The translations are based on Ṭūsī's edition of the *Middle Books*, but Ṭūsī's name is not mentioned in his new edition of the *Lemmas*, so his name does not occur in the Latin versions either.

In 1675, Isaac Barrow published a new version of the *Lemmas* in his edition of the works of Archimedes and Apollonius. Barrow had access to the two Latin translations of 1659 and 1661, and he added some commentaries of his own. He often changed the labels of points in geometrical figures, and used some mathematical symbols in his translation (such as +, ×). He applied the same treatment to Kūhī's theorems. In 1695, The Dutch geometer C.J. Voogt (on C.J. Voogt almost nothing is known: see Van der Aa, A.I, V 109) published a complete Dutch edition of the *Elements* of Euclid. To Euclid's Book 6, Voogt added an appendix which included, among other things, a reworking of the entire contents of the *Lemmas* of "Archimedes". Thus, proposition 24 of this appendix is a paraphrase of proposition 5 of "Archimedes" together with the two theorems by Kūhī. We will see

1. The references are: *Lemmata Archimedis apud graecos et latinos jam pridem desiderata, e vetuste codice M.S. arabico a Johanno Gravio traducta et nunc primum cum arabum scholis publicata, revisa et pluribus mendis expurgata a Samuele Foster*, which was publised in S. Foster, *Miscellanea sive lucubrationes mathematicae*, Londini 1659. I have consulted the copy in the University Library in Leiden. The 1661 translation by Ecchellensis and Borelli is found in *Apollonii Pergaei Conicorum Lib. V. VI. VII. paraphraste Abalphato Asphahanensi nunc primum editi, Additus in calce Archimedes Assumptorum Liber ex codicibus arabicis mss. ... Abrahamus Ecchellensis Maronita ... latinos reddidit Io.[hannes] Alfonsus Borellus in Geometricis versione contulit*, Florentiae 1661.

in Section 4 that Voogt based his paraphrase on Barrow's edition, but deleted Barrow's mathematical symbolism. Voogt added some new elements, which were not always improvements.

In the nineteenth century, Kūhī's two theorems appeared in a footnote in the 1824 German translation of the works of Archimedes by Nizze, and in a brief article which appeared in 1869 in London. Needless to say, Kūhī's two theorems were not included in the standard editions and translations by Heiberg (II, 516, note 3), Heath (307) and Ver Eecke (II, 529, footnote 2), whose main interest was the "restoration" of the mathematical work of the Greeks.

We now turn to the way in which the theorems and their author were judged. Ecchellensis and Borelli seem to have been prejudiced with respect to Islamic mathematicians. They write that the theorems by Kūhī are "indeed easy"[1] They do not pass judgement of Kūhī, at least not explicitly,[2] but elsewhere they point out (to my mind incorrectly) that Nasawī was "not quite experienced in geometry."[3] Isaac Barrow, on the other hand, was more positive with respect to the Islamic scientific tradition, and at one point he intended to study Arabic. He apparently learned the Arabic alphabet, for his edition of the *Lemmas* contains a few names and technical terms in Arabic. Barrow introduced Kūhī's theorems by the words: "Then, the commentator Nasvaeus explained the other cases of this fifth Theorem according to Abi Sahl Cuhensis, the famous Mathematician, somehow as follows."[4] In his translation, Voogt uses "de doorlugtige wiskonstenaar Abi Sahl Cuhensis" (the illustrious mathematician Abū Sahl Kūhī), and we have no reason to doubt that this was Voogt's own judgement as well. Elsewhere in his work, Voogt (Introduction, p. 3) also praises Islamic improvements in arithmetic: "Pythagoras ..., and his successors, as well as the Egyptians, and after them the Greeks and

1. Ecchellensis and Borelli, p. 383: Reliquae duae propositiones superadditae ad Arabibus faciles quidem sunt.

2. It can be shown that their implicit judgement of Kūhī was negative, see my paper "Kuhi Latinus", to appear.

3. Ecchellensis and Borelli, p. 396: … Almochtasso non admodum in Geometris versati.

4. Deinde Adnotator *Nasvaeus* caeteros casus hujusce quinti Theorematis ad mentem *Abi Sahl Cuhensis* percelebris athematici, hoc fere modo exponit (Barrow, 269).

the Arabs have notably increased arithmetic.”

2. The *Ornamentation of the Lemmas* of Abū Sahl Kūhī; Arabic text and English translation.

This section contains an English translation of “Proposition 5” of the *Lemmas* of “Archimedes”, the two theorems by Kūhī, and two intermediary theorems by Nasawī. I have inserted numbers in square brackets [1], [2], … to facilitate comparison with Voogt’s paraphrase in Section 3. These numbers need not always be consecutive. The translation is based on the uncritical Hyderabad edition[1] of the *Middle Books*. The text has been compared to the recent facsimile of the *Middle Books* published by Dr. J. Aghayani Chavoshi (Tehran 2005, 192, 194-197). A table of contents of Dr. Chavoshi’s facsimile is presented at the end of this paper. Arabic letters indicating points in the geometrical figures have been transcribed in the translation as follows: *alif* = *A*, *bā* =*B*, *jīm* =*G*, *dāl* =*D*, *hā* = *E*, *zā* = *Z*, *ḥā* = *H*, *ṭā* =*T*, *kāf* = *K*, *lām* =*L*, *mīm* = *M*, *nūn* = *N*, *ʿayn* = *O*, *sīn* = *S*.

I include an English translation of the preface to the *Lemmas*, in which Kūhī is mentioned. This preface is of additional interest because there are (strange) references to other works by “Archimedes”. None of these works, if they ever existed, have come down to us, and to my mind, these references make Archimedes’s authorship of the *Lemmas* very unlikely.

The Latin translations by Greaves and Ecchellensis correspond closely to the Arabic original. The reader may find the Latin translations by Ecchellensis of proposition 5 and of the introduction to the *Lemmas* in vol. 2 of Heiberg’s edition (514-516, 511 footnote).

1. *Kitāb Makhūdhāt Arshimīdis, Taḥrīr Naṣīr al-Dīn al-Ṭūsī*, Hyderabad: Osmania Oriental Publications Bureau, 1359 A.H. (lunar), reprinted in: F. Sezgin, ed., *A Collection of Mathematical and Astronomical Treatises as Revised by Naṣīraddīn aṭ-Ṭūsī,* Frankfurt: Institute for the History of Arabic-Islamic Science, 1998, Islamic Mathematics and Astronomy, vol. 48, pp. 100-101, 108-115. Note that the following changes have to be made to the Hyderabad edition: p. 108, line 10, omit the second word *wa-naṣil*; line 11 change *ʿamūd* to *wa-ʿamūdun ʿalā*. A few self-evident changes have to be made to the labels of points in the geometrical figures (from *jīm* to *ḥāʾ* etc.).

Translation of the Preface to the *Lemmas*

Edition[1] of the Book of *Lemmas* of Archimedes, translation of Thābit ibn Qurra, and commentary by the Competent Scholar Abū al-Ḥasan ʿAlī ibn Aḥmad Nasawī. Fifteen Propositions.

The Competent Scholar (= Nasawī) said: This treatise is attributed to Archimedes. It contains beautiful proposition, few in number but with many benefits, on the principles of Geometry; (they are) extremely good and subtle. The contemporaries have added them to the collection of middle books which have to be read between the book of Euclid (the *Elements*) and the *Almagest*. But in some of its propositions are places which require other propositions, with which the proof of that proposition is completed. In some of them, Archimedes referred to propositions which he had presented in other works by him. Thus he said: "as we have proved in the Right-Angled Figures, and as we have proved in our Commentary on the Comprehensive Treatise on Triangles, and as has been proved in our Treatise on Quadrilateral Figures." And in the fifth proposition he (Archimedes) presented a proof in a way in which is (only) a special aspect. Then after that, Abū Sahl Qūhī [2]wrote a treatise which he calls *Ornamentation of the Book of Archimedes on Lemmas*. In it, he presented a proof of this proposition in a more general and more beautiful way, together with the addition and compositon of ratios involved in it (the proof).

When I found the situation like this, I (= Nasawī) made a commentary to the obscure places in this work, by way of notes appended to the text. I have explained the things to which he referred by means of propositions which I invented. Of the propositions of Qūhī, I have presented two propositions which are necessary in the fifth proposition (by "Archimedes"), and I have omitted the rest because I did not want to be too lengthy and because I did not need it. With God is success.

1. The name of the editor, Naṣīr al-Dīn Ṭūsī, is not mentioned here.
2. Qūhī is an alternative spelling of Kūhī, often found in Arabic geometrical texts.

Translation of proposition 5 by "Archimedes," the two intermediary theorems by Nasawī and the two theorems by Kūhī

(Figure 4) [1] If there is a semicircle *AB* and a point *G* is marked arbitrarily on its diameter, and two semicircles *AG*, *GB* are constructed on the diameter, and from *G* a perpendicular *GD* is drawn to *AB*, and on both sides of it, two circles are drawn, which are tangent to it and tangent to the semicircles, then the(se) two circles are equal.

[2] Proof: Let one of the circles touch *GD* at *Z* and the semicircle *AB* at *H* and semicircle *AG* at *K*. [3] We draw the diameter *ZE*, then it is parallel to the diameter *AB*, since the two angles *EZG*, *AGZ* are right (angles). [4] We join *HE*, *EA*, then line *AH* is straight, because of what has been explained in the first proposition.[1] [5] Let *AH* and *GZ* meet at *D*; (they will meet) since they are drawn from *AG* at (angles whose sum is) less than two right angles.

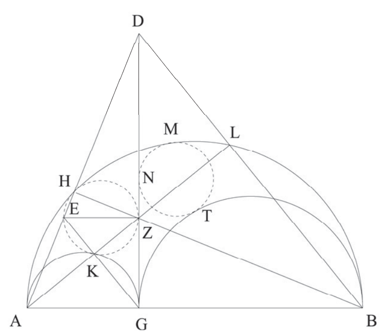


Figure 4

[6] We also join *HZ* and *ZB*. Then *BH* is also a straight line, because of what we have mentioned, and it is a perpendicular to *AD*, since angle *AHB* is a right angle because it is located in semicircle *AB*. [7] We join *EK* and *KG*, then *EG* is also a straight line. We join *ZK* and *KA*, then *ZA* is a straight line. [8] We extend it towards *L* and we

1. In the first proposition of the *Lemmas* the following is proved (in the notation of the present proposition): If *EZ* and *AB* are parallel diameters of circles which are tangent at *H*, then *HEA* and *HZB* are straight lines.

join *BL*, and this (line) is also perpendicular to *AL*. We join *DL*.

Since *AD* and *AB* are two straight lines, and from *D* a perpendicular *DG* has been drawn towards *AB*, and from *B* a perpendicular *BH* has been drawn towards *DA*, which (perpendiculars) intersect at *Z*, and *AZ* has been drawn towards *L*, and it is perpendicular to BL, therefore *BLD* is a straight line; as we have proved in the propositions which we have made in the commentary of the *Treatise on the Right-Angled Triangles*.[1]

Since the two angles *AKG* and *ALB* are right angles, [9] *BD* and *GE* are parallel. [10] So the ratio of *AD* to *DE*, which is equal to the ratio of *AG* to *EZ*, is equal to the ratio of *AB* to *BG*. [11] Thus the rectangle *AG* by *GB* is equal to the rectangle *AB* by *EZ*. [12] In the same way it can be proved for the circle *TMN* that the rectangle *AG* by *GB* is equal to the rectangle *AB* by its diameter. [13] It is proved by this that the two diameters of the circles *ZHK* and *TMN* are equal, and therefore the two circles are equal. That is what we wanted.

[14] The Scholar (Nasawī) said: What he took from the commentary of the Right-Angled Triangles can be proved by means of a lemma, which is a useful proposition in the original (meaning: in its own right?), and especially for acute-angled triangles. We also need it in the sixth proposition of this book. It is as follows:

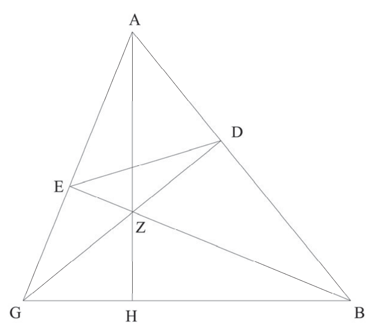


Figure 5

1. Here "Archimedes" is speaking. *The Treatise on the Right-Angled Triangle*s has not come down to us. Below, Nasawī proves in his two intermediate theorems that the two lines *BL* and *LD* are on one straight line. These two theorems boil down to the statement that the three altitudes of a triangle (*ABD*) pass through one point (*Z*).

(Figure 5) In triangle *ABG*, the two perpendiculars (i.e., altitudes) *BE*, *GD* have been drawn, intersecting at *Z*. *AZ* as been joined and extended towards *H*. Then it is perpendicular to *BG*. (Proof:) So we join *DE*. Then the two angles *DAZ*, *DEZ* are equal, because the circle which circumscribes triangle *ADZ* passes through point *E*, since angle *AEZ* is a right angle, and they (the two angles *DAZ*, *EDZ*) stand in it (the circle) on the same arc. Again, angle *DEB* is equal to angle *DGB* since the circle which circumscribes triangle *BDG* also passes through point *E*. So in the two triangles *ABH*, *GBD*, the two angles *BAH*, *BGD* are equal and angle *B* is common, so angle *AHB* is equal to the right angle *GDB*. So *AH* is perpendicular to *BG*.

(Figure 6) And since this preliminary has now been proved, let us repeat from the figure which Archimedes presented (Figure 4) the two lines *DA*, *AB* and the perpendiculars *DG*, *BH*, *AZ*, *BL* and the line *DL*. We say: if line *BLD* is not a straight line, let us join the straight line *BSD*. Then angle *BSA* is (a) right (angle) by the above-mentioned preliminary. But angle *BLA* was (shown to be) a right angle. Then the interior angle in triangle *BLS* is equal to the exterior angle opposite to it. This is absurd. Therefore line *BLD* is a straight line.

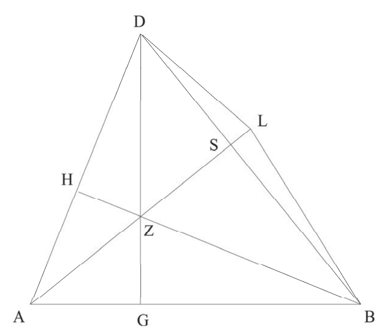


Figure 6

(Ṭūsī is speaking here). [15] Then he (= Nasawī) presented two propositions by Abū Sahl Qūhī. [16] The first of them is as follows. If the two semicircles are not tangent but intersecting, and the perpendicular (is drawn) from the point of intersection, the statement is as before.

(Figure 7) Thus let there be semicircles *ABG*, *ADE* and *ZDG*. The two semicircles intersect at *D*. *BH* is drawn perpendicular to *AG* from *H*. Circle *TKL* is tangent to circle *AKG* at *K*, to circle *ZLG* at *L*, and to the perpendicular at *T*. We say: it is equal to the circle which is at the other side according to the same description.[1]

(Proof:) Thus let us draw *TS* parallel to *AG*, and let us join *GK*, then it passes through *S*, as Archimedes proved.[2] We extend it until it meets *HB* at *N*. We join *TG*, then it passes through *L*, and we extend it towards *M*. We join *AM* and *MN*, then they are one straight line. We join *SZ*, then it passes through *L*. We join *AK*, then it passes through *T*.

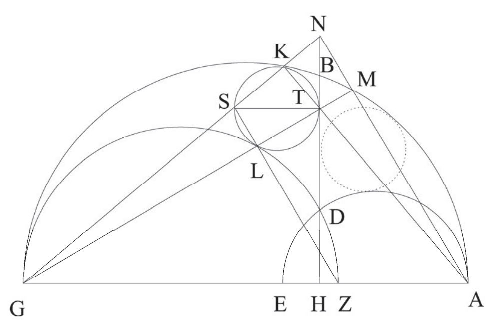


Figure 7

[17] Line *AMN* is parallel to line *ZS*. Thus the ratio of *GN* to *NS*, I mean the ratio of *GH* to *TS*, is equal to the ratio of *GA* to *AZ*. [18] So the rectangle *GH* by *AZ* is equal to the rectangle *GA* by *TS*. [19] Since in the two circles *GDZ*, *EDA*, *HD* is perpendicular to the chords[3] *GZ* and *EA*, the rectangle *GH* by *HZ* is equal to the square of *HD*, and the rectangle *AH* by *HE* is also equal to it. So the rectangle *GH* by *HZ* is equal to the rectangle *AH* by *HE*. [20] Thus the ratio of *GH* to *HA* is equal to the ratio of *EH* to *HZ*, that is, equal to the ratio of the

1. I have added the dotted circle to the figure for sake of clarity.
2. Kūhī uses all the time the first proposition of the *Book of Lemmas* of Archimedes, see my footnote above.
3. It would be more correct to say that *GZ* and *EA* are the two diameters.

remainder *GE* to the remainder *ZA*, so the rectangle *GH* by *ZA*, which is equal to the rectangle *GA* by *TS*, is equal to the rectangle *HA* by *GE*. [21] If there is on the other side a circle according to the same description, we can also prove by this argument that the rectangle *GA* by the diameter of that circle is equal to the rectangle *HA* by *GE*. Thus it is proved that the diameters of the two circles are equal.

[22] The second (proposition) is this: He (Kūhī) said: If the two semicircles are neither tangent nor intersecting, but removed from one another, and the perpendicular passes through the meeting point of two equal tangents to them, the statement is also like this.

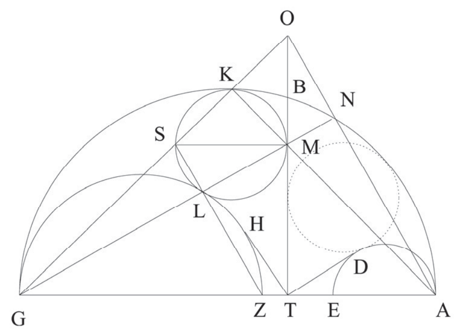


Figure 8

(Figure 8) Thus let the semicircles *ABG*, *ADE*, *ZHG* be as we have described. Lines *TD* and *TH* are tangent to the semicircles at *D* and *H*, and equal, and they meet at *T* (on diameter *AB*). Line *BT* is a perpendicular passing through point *T*, erected to *AG*. Let circle *MS* touch it at *M*, and let circle *MS* touch circle *ABG* at *K* and circle *ZLG* at *L*. [23] We draw diameter *MS* parallel to *AG* and we join *GK*, then it passes through *S* and meets perpendicular *TB* at *O*. We join *AK*, then it passes through *M*. We join *SZ*, then it passes through *L*. We join *GM*, then it passes through *L* and we extend it toward *N*. We join *AO*, then it passes through *N* and [24] it is parallel to *ZS*. Thus the ratio of *GO* to *OS*, I mean the ratio of *GT* to *MS*, is equal to the ratio of *GA* to *AZ*. [25] Therefore the rectangle *GT* by *AZ* is equal to the rectangle *GA* by *MS*. [26] By the same argument it is proved that the rectangle

AT by EG is equal to the rectangle *GA* by the diameter of the circle which is on the other side (of the perpendicular *BT*).

[27] Since the rectangle *AT* by *TE* is equal to the square of *TD*, which is equal to the square of *TH*, which is equal to the rectangle *GT* by *TZ*, the rectangle *AT* by *TE* is equal to the rectangle *GT* by *TZ*, [28] so the ratio of *AT* to *GT* is equal to the ratio of *TZ* to *TE*, and equal to the ratio of the sum *AZ* to the sum *GE*. So the rectangle *GT* by *AZ* is equal to the rectangle *AT* by *EG*. [30] But it has been proved that the rectangle *GT* by *AZ* is equal to the rectangle *GA* by *MS*, and that the rectangle *AT* by *EG* is equal to the rectangle *GA* by the diameter of the other circle. So the two diameters are equal, and the two circles are equal. That is what was desired. [31]

3. The Dutch paraphrase of the extant fragment of Kūhī's Ornamentation of the Lemmas.

I now present the relevant Dutch passages from the work *Euclidis Beginselen der Meet-Konst* (Foundations of Geometry by Euclid) by C.J. Voogt (Amsterdam 1695), followed by an English translation. Pages have been indicated between square brackets, thus [p. 218] for page 218.

[p. 189] 't Aanhangsel des zesten Boeks.

Wy hebben uyt lust, veelvuldig gebruik, en aangemerkte nut des Meet-konsts hier aangehangen deze drie-en-dertig Voorstellen, onder de welke in rang gaan de vijfthien voorbewijsen des grooten Wis-konstenaars *Archimedis* van Siracusen, sijnde 't twintigste Voorstel deses Aanhangṣẹls sijn eerste …

Translation:

Appendix to the sixth book.

We have appended here these thirty-three Propositions, because of the delight, the many uses, and the above-mentioned utility of Geometry. They include the fifteen Lemmas of the great Mathematician Archimedes of Syracuse in their proper order. The twentieth Proposition of this Appendix is his first (the first proposition of the Lemmas) …

What follows is the Dutch text and translation of "proposition 24" in the Appendix. Some printer's errors in the edition have been corrected; the corrections have been indicated by an underdot. Example: the error *OPN* on page 219 has been corrected to *ẠPN*. In his text, Voogt prints numerous marginalia with references to theorems, which marginalia are indicated with superscipt lower-case letters (*a*, *b*, *c* and so on) in his main text. For example, there is a superscript reference d after the line segment *AC* in the first line of page 219; in the margin next to the line there is the reference: d. 31 prop. 1 b. (meaning by Proposition 31 of Book 1 of the *Elements*). All these superscript references and marginalia have been omitted in the text and translation below. The references to the figures have been added by me. The dotted lines in my figures are also drawn as dotted lines in Voogt's figures, and italicized words in my text and translation were also printed in italics by Voogt. In my translation I have inserted numbers in square brackets [1], [2], … in order to facilitate comparison with the English translations in Section 2.

[p. 218] 't Vier-en-twintigste Voorstel.

(Figure 9)[1] *Indien op een rechte streep AC en deszelfs stukken AD en DC drie halve ronden ABC, AED en DFC beschreven worden, enop de rechte AC word uyt de scheyding D gerecht een loodryhangende GD, soo sullen de ronden BHE en LFM in 't seynstuk beschreven, soodanig datze soo de loodryhangende DG, als de halfronden raken, malkanderen gelijk sijn.* [p. 219]

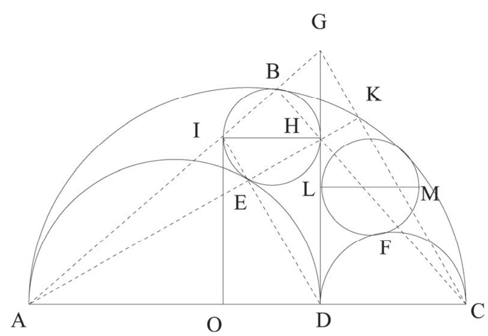


Figure 9

't Bewijs. Trek de midstreep *HI* evenwijdig met *AC*, daarom *H* de raking, en de getogene *AI* en *BI*, *B* de raking sijnde, een rechte. Nu nadien de hoek *ABC* recht is, soo sijn beyde hoeken *BAC* en *ACB*

1. Point *O* and line *IO* in Figure 9 are not used in the text.

gelijk een rechte, dat is, de hoek *BAC* minder dan een rechte. Maar de hoek *ADH* recht sijnde door 't opstel, soo sijn beyde hoeken *ADH* en *BAD* minder dan twee rechte, en vervolglijk *AB* en *DH* komen in *G* 't samen, maar *BH* en *CH* is een rechte, loodryhangende op *AG*, ook sijn *IE* en *ED* een rechte, als ook *AH* en *KH* een rechte. Trekkende *CK*, soo sal om de rechte hoeken *AED* en *AKC*, die malkanderen gelijk sijn, *CK* evenwijdig met *DI* sijn, makende alsoo *CG* evenwijdig met *DI*. Waar door *AD* tot *HI* is, als *AG* tot *GI*, en *AG* tot *GI* als *AC* tot *CD*, dat is, *AD* tot *HI*, als *AC* tot *CD*, of 't rechthoek *ADC* gelijk 't rechthoek *AC*, *HI*. Met dezelve swier word ook aan 'd andere kant bewesen 't rechthoek *ADC* gelijk 't rechthoek *AC*, *LM*, of *HI* gelijk *LM*, en 't vierkant *HI* gelijk 't vierkant *LM* Maar nadien de ronden tot malkanderen sijn, als de vierkanten hunner midstreepen [gelijk hier na in 't 2de Voorstel des 12den Boeks sal gethoont worden] daarom sijn de ronden *BHEI* en *LFM* malkanderen gelijk: dat te bewijṣen was.

Byvoegsel.

Dat GC een rechte streep is, heeft die griek, die dit gevonden heeft, of eenige Arabiers gethoont, dat Ali Abul Hasan *tot sijn behulp genomen heeft. Wy zullen't dus thoonen. (Figure 9)*

Trekkende CG. Nu is om de gelijke hoeken ABC en CDH, de hoek BAC gelijk de hoek DHC, dat is, gelijk de hoeken DGC en GCH: waar uyt volgt, om de gelijke hoeken GAH en GCH, de hoeken CAH en HGK malkandere.n gelijk te sijn. Maar de hoeken AHD en GHK malkanderen gelijk sijnde, soo volgt de hoeken ADH en HKG malkanderen gelijk te sijn , dat is, de hoek HKG of AKC recht, en vervolglijk AK ontmoet d'omring ABC in K, en voort om de gelijke hoeken AED ẹn AKC, de rechten DI en CG evenwijdigen.

Voorts brengt Nasvaeus *hier noch twee voorvallen op na't ontwerp van den doorlugtigen Wiskonstenaar* Abi Sahl Cuhensis*, die dese sijn.*

Indien de halfronden ẠPN en OPC malkander in P snijden, waar door de loodryhangende DG op AC gerecht is. Sijnde AC en HI evenwijdige. Trekkende alles als voren. Om 'd evenwijdige CK en IN , is AC tot CN , als AG tot GI.

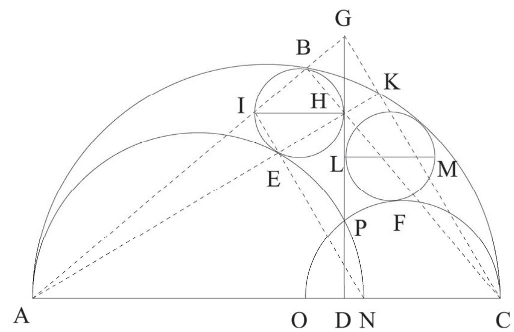


Figure 10

[*p. 220] Maar AG tot GI sijnde, als AD tot HI, soo is AC tot CN , als AD tot HI, en vervolglijk 't rechthoek CN, AD gelijk 't rechthoek AC, HI. Wederom 't rechthoek CDO is gelijk 't vierkant DP, welke vierkant DP is gelijk 't rechthoek ADN, daarom 't rechthoek CDO gelijk 't rechthoek ADN, dat is, beyde van 't rechthoek ADC nemende, 't rechthoek AD, NC gelijk 't rechthoek CD, AO, en de rechthoeken AD, NC en AC, HI malkanderen gelijk sijnde door 't gethoonde't. rechthoek AC, HI gelijk 't rechthoek CD, AO. Desgelijks bewijst men aan d' andre kant de rechthoeken CD, AO en AC, LM gelijk te sijn, makende alsoo HI gelijk LM.*

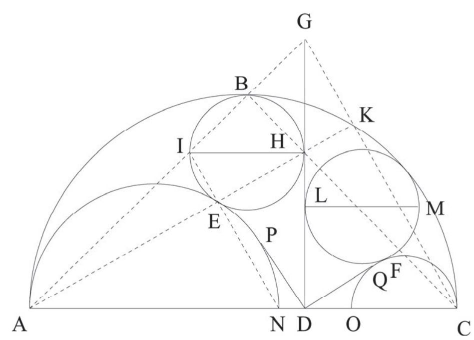


Figure 11

Indien de halfronden AEN en CFO malkanderen niet raken, maar de gelijk rakende DP en DQ malkanderen in D ontmoeten, en dan DG *loodry-hangende op AC. Sijnde voorts alles als boven. Om 'd evenwijdige CG en IN , is AD tot HI , als AG tot GI, en AG tot GI, als AC tot CN, dat is, AD tot HI, als AC tot CN, of 't rechthoek AD, CN gelijk 't rechthoek AC, HI. Wederom nadien 't vierkant DP gelijk 't rechthoek AND, en 't vierkant DQ gelijk 't rechthoek CDO is, soo sal, om de gelijke DP en DQ door 't opstel, dat is, om de gelijke vierkanten DP en DQ, 't rechthoek AND gelijk 't rechthoek CDO sijn. Nemende dese beyde van 't rechthoek ADC, soo blijft 't rechthoek AD, CN gelijk 't rechthoek CD, AO, en om de gethoonde gelijke rechthoeken AD, CN en AC, HI, 't rechthoek CD, AO gelijk 't rechthoek AC, HI. Desgelijks word aan d' andere kant bewesen de rechthoeken CD, AO en AC, LM malkanderen gelijk te sijn, makende also HI gelijk LM.*

Nadien 't nodig is om 't stip D te vinden, soo lust ons dat na te vorschen. Door 't getoonde sijn de rechthoeken AND en CDO malkanderen gelijk, dat is, AD tot CD als DO tot DN, en 't samensettende AO tot CN als DO tot DN, en verwisselende AO tot DO als CN tot DN, en 't samensettende AO en CN makende AC en NO tot ON, als CN tot DN waar door 't stip D ook gegeven is.

English translation

(In the following translation, "rectangle *ADC*" means the rectangle whose length and breadth are equal to *AD* and *DC* respectively. My own explanatory additions are in parentheses.)

[p. 218] The twenty-fourth proposition.

[1, 2] *If on a straight line AC and its parts AD and DC three semicircles ABC, AED and DFC are described, and on the straight line AC from the point of separation D a perpendicular GD is erected, then the circles BHE and LFM which are described in the sickle, in such a way that they are tangent to the perpendicular DG and the semicircles, will be equal. [p. 219].*

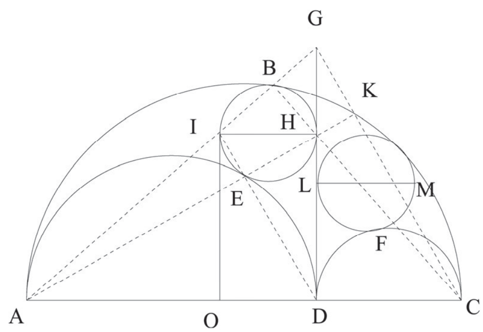


Figure 9

Proof. [3] Draw the diameter *HI* parallel to *AC*, then *H* is the point of tangency, [4] and since *B* is the point of tangency, the lines *AI* and *BI* are one straight line. [5] Now since the angle *ABC* is a right angle, both angles *BAC* and *ABC* are right angles, that is to say that the angle *BAC* is less than a right angle. But since the angle *ADH* is a right angle by hypothesis, both angles *ADH* and *BAD* are less than two right angles, and consequently *AB* and *DH* meet at *G*, [6] but *BH* and *CH* are a straight line, perpendicular to *AG*; [7] and *IE* and *ED* are also a straight line, [8] and also *AH* and *KH* are a straight line. If we draw *CK*, then, because of the right angles *AED* and *AKC*, which are equal to one another, [9] *CK* will be parallel to *DI*, so *CG* will be parallel to *DI*. [10] Therefore, as *AD* is to *HI*, so is *AG* to *GI*, and as *AG* is to *GI*, so is *AC* to *CD*, that is, as *AD* is to *HI* , so *AC* is to *CD*, [11] or the rectangle *ADC* is equal to the rectangle *AC*, *HI*. [12] In the same way it is proved that, on the other hand, the rectangle *ADC* is equal to the rectangle *AC*, *LM*, [13] or *HI* equal to *LM*, and the square of *HI* equal to the square of *LM*. But since the circles have the same ratio as the squares of their diameters, as will be proved below, in the second proposition of the 12th Book (of Euclid's *Elements* in Voogt's translation), therefore the circles *BHEI* and *LFM* are equal: which was to be proved.

Appendix

[14] *That GC is a straight line has been shown by that greek, who has found this (proposition), or by some Arab; which* Ali Abul Hasan (Nasawī) *has taken as an auxiliary. We will show it in this way*.

We draw CG. Now, because of the equal angles ABC and CDH, angle BAC is equal to angle DHC, that is, equal to the angles DGC and GCH: from which follows, because of the equal angles GAH and GCH, that the angles CAH and HGK are equal to one another. But since the angles AHD and GHK are equal to one another, it follows that the angles ADH and HKG are equal to one another, that is, the angle HKG or AKC is a right angle, and as a consequence, AK meets the circumference ABC at K. Further, because of the equal angles AED and AKC, the straight lines DI and CG are parallel.

[15] *Further*, Nasvaeus (Nasawī) *presents two more cases here, as designed by the illustrious Mathematician* Abi Sahl Cuhensis (Abū Sahl Kūhī), *as follows*.

[16] (Figure 10) *If the semicircles APN and OPC intersect one another at P, through which the perpendicular DG is drawn to AC. AC and HI are parallel. We draw everything as before.* [17] *Because of the parallels CK and IN: as AC is to CN, so is AG to GI.*

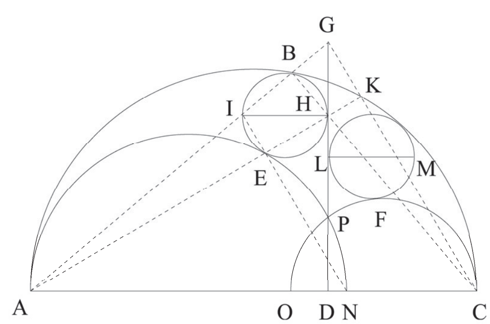


Figure 10

[p. 220] *But since, as AG is to GI, so is AD to HI, therefore as AC is to CN , so AD is to HI,* [18] *and as a consequence the rectangle CN, AD is equal to the rectangle AC, HI.* [19] *Again, the rectangle CDO is* *equal to the square of DP, which square DP is equal to the rectangle ADN, so the rectangle CDO is equal to the rectangle ADN,* [20] *that is, if we subtract each of them from the rectangle ADC the rectangle AD, NC is equal to the rectangle CD, AO. The rectangles AD, NC and AC, HI are equal as has been shown, so the rectangle AC, HI is equal to the rectangle CD, AO.* [21] *Similarly one can prove on the other hand that the rectangles CD, AO and AC, LM are equal. Thus HI is equal to LM.*

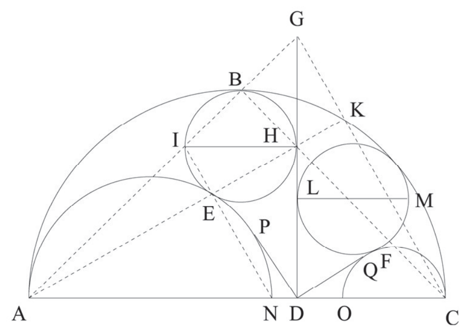


Figure 11

(Figure 11) [22] *If the semicircles AEN and CFO do not touch, but the equal tangents DP and DQ meet at D, and then DG is perpendicular to AC. Further everything is as above.* [24] *Because of the perpendiculars CG and IN, as AD is to HI, so is AG to GI, and as AG is to GI, so is AC to CN, that is, as AD is to HI , so is AC to CN,* [25] *or the rectangle AD, CN is equal to the rectangle AC, HI.* [27] *Again, since the square DP is equal to the rectangle AND, and the square of DQ is equal to the rectangle CDO, therefore, since DP and DQ are equal by hypothesis, that is to say, because of the equal squares of DP and DQ, the rectangle ADN will be equal to the rectangle CDO.* [28] *If we subtract these two from the rectangle ADC, then the remainders, rectangle AD, CN and rectangle CD, AO are equal*. [29] *Since rectangles AD, CN and AC, HI were shown to be equal, the rectangle CD, AO is equal to the rectangle AC, HI*. [26] *Similarly it is proved on the other side that the rectangles CD, AO and*

AC, LM are equal [30] *Thus HI is equal to LM .*

Since it is necessary to find the point D, we [i.e., Voogt] like to investigate this. Because of what has been shown, the rectangles AND and CDO are equal to one another, that is, as AD is to CD, so is DO to DN, and, putting together, as AO is to CN , so is DO to DN, and, exchanging, as AO is to DO, so is CN to DN, and, putting together, as AO and CN, which makes AC and NO, is to ON , so is CN to DN, by which the point D is also given.

4. Comparison of the Arabic original with the Dutch paraphrase by Voogt.

In the following comparison between the originals in Section 2 by "Archimedes", Nasawī and Kūhī and the paraphrase by Voogt in Section 3, we will use the nubmers in square brackets [1], [2], etc., which I have inserted in the English translations.

The reader may have noticed that the Dutch paraphrase by Voogt in Section 3 differs to some extent from the Arabic original in Section 2. The difference is explained by the fact that Voogt used as his main source the paraphrase by Isaac Barrow, although he may have consulted the Ecchellensis-Borelli translation as well. The close connection between Voogt and Barrow can be shown by the following arguments:

1. For labeling points in the geometrical figures (9, 10, 11), Voogt uses exactly the same letters as Barrow, which are very different from the letters in the Ecchellensis-Borelli translation (and also different from the letters in the Latin translation by Greaves). Voogt's figure 9 includes line *IO* which is redundant in Voogt's own text. The same line *IO* occurs in Barrow's figure 267 and is used by Barrow further on in a remark of his own after the sixth proposition of his edition of the *Lemmas*.

2. The first sentence of [14] is not very intellegible in Voogt's edition. We can explain it as a sloppy translation by Voogt of the following passage in Barrow: "Either that Greek, who first collected these lemmas, or rather some Arab cited his work on right-angled triangles (in the passage) where *CG* is shown to be a straight line. Hence Ali Abu'l-Hasan took this (i.e., the following, namely Barrow's paraphrase of the theorems of Nasawī, see Figures 5, 6) in the way of auxiliary."[1] The author to whom Barrow refers as "that Greek or rather some Arab" is our "Archimedes."

3. Broken lines in Barrow's figures are displayed as broken lines in

1. Sive *Graecus* ille, qui hec lemmata primus collegit, sive potius *Arabum* aliquis, quo *CG* rectam lineam esse ostenderet, citat Opusculum suum de Trigonis Rectangulis. Inde vero *Ali Abu'l-Hasan* hoc adjumenti accepit."

Voogt's figures (except the diameters of the complete circles). In Arabic manuscripts the figures were all drawn by hand so this technique was not available to Kūhī. Even in the Latin translation, all lines in the figures are continuous.

4. In [16] and [22], neither Voogt nor Barrow explains Kūhī's two theorems too clearly. The reader only finds out in the end what exactly Kūhī wanted to prove. The description in the originals is much clearer.

5. The marginalia in Voogt's edition resemble the marginalia in Barrow's, and are also indicated by superscript lowercase letters in the text. Voogt has even more marginalia than Barrow.

But Voogt's paraphrase is not a direct translation from Barrow's edition. Voogt deleted Barrow's mathematical symbolism, such as $IE + ED$ for line *IED*; $GD \perp DA$ for *GD* perpendicular to *DA*; *AD.IH*:: *AG.GI* for the ratio of *AD* to *IH* is the ratio of *AG* to *GI*; $AD \times IH$ for the rectangle contained by *AD* and *IH*; *DP*q for the square of *DP* , and so on. See below for an example. In this sense, Kūhī's original is closer to the Dutch version than to Barrow's Latin version.

Passage [8]-[9] is interesting because of the errors that were made in its transmission. "Archimedes" first states that *AE* and *HE* are one straight line (this is correct and proved in proposition 1 of his *Lemmas*.) He or she then introduces *K* as the point of intersection of *AH* extended and the circle *ABC*. Then *AK* is a straight line by definition. "Archimedes" draws *CK* and *GK* and says that they are a straight line, according to a theorem which he proved in his commentary to the *Treatise on the Right-Angled Triangles*.

Here Barrow is less clear than the original because he implicitly defines *K* as a point on *AH* extended. His text reads (in my translation).

"$IE + ED$ & $AE + EK$ are straight lines. But $GD \perp DA$, & if one draws $CK \perp KA$, then the extension *CKG* will be a straight line. Because *ED*||*CG*, because of the right angles *AED*, *AKC*, we will have *AD.IH*:: (*AG.GI* ::)*AC.CD*…"[1]

---

1. $IE+ ED$, & $AE + EK$ etiam rectae. Est autem $GD \perp DA$, & juncta $CK \perp KA$, quare producta *CKG* recta erit. Quoniam vero *ED*||*CG*, propter rectas *AED*, *AKC*, erit *AD.IH*:: *(AG.GI* ::) *AC.CD*.

Voogt is even less clear than Barrow. Voogt does not say that *AE* and *EH* are straight lines, nor does he define point *K*. In [8] he mentions segments *AH* and *KH* and says that they are a straight line, with a reference *l* to proposition 31 of Book III of the *Elements*. This proposition shows that the angle in a semicircle is a right angle, but the reference is useless because *AHK* is a straight line by definition (that point *E* lies on *AHK* has to be proved). Voogt then implicitly assumes in [9] that *CKG* is a straight line.

Nasawī provides two intermediary theorems in [14] (Figures 5, 6) which solve the difficulty. They boil down to the fact that point *H* is the intersection of the two altitudes *GD* and *CB* in triangle *ACG*. Then it can be proved that *H* is also on the third altitude, so *AH* extended meets *CG* at right angles in the point of intersection *K*.

Barrow repeats the proof by Nasawī. But Voogt provides a different proof in his *Byvoegsel* (Appendix) in [14] (Figure 9). Here it is assumed that *CB* and *GD* are altitudes in triangle *ACG*. Voogt does not care to tell his reader how point *K* should be defined. Let us try to derive the implicit definition from the proof. Voogt first notes (correctly) that (because *B* and *D* are right angles), $\angle BAC = \angle DHC = \angle DGC + \angle HGC$ (the exterior angle of a triangle is the sum of the two non-adjacent interior angles). Then he remarks that $\angle GAH = \angle GCH$. The text has a reference to Euclid's *Elements* III: 22, to the effect that the sum of opposite angles of a concyclic quadrilateral is equal to two right angles. This theorem is irrelevant, so it is likely that Voogt wanted to refer to *Elements* III: 21, stating that the two angles $\angle GAH$ and $\angle GCH$ are equal because they stand on the same arc of a circle. Figure 9 shows that the circular arc in question must be arc *BK* of circle *ABC*; *Elements* III: 21 can only be used if point *K* is on the circle and line *CKG* is a straight line. Voogt assumes the result which he has to prove, so his proof is a failure. Thus the transmission led from a theorem by "Archimedes", which was clarified by Nasawī, via an unclear exposition by Barrow, to an incorrect proof by Voogt.

Voogt made an interesting addition, namely the construction in [31] of the point on the diagonal from which the two equal tangents can be drawn to the small semicircles. This explanation is found neither in the extant fragment of Kūhī's *Ornamentation of the Lemmas*, nor in one of the Latin translations, nor in Barrow's edition.

Thus, Kūhī's two theorems were fascinating to a whole series of mathematicians in the Islamic and the European traditions.

References

Archiemedes, *Archimedes von Syrakus vorhandene Werke aus dem Griechischen übersetzt und mit erlāuternden und kritischen Anmerkunden begleitet* von Ernst Nizze, Stralsund 1824**.**

Barrow, I., Archimedis Opera; Apollonii Pergaei Conicorum libri IIII; Theodosii Sphrica, methodo nova illustrata, et succinct demonstrata, per Is. Barrow, London 1675.

Berggren, J.L., "Tenth-century mathematics through the Eyes of Abū Sahl al-Kūhī", in J.P. Hogendijk, A.I. Sabra, eds., *The Enterprise of Science in Islam, New Perspectives*, Cambridge Mass., 2003.

C.G.Gillispie, ed., *Dictionary of Scientific Biography*, vol. I (New York 1970).

Heath, T.L., *The Works of Archimedes*, Cambridge 1896.

Heiberg, J.L., *Archimedis Opera Omnia com commentariis Eutocii*, 1910.

Merrifield, C.W., "On a geometrical proposition, indicating that the property of the radical axis was probably discovered by the Arabs", *Proceedings of the London Mathematical Society* 2, 1866-1869, pp. 175-177, reprinted in F. Sezgin, *Islamic Mathematics and Astronomy* vol. 79, Frankfurt 1998, pp. 221-223, also found on the internet at:
http://plms.oxfordjournals.org/cgi/reprint/s1-2/1/175.pdf

Rosenfeld, B.A.; Ihsanoğlu, E., *Mathematicians, Astronomers and Other Scholars of Islamic Civilization and their Works (7th-19th c.)*, Istanbul 2003.

Sezgin, F., *Geschichte des arabischen Schrifttums*, Leiden, vol. 5 (1974), 6 (1978).

Toomer, G.J., *Eastern Wisedom and Learning*, Oxford 1996.

Van der Aa, A.J., *Biographisch Woordenboek der Nederlanden*, Haarlem 1852-1878.

Ver Eecke, Paul, *Les Oeuvres Complètes d'Archimède, suivies des commentaires d'Eutocius d'Ascalon*, Bruges 1921.

Voogt, C.S., *Euclidis beginselen der Meet-Konst Waar bij 't 16 Boeck Flussatis Candalla, door C.J. Voogt Geometra*. Amsterdam: Johannes van Keulen, 1695.

Ecchellensis and Borelli is found in *Apollonii Pergaei Conicorum Lib. V. VI. VII. paraphraste Abalphato Asphahanensi nunc primum editi, Additus in calce*

Archimedes Assumptorum Liber ex codicibus arabicis mss. ... Abrahamus Ecchellensis Maronita ... latinos reddidit Io.[hannes] Alfonsus Borellus in Geometricis versione contulit, Florentiae 1661.

Appendix

List of the *Middle Books* in the Recension of Nasīr al-Dīn Ṭūsī in the Facsimile Edition of Ms. Tabriz, Melli Library, no. 3484, edited by Dr. Jafar Aghayani Chavoshi (Tehran, Institute for Humanities and Cultural Studies, 2005), with references to vols. 5 and 6 of *GAS* = F. Sezgin, *Geschichte des arabischen Schrifttums* (Leiden 1974-1978).

1. p. 1-22, Euclid, *Data*, *GAS* V, 116.
2. p. 27-28, 25-26, 23-24, gap, 29-56, Theodosius, *Spherics*, *GAS* V, 154. The gap begins with *Spherics* book I, the end of prop. 19 and ends with *Spherics*, Book II, beginning of prop. 8.
3. p. 58-63, Autolycus, *Moving Sphere*, *GAS* V, 82.
4. p. 66-84. Euclid, *Optics*, *GAS* V, 117.
5. p. 88-95, Theodosius, *Inhabited Places*, *GAS* V, 155.
6. p. 98-117, Autolycus, *Risings and Settings*, *GAS* VI, 73.
7. p. 122-145, Euclid, *Phaenomena*, *GAS* V, 118.
8. p. 147-168, Theodosius, *Days and Nights*, *GAS* V, 156, dated.
9. p. 171-184, Aristarchus, *Sizes and Distances of the Sun and Moon*, *GAS* VI, 75, dated.
10. p. 187-189, Hypsicles, *Ascensions*, *GAS* V, 145.
11. p. 192-203, "Archimedes," *Lemmata*, *GAS* V, 131.
12. p. 205-214, Thabit ibn Qurra, *Assumed Things* (Mafrūḍāt), *GAS* V, 271 no. 19.
13. p. 221-331, Menelaus, *Spherics*, *GAS* V, 162 no. 5. The last three pages (328-331) are not found in the Hyderabad edition.
14. p. 331-332, Ibn al-Haytham, *Division of the Line which Archimedes used in the second Book On the Sphere and Cylinder*. *GAS* V, 371 no. 31.
15. p. 335-442, Naṣīr al-Dīn Ṭūsī, *On the Transversal Theorem*. Rosenfeld and Ihsanoğlu, p. 214 no. M 14.
16. p. 447-532, Archimedes, *On the Sphere and Cylinder*, with the commentary of Eutocius, *GAS* V, 129b.
17. p. 532-541, Abū Sahl Kūhī, *Additions to the Book On the Sphere and Cylinder of Archimedes*, *GAS* V, 320 no. 25.
18. p. 541-545, Archimedes, *Measurement of the Circle*, *GAS* V, 130 no. 2.

Appendix 2: Arabic Texts

This appendix contains an Arabic text of "Proposition 5" of the *Lemmas* of "Archimedes", the two related proposition by Kūhī, and two intermediary theorems by Nasawī. My edition is based on the Hyderabad edition of the *Middle Books*. The text has been compared to the recent facsimile of the *Middle Books* published by Dr. Aghayani Chavoshi. The Arabic texts in this section are not intended as critical edition.

Arabic text of the preface to the *Lemmas*

تحرير كتاب مأخوذات أرشميدس ترجمة ثابت بن قرة وتفسير الأستاذ المختصّ أبي الحسن علي بن احمد النسوي خمسة عشر شكلاً.

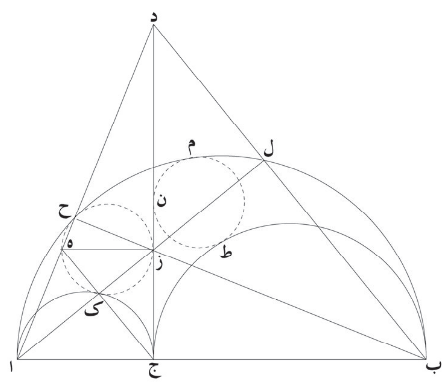
قال الاستاذ المختصّ هذه المقالة منسوبة إلى أرشميدس وفيها أشكال حسنة قليلة العدد كثيرة الفوائد في أصول الهندسة في غاية الجودة واللّطافة قد أضافها المحدثون إلى جملة المتوسطات التي يلزم قراءها فيما بين كتاب أقليدس و المجسطي إلا أن في بعض أشكاله مواضع تحتاج إلى أشكال أخر يتّم بها بيان ذلك الشكل وقد أشار في بعض ذلك أرشميدس إلى أشكال أوردها في سائر مصنفاته وقال كما بيّنّا في الأشكال القائمة الزوايا وكما في تفسيرنا في جملة القول في المثلثات وكما قد تبيّنّ في قولنا في الأشكال ذوات الأضلاع الاربعة. و أورد في الشكل الخامس برهاناً على طريق فيه نظر أخصّ ثم من بعد ذلك عمل أبوسهل القوهي مقالة سمّاه **تزيين كتاب أرشميدس في المأخوذات** وأورد برهان ذلك الشكل بطريق أعمّ وأحسن مع ما يتعلق به من تركيب النسبة وتأليفها فلما وجدت الحالة على هذه جعلت للمواضع الغامضة من هذه المقالة شرحاً على سبيل تعليق الحواشي و بينت ما إشار إليه بأشكال اتّجه إليها خاطري وأوردت من أشكال أبي سهل شكلين يحتاج إليها

في الشكل الخامس وتركت الباقي اجتناباً من التطويل واستغناءً عنه وبالله التوفيق

Arabic text of Proposition 5 by "Archimedes" (Figure 4), two intermediary theorems by Nasawī (Figures 5, 6) and the two theorems by Kūhī (Figures 7, 8).

إذا كان نصف دائرة عليه اب وتعلمت على قطرها نقطة جـ كيف وقعت وعمل على القطر نصفا دائرتين عليهما اجـ جب وأخرج من جـ عمود جد على اب وترسم على جنبيه دائرتان تمسانه وتمسان أنصاف الدوائر فإن الدائرتين متساويتان.

برهانه لتكن إحدي الدائرتين تماس جد على ز ونصف دائرة اب على ح ونصف دائرة اجـ على ك ونخرج قطر زه فهو مواز لقطر اب لكون زاويتي هزجـ اجز قائمتين ونصل حه ها فخط اح مستقيم لما مرّ في الشكل الأول وليلق اح جز على د لخروجهما من اجـ على أقلّ من قائمتين.



ونصل أيضًا حز زب فحب ايضاً مستقيم لما ذكرنا وعمود على اد لكون زاوية احب قائمة لوقوعهما في نصف دائرة اب ونصل مك كجـ فهجـ ايضاً مستقيم

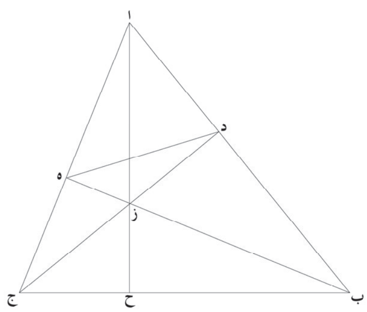
ونصل زك كا فزا مستقيم ونخرجه إلى ل ونصل بل وهو ايضاً عمود على ال ونصل دل.

ولأن اد اب مستقيمان وأخرج من د إلى اب عمود دجـ ومن ب إلى دا عمود بح فيتقاطعان على ز وأخرج از إلى ل وكان عمودًا على بل يكون بلد مستقيمًا كما بيّنا في الأشكال التي عملناها في شرح القول في المثلثات القائمة الزوايا.

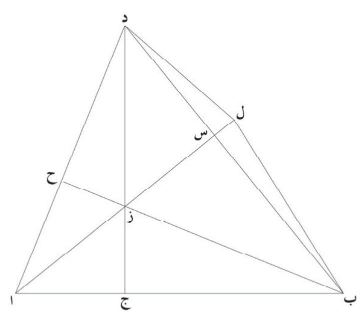
ولأن زاويتي اكجـ الب قائمتان فـ بد جه متوازيان فنسبة اد إلى ده التي هي كنسبة اج إلى هز كنسبة اب إلى بجـ فسطح اجـ في جب مساو لسطح اب في هز وبمثل ذلك تبيّن في دائرة طمن أن سطح اجـ في جب مساو لسطح اب في قطرها وتبيّن من ذلك أن قطري دائرتي زحك طمن متساويان فإذًا الدائرتان متساويان وذلك ما اردناه.

قال الاستاذ ويتبيّن ما أحاله على شرح المثلثات القائمة الزوايا من مقدمة وهي شكل مفيد في الأصل وخاصّة في المثلّثات حاد الزوايا ونحتاج إليه في الشكل السادس من هذا الكتاب وهي هذه

مثلث ابجـ أخرج فيه عمودا به جد المتقاطعين على ز ووصل از وأخرج إلى ح فهو عمود على بجـ. فنصل ده فيكون زاويتا داز دهز متساويتين لأن الدائرة التي تحيط لمثلث ادز يمرّ بنقطة ه لكون زاوية اهز قائمة وهما يقعان فيها على قوس واحدة وأيضًا زاوية دهب مثل زاويه دجب لأن الدائرة التي تحيط بمثلث بدجـ تمرّ بنقطة ه ايضاً ففي مثلثي ابح جبد زاويتا باح بجد متساويتين وزاوية ب مشتركة فزاوية احب مثل زاوية جدب القائمة فاح عمود على بجـ.



وإذا تقدمت هذه المقدمة فلنعد من الشكل الذي أورده أرشميدس خطي دا اب وأعمدة دجـ بح از بل وخط دل. ونقول إن لم يكن بلد خطًا مستقيمًا فنصل بسد المستقيم وتكون زاوية بسا قائمة للمقدمة المذكورة وكانت زاوية بلا قائمة فالداخلة في مثلث بلس مساوية للخارجة المقابلة له هذا خلف فإذا خط بلد مستقيم.

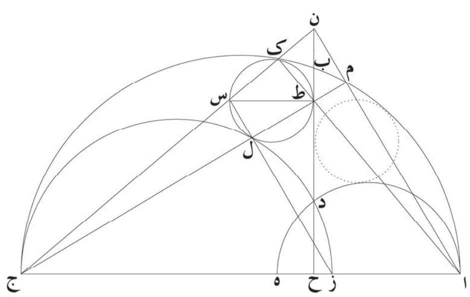


ثم أورد شكلين لأبي سهل الكوهي أولهما هذا فأن لم يكن نصفا الدائرتان متماسين ولكن متقاطعين والعمود من موضع التقاطع كان الحكم كما مرّ.

فلتكن انصاف الدوائر ابجـ اده زدجـ ونصفا الدائرتين متقاطعان على د فـ بح عمودًا على اجـ خارجًا من ح ودائرة طكل مماسّة لدائرة اكجـ على ك ولدائرة زلجـ على ل وللعمود على ط ونقول فهي مساوية للدائرة التي تكون

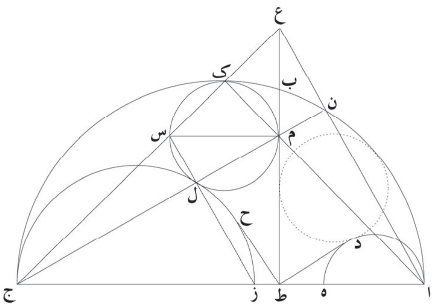
في الجانب الآخر بهذه الصفة.

فلنخرج طس موازيًا لــاج ولنصل جك فهي يمرّ بـس كما بــيّن أرشميــدس. ونخرجه إلى أن تلقى عمود حب على ن ونصل طجــ فيمرّ بـل ونخرجه إلــى م ونصل ام من فهو خط مستقيم ونصل سز فهو يمرّ بـل ونصل اك فيمرّ بـط



وخط امن مواز لخط زس ونسبة جن إلى نس أعني نسبة حج إلى طس كنسبة جا إلى از فسطح حج في از مساو لســطح جا في طس ولأن حد عمــود في دائرتي هدا جدز على وتري جز ها يكون سطح حج في حز مســاويًا لمربــع حد وسطح اح في حه أيضاً مساوياً له فسطح حج في حز مساو لســطح اح في حه ونسبة حج إلى حا كنسبة هح إلى حز بل كنسبة جه الباقي إلى زا الباقي فسطح حج في زا المساوي لسطح جا في طس مساو لسطح حا في جه وإذا كانــت في الجانب الآخر دائرة بالصفة المذكورة بيّنا بهذا التدبير ايضاً أن سطح جا في قطــر تلك الدائرة كسطح حا في جه فيتبيّن أن قطري الدائرتين متساويان.

وامّا الثاني فهو هذا قال وإن لم يكن نصفا الدائرتين متماسّين ولا متقاطعين لكن متباعدين والعمود يمرّ بالتقاء الخطين المماسّين لهما المتساويين كان الحكم كــذلك ايضاً.



فليكن أنصاف الدوائر ابجـ اده زجح على ما وصفنا وخطا طد طح مماسّين لنصفي الدائرتين على د ح ومتساويين ومتلاقيين على ط وخط بط عمود مارّ بنقطة ط قائم على اجـ ولتماسه دائرة مس على م ولتماس دائرة مس دائرة ابجـ على ك ودائرة زلجـ على ل ونخرج قطر مس موازيًا لـ اجـ ونصل جك فيمّر بـس ويلقى عمود طب على ع ونصل اك فيمرّ بـم ونصل سز فيمرّ بـل ونصل جم فيمرّ بـل ونخرجه إلى ن ونصل اع فيمرّ بـن ويكون موازياً لـ زس وتكون نسبة جع إلى عس أعني نسبة جط إلى مس كنسبة جا إلى از فسطح جط في از مساوياً لسطح جا في مس وبمثل هذا التدبير تبيّن أن سطح اط في هجـ يكون مساوياً لسطح جا في قطر الدائرة التي تكون من الجانب الآخر ولأن سطح اط في طه مساو لمربع طد وهو مساو لمربع طح المساوي لسطح جط في طز يكون سطح اط في طه مساوياً لسطح جط في طز ونسبة اط إلى جط كنسبة طز إلى طه وكنسبة جميع از إلى جميع جه فسطح جط في از مساو لسطح اط في هجـ وقد تبيّن أن جط في از مساو لسطح جا في مس وأن سطح اط في هجـ مساو لسطح جا في قطر الدائرة الأخرى فإذاً القطران متساويان والدائرتان متساويتان وهو المطلوب.