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The Symmetries of Equivalent Lagrangian Systems and Constants of Motion

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Abstract

In this paper Mathematical structure of time-dependent Lagrangian systems and their symmetries are extended and the explicit relation between constants of motion and infinitesimal symmetries of time-dependent Lagrangian systems are considered. Starting point is time-independent Lagrangian systems ,then we extend mathematical concepts of these systems such as equivalent lagrangian systems to the case of time-dependent Lagrangian systems. Also some new theorems and corollaries will be proved. Finally we make a 1-1 correspondence between the symmetries of equivalent time-dependent lagrangian systems and constants of motion by the new geometric concept of Galilean space-time.

Keywords: Lagrangian system; Hamiltonian system; Constant of motion; Symmetries of Lagrangian systems; Infinitesimal symmetries

Introduction

Lagrangian and Hamiltonian are of foundamental concepts in classical mechanics and there are many researches about them[1]. Noether's theorem shows that the infintisimal symmetries of Lagrangian systems and constants of motion (conservsd quantity) are related to each other[1]. For example, conservation of the linear and angular momenta are due to the symmetries, translations and rotations of the space, and energy conservation is due to the timereversal symmetry. Symmetries can be used to decrease the number of degrees of freedom of systems. Newton, in 1687 was the first who find symmetry in solution of Kepler problem.

The Mathematictions have worked on these subjects from geometrical view and have gotten some theorems about the relation between the symmetries and the constants of motion of a Lagrangian system.

In preliminary section we review some basic concepts and notations.

The second and third sections contain some standard definitions and theorems that are brought in references completely. The concept of equivalent Lagrangian systems in section 2 is new and useful to extend some concepts in later sections.

In section 4 it seems we need a suitable structure for classical mechanics, named Galilean space-time. Some physical concepts are rewritten in this frame work.

Result section contains some new relations between the symmetries of Lagrangian systems and constants of motion in time-dependent Lagrangian systems using the concept of equivalent Lagrangian systems.

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Preliminary

In this paper, M is a real C^∞ manifold and (x,U) is a coordinates system on M. (x,U) induces a coordinates system on TM which is denoted by $(\bar{x}^i, \dot{x}^i, TU)$. If $\pi : TM \to M$ be theprojection map, then $\bar{x}^i = x^i \circ \pi$, $\dot{x}^i = dx^i$ .The set of vector fields on M is denoted by $X(M)$.

Any C^∞ function $L : TM \to I\!R$ is called a Lagrangian on M.

A hamiltonian system is a triple (M, ω, H) in which (M, ω) is a symplectic manifold and $H \in C^\infty(M)$. For any function $f \in C^\infty(M)$, it's associated vector field, denoted by X_f, satisfies the following equation:

$$\omega(X_f, Y) = Y(f), \forall Y \in X(M)$$

Integral curves of X_H are called motions of the Hamiltonian system (M, ω, H). A function $f \in C^\infty(M)$ is called a constant of motion, if f is constant on the motions of the system, i.e. $X_H(f) = 0$.

A diffeomorphism $f : M \to M$ is called a symmetry of the Hamiltonian system (M, ω, H) if $f^*(\omega) = \omega$ and $f^*(H) = H \circ f = H$.

The vector field $X \in \mathrm{X}(M)$ is called an infinitesimal symmetry of the Hamiltonian system (M, ω, H) if $L_X \omega = 0$ and $X(H) = 0$.

It is well known that[9]:

1) A vector field $X \in \mathrm{X}(M)$ whose flow is $\{\varphi_t\}$, is an infinitesimal symmetry of the Hamiltonian system (M, ω, H) if and only if each φ_t is a local symmetry of the hamiltonian system.

2) A vector field $Z \in \mathrm{X}(M)$ is an infinitesimal symmetry of a Hamiltonian system (M, ω, H) if and only if for some constant of motion f, $Z = X_f$ locally.

3) If The vector field $Y \in \mathrm{X}(M)$ is an infinitesimal symmetry of the Hamiltonian system (M, ω, H), then $[Y, X_H] = 0$.

For any $u, v \in T_pM$, vertical lift of v at u is denoted by $I_u v$ and is defined as follows[8]:

$$\Im_u v = \frac{d}{dt}\Big|_{t=0} (u + tv) \in (VTM)_u$$

There is a natural vertical vector field on TM that is defined as follows:

$$\Delta_v = \Im_v v, \forall v \in TM$$

Δ is called Liouville vector field of TM. There exists a canonical $1-1$ form on TM which is called liouville $1-1$ form of TM. This form is denoted by J and defined as follows:

$$J(\mathbf{v}) = \Im_u \pi_*(\mathbf{v}), \forall \mathbf{v} \in T_u TM$$

In coordinates systems the $1-1$ form J, and Liouville vector field Δ have the following representations:

$$\Delta = \dot{x}^i \frac{\partial}{\partial \dot{x}^i} \quad , \quad J = d\bar{x}^i \otimes \frac{\partial}{\partial \dot{x}^i}$$

Any $1-$form $\alpha \in A^1(M)$ can be considered as a function on TM, therefore the following hold:

$$d\alpha \circ J = \pi^*(\alpha) \quad , \quad \Delta(\alpha) = \alpha$$

A vector field $\mathbf{X} \in \mathrm{X}(TM)$ is called a semi-spray if its integral curves be in the form of α' for some curve α in M. By abuse of language, we call α an integral curve of $\mathbf{X}$. The following propositions are equivalent:

i) $\mathbf{X}$ is a semi-spray.

ii) for each $v \in TM$, $\pi_*(\mathbf{X}_v) = v$

iii) In any coordinates system we have: $\mathbf{X} = \dot{x}^i \frac{\partial}{\partial \bar{x}^i} + g^i \frac{\partial}{\partial \dot{x}^i}$

iv) $J(\mathbf{X}) = \Delta$.

If $X \in \mathrm{X}(M)$ be a vector field on M, it's complete lift, denoted by X^c, is a vector field on TM. If the flow of X be $\{\phi_t\}$, then by definition the flow of X^c is $\{\phi_{t*}\}$. If α be a $1-$form on M, considering α as a function on TM, $X^c(\alpha)$ is equal to $L_X \alpha$.

Time-Independent Lagrangians Systems

Let $L : TM \to I\!R$ be a Lagrangian ($L \in C^\infty(TM)$).The 1- form Θ_L on TM, associated to L, is defined by $\Theta_L = dL \circ J$. The reprsentation of Θ_L in coordinates system is $\Theta_L = \frac{\partial L}{\partial \dot{x}^i} d\bar{x}^i$. ω_L is defined as $\omega_L = -d\Theta_L$. If ω_L be a nondegenerate 2-form, (TM, ω_L) is a symplectic manifold .In this case L is called a regular Lagrangian. So L is regular if and only if, in coordinates systems, $(\frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j}(v))$ be an invertible matrix at every $v \in TM$. Also an energy function H_L on TM can be defined by the following relation:

$$H_L = \Delta(L) - L = \dot{x}^i \frac{\partial L}{\partial \dot{x}^i} - L .$$

If L be a regular Lagrangian, then the triple (TM, ω_L, H_L) is a Hamiltonian system and it is called the Hamiltonian system associated to L. Let X_L be the associated vector field to H_L on TM ,then it is well known that X_L is a semi-spray and its integral curves on M are exactly the critical paths of L [9]. In other words, a curve α on M is an integral curve of X_L if and only if α satisfies Euler-Largrange equations.

The motions of the Hamiltonian system (TM, ω_L, H_L), are also called the motions of the Lagrangian system (M, L).

Two different Lagrangians may produce the same dynamical systems, so we need to know in what conditions, two Lagrangian define the same dynamical systems[1].

Definition 1. Two regular lagrangian $L, L' : TM \to I\!R$ are called equivalent, if $\omega_L = \omega_{L'}$ and $H_L = H_{L'}$.

Theorem 1. Two lagrangians $L, L' : TM \to I\!R$ are equivalent if and only if $L - L'$ is a closed 1-form on M .

Symmetries of Lagrangian Systems

Definition 2. A diffeomorphism $f : M \to M$ is called a symmetry in a Lagrangian system (M, L), if L and $L \circ f_*$ be equivalent.

Theorem 2. Let (M, L) be a Lagrangian system, then a diffeomorphism $f : M \to M$ is a symmetry of (M, L) if and only if $f_* : TM \to TM$ be a symmetry of (TM, ω_L, H_L).

Definition 3. In a Lagrangian system (M, L), a vector field $Y \in \mathfrak{X}(M)$ whose flow is $\{\phi_t\}$, is called an infinitesimal symmetry if for every t , ϕ_t is a local symmetry of the system.

Corollary 1. A vector field $Y \in \mathfrak{X}(M)$ is an infinitesimal symmetry of a Lagrangian system (M, L) if and only if Y^c is an infinitesimal symmetry of the Hamiltonian system (TM, ω_L, H_L).

Theorem 3. A vector field $Y \in \mathfrak{X}(M)$ is an infinitesimal symmetry of a Lagrangian system (M, L) if and only if the function $Y^c(L)$ on TM be a closed $1-$form on M .

Constants of Motion

Theorem 4. If (M, L) be a Lagrangian system, and $f_Y = \Theta_L(Y^c), Y \in X(M)$,then $X_L(f_Y) = Y^c(L)$.

Proof. Note if $L : TM \to IR$ be a regular Lagrangian, then $L_{X_L}(\Theta_L) = dL$, and if $h \in C^\infty(M)$, $D \in \mathfrak{X}(TM)$ be respectively a function and a semi-spray,then $D(h \circ \pi) = dh$. A simple local computation shows that $[D, Y^c]$ is vertical.

$$\begin{aligned} X_L(f_Y) &= X_L(\Theta_L(Y^c)) = L_{X_L}(\Theta_L(Y^c) + \\ &\Theta_L([X_L, Y^c]) = dL(Y^c) + dL(J([X_L, Y^c])) \\ &= Y^c(L) + dL(0) = Y^c(L) \end{aligned}$$

■

Theorem 5. If $Y \in \mathfrak{X}(M)$ be an infinitesimal symmetry of a Lagrangian system (M, L) and g be a local function that $Y^c(L) = dg$, then the function $C_Y = f_Y - g \circ \pi$ is a constant of motion.

Proof.

$$\begin{aligned} X_L(C_Y) &= X_L(f_Y - g \circ \pi) = X_L(f_Y) - X_L(g \circ \pi) \\ &= Y^c(L) - dg = dg - dg = 0 \end{aligned}$$

If L' , L be equivalent Lagrangians, then there is a closed $1-$form α ,which $L' = L + \alpha$. So the following holds:

$$\Theta_{L'} = dL \circ J + d\alpha \circ J = \Theta_L + \pi^*(\alpha) .$$

If h be a local function ,which $\alpha = dh$, then

$$\begin{aligned} Y^c(L') &= Y^c(L) + Y^c(dh) = Y^c(L) + L_Y(dh) \\ &= dg + d(L_Y h) = d(g + Y(h)) \end{aligned}$$

With approximation of a constant, we can choose $g' = g + Y(h)$,which $Y^c(L') = dg'$. Now, the constant of motion related to L' is $C_{Y'}$ and is infered from the following computations:

$$C_{Y'} = \Theta_{L'}(Y^c) - g' \circ \pi = (\Theta_L + \pi^*(\alpha))(Y^c) - (g + Y(h)) \circ \pi = \Theta_L(Y^c) + \pi^*(\alpha)(Y^c) - g \circ \pi - Y(h) \circ \pi = C_Y + \alpha(Y) \circ \pi - Y(h) \circ \pi = C_Y + dh(Y) \circ \pi - Y(h) \circ \pi = C_Y + Y(h) \circ \pi - Y(h) \circ \pi = C_Y$$

■

In above theorem, the function C_Y is called constant of motion associated to the symmetry Y.

Theorem 6. If $Y \in \mathfrak{X}(M)$ be an infinitesimal symmetry of the Lagrangian system (M, L), then X_{C_Y} equals Y^c in the associated Hamiltonian system (TM, ω_L, H_L).

Corollary 2. If f be a constant of motion of the Lagrangian system (M, L), in which $X_f = Y^c$ for some $Y \in X(M)$, then Y is an infinitesimal symmetry of (M, L) and $C_Y = f$.

Example: Let $M = \boldsymbol{R}^n$ and

$$L(\bar{x}^i, \dot{x}^i) = \bar{x}^1 \dot{x}^1 + \sum_j \dot{x}^j \dot{x}^j \ , \ Y = \frac{\partial}{\partial x^1}$$

Then $Y^c = \dfrac{\partial}{\partial \bar{x}^1}$ and $Y^c(L) = \dot{x}^1 = dx^1$. Since

$$\Theta_L = (\bar{x}^1 + 2\dot{x}^1) d\bar{x}^1 + 2\sum_{1<j} \dot{x}^j d\bar{x}^j$$

And $\Theta_L(Y^c) = \bar{x}^1 + 2\dot{x}^1$, then we find $C_Y = \bar{x}^1 + 2\dot{x}^1 - \bar{x}^1 = 2\dot{x}^1$ is a constant of the motions hn the system $(\boldsymbol{R}^n, L)$

Time-Dependent Lagrangian Systems

To have a good framework for discussing about time-dependent Lagrangian systems, we need to define a suitable mathematical structure, named Galilean space-time.

Definition 4. A fiber bundle $\pi : E \to I\!R$ with a standard fiber M is called a Galilean space-time. For any $t \in I\!R$, $M_t = \pi^{-1}(t)$ is called space at time t.

Definition 5. Every section of a Galilean space-time $\pi : E \to I\!R$ is called a motion of system.

In this section $\pi : E \to I\!R$ is a fixed Galilean space-time.

Definition 6. If $S : I\!R \to E$ be a motion, then $S'(t)$ is called the world velocity of S at time t.

Clearly, $d\pi(S'(t)) = 1$ and all world velocities of particles lies in a submanifold of TE which will be defined later. Since $I\!R$ is contractible, every Galilean space-time is isomorphic to the trivial bundle $I\!R \times M \xrightarrow{pr_1} I\!R$.

Definition 7. Every bundle isomorphism

$$\begin{array}{ccc} E & \xrightarrow{\phi} & I\!R \times M \\ \downarrow \pi & & \downarrow pr_1 \\ I\!R & \xrightarrow{h} & I\!R \end{array}$$

in which h has the form $h(t) = t + t_0$, is called an observer of the Galilean space-time $\pi : E \to I\!R$.

An observer sees all spaces M_t like M, i.e if $S : I\!R \to E$ be a motion of a particle, then for any observer ϕ there exists some curve $\alpha : I\!R \to M$ such that $\phi(S(t)) = (h(t), \alpha(h(t)))$. α is called the motion of the particle relative to the observer ϕ, and α' is called velocity of the particle relative to the observer ϕ. There exists a unique vector field $X_\phi \in \mathfrak{X}(E)$ that is $\phi-$related to $\dfrac{\partial}{\partial t}$ and is called the vector field of the observer ϕ.

The 1-jet bundle of the Galilean space-time $E \xrightarrow{\pi} I\!R$ can be described as the following:

$$J^1E = \{v \in TE \mid d\pi(v) = 1\}$$

$J^1(E)$ is an affine subbundle of the bundle $\pi_E : TE \to E$, modeled on the vertical subbundle VE. Since the world velocities of all particles lies in J^1E, then we need the restriction of π_E to J^1E ,that denote by $\pi_1 : J^1E \to E$.

Definition 8. A vector field $\mathbf{X} \in \mathfrak{X}(J^1E)$ is called a time-dependent semi-pray, if it's integral curves be in the form of α' in which α is a motion of E.

Time-dependent semi-sprays are similar to ordinary semi-sprays . $\mathbf{X} \in \mathfrak{X}(J^1E)$ is time-dependent semi-spray if and only if $\pi_{1*}(\mathbf{X}_v) = v$ for any $v \in J^1E$.

Definition 9. In a Galilean space-time $\pi : E \to I\!R$, every smooth function $L : J^1E \to I\!R$ is called a Lagrangian on E.

Let $\phi : E \to I\!R \times M$ be a trivialization of E over an open set U in M (an observer),then ϕ equals a pair of functions (π,ψ) .Set $U'=\psi^{-1}(U)$,$t=\pi$ and $x'^i=x^i\circ\psi$. Therefore (x'^i,t) is a bundle chart on E as the following:

$$U'\to x(U)\times IR$$

$$\xi\mapsto(x\circ\psi(\xi),\pi(\xi))$$

This bundle chart induces a bundle chart on TE that is restrictable to J^1E . The component functions of the induced bundle chart on J^1E are the followings:

$$\bar{x}^i=x'^i\circ\pi_1=x^i\circ\psi\circ\pi_1 \quad ,$$

$$\bar{t}=t\circ\pi_1=\pi\circ\pi_1 \quad ,\dot{x}^i=dx'^i=dx^i\circ\psi_*$$

Note that the functions t and $\bar{t}$ do not depend on (x,U) and ϕ .

For $\xi\in E$ and $u\in(J^1E)_\xi$ and $v\in(VE)_\xi$, the vertical lift of v at u is a vertical vector in $T_\xi J^1E$, denoted by $\Im_u v$, is defined as follows :

$$\Im_u v=\frac{d}{dt}|_{t=0}(u+tv)$$

Note that for any $\hat{w}\in T_u(J^1E)$, the vector $\pi_{1*}(\hat{w})-d\bar{t}(\hat{w})u$ lies in VE ,so we can construct it's vertical lift at u . This is a bundle map on TJ^1E , denoted by v ,and is called the Liouville $1-1$ form of J^1E and is defined as follows:

$$v:TJ^1E\to TJ^1E$$
$$\hat{w}\in T_u(J^1E)\mapsto\Im_u(\pi_{1*}(\hat{w})-d\bar{t}(\hat{w})u)$$

v has the following representation in bundle chart[5]:

$$v=(d\bar{x}^i-\dot{x}^i d\bar{t})\otimes\frac{\partial}{\partial\dot{x}^i}$$

Let L be a Lagrangian on E , a $1-$form on J^1E , denoted by Θ_L , is constructed in [5] as follows:

$$\Theta_L=dL\circ v+Ld\bar{t}$$

Θ_L has the following form bundle chart:

$$\Theta_L=\frac{\partial L}{\partial\dot{x}^i}d\bar{x}^i-(\dot{x}^i\frac{\partial L}{\partial\dot{x}^i}-L)d\bar{t}$$

An observer can show the relations between time-dependent and time-independent Lagrangian systems. In this case we may assume E is the trivial bundle , so $J^1E=IR\times TM$ .For a Lagrangian $L:J^1E\to IR$, by fixing t , we can define the time-independent Lagrangian $L_t:TM\to IR$. The associated energy function af L_t is $H_{L_t}=\Delta(L_t)-L_t$. Now Θ_L as a $1-$form on $IR\times M$ can be written as follows:

$$\Theta_L=dL_t\circ J-H_{L_t}d\bar{t}$$

Critical paths of a time-dependent Lagrangians are defined similar to the case time-independent Lagrangians and for these paths Euler-Lagrange equations must hold.

The $2-$form $\omega_L=-d\Theta_L$ can be used to describe the critical paths of L . Since dimension of J^1E is odd, ω_L is degenerate and we can not construct a Hamiltonan system.

Definition 10. A Lagrangian $L:J^1E\to IR$ is called regular, if ω_L has maximum rank.

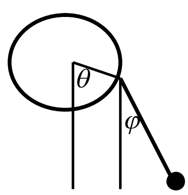
For a regular Lagrangians L , kernel of $(\omega_L)_u$ is a one dimensional subspace of $T_u(J^1E)$, at any $u\in J^1E$, so there exists a unique vector field $X_L\in\mathfrak{X}(J^1E)$ such that

$$i_{X_L}\omega_L=0 \quad , \quad d\bar{t}(X_L)=1$$

X_L is a semi spray and the integral curves of X_L are exactly the critical paths of the Lagrangian L [5]. The integral curves of semi-spray X_L are called motions of the system (E,L). A C^∞ function $f:J^1E\to IR$ is called a constant of motion of the Lagrangian system (E,L), if for any motion α of this system $f\circ\alpha'$ be constant, i.e. $X_Lf=0$.

A $1-$form on E can be considered as a function on TE , so we can consider it's restriction to J^1E . A $1-$form on E , completely is determined by it's restriction to J^1E . A function on J^1E is called a $1-$form on E , if it is the restriction of a $1-$form of E to J^1E .

Example:**(Kapitza pendulum)**. Consider the pendulum suspended from a rotating disk. The disk has diameter d and the pendulum has length _. At the end of the pendulum there is a mass m. The rotation of the disk is forced to be at constant angular speed θ(t) = ωt. φ is the angle of the pendulum relative to the vertical.



In this case the Lagrangian is time-dependent and $E=\mathbb{R}\times S^1$. We can find the Lagrangian of the system with respect to ϕ that represent a coordinates system on S^1

$$J^1E = \boldsymbol{R}\times TS^1 \to \boldsymbol{R}$$

$$(t,\phi,\dot{\phi})\mapsto L(t,\phi,\dot{\phi})$$

The $\boldsymbol{x}$ and $\mathbf{y}$ position of the mass is
$x=d\sin\omega t+l\sin\phi$, $y=d\cos\omega t+l\cos\phi$
so the velocity of the mass is

$$\dot{x}=d\,\omega\cos\omega t+l\,\dot{\phi}\cos\phi$$

$$\dot{y}=-d\,\omega\sin\omega t-l\,\dot{\phi}\sin\phi$$

so the kinetic energy of the pendulum is

$$\frac{1}{2}m(\dot{x}^2+\dot{y}^2)=$$

$$\frac{m}{2}(d^2\omega^2+l^2\dot{\phi}^2+2dl\,\omega\dot{\phi}\cos(\omega t-\phi))$$

Therefore the Lagrangian is

$$L=\frac{m}{2}(d^2\omega^2+l^2\dot{\phi}^2+2dl\,\omega\dot{\phi}\cos(\omega t-\phi))$$

$$mg(d\cos\omega t+l\cos\phi)$$

Definition 11. Two Lagrangians $L,L':J^1E\to I\!R$ are called equivalent, if $\omega_L=\omega_{L'}$.

Theorem 7. Two lagrangians $L,L':J^1E\to I\!R$ are equivalent if and only if $L-L'$ is a closed $1-$form on E.

Proof. Without loss of generelity assume E is trivial. First suppose L and L' are equivalent, i.e $\omega_L=\omega_{L'}$. Set $L'=L+h$, so

$$\Theta_{L'}=dL'_t\circ J-H_{L'_t}d\overline{t}=\Theta_L+dh_t\circ J-H_{h_t}d\overline{t}$$

Since $\omega_L=\omega_{L'}$, then $dh_t\circ J-H_{h_t}d\overline{t}$ is a closed $1-$form on $I\!R\times TM$ that in bundle chart has the following representation:

$$\frac{\partial h_t}{\partial \dot{x}^i}d\overline{x}^i-(\dot{x}^i\frac{\partial h_t}{\partial \dot{x}^i}-h_t)d\overline{t}$$

Since the exterior differential of this form is zero, then

$$\frac{\partial^2 h_t}{\partial \dot{x}^i\partial \dot{x}^j}=0,\quad \frac{\partial^2 h_t}{\partial \dot{x}^i\partial \overline{x}^j}=\frac{\partial^2 h_t}{\partial \dot{x}^j\partial \overline{x}^i}=0(i\neq j)$$

$$,\frac{\partial^2 h_t}{\partial \dot{x}^i\partial t}+\dot{x}^j\frac{\partial^2 h_t}{\partial \dot{x}^j\partial \overline{x}^i}-\frac{\partial h_t}{\partial \overline{x}^i}=0$$

From the first equation it is infered that $h_t=h_t^i\circ\pi.\dot{x}^i+g_t\circ\pi$, in which g_t,h_t^i are local functions on M. The second equation yields $\frac{\partial h_t^i}{\partial x_j}=\frac{\partial h_t^j}{\partial x_i}$. So, the local $1-$form $\sum h_t^i dx^i$ on M is closed form and equals df_t for some local function f_t on M i.e $h_t^i=\frac{\partial f_t}{\partial x_i}$.

The result of third equation is

$$\frac{\partial h_i}{\partial t}+\dot{x}^j\frac{\partial h_j}{\partial x^i}-(\dot{x}^j\frac{\partial h_j}{\partial x^i}+\frac{\partial g}{\partial x^i})=0$$

$$\Rightarrow\quad \frac{\partial}{\partial x^i}(\frac{\partial f}{\partial t}-g)=0$$

$$\Rightarrow\quad \frac{\partial f}{\partial t}=g_t+k(t)$$

If f be replaced by $f_t-k(t)$, then all equations hold and $g_t=\frac{\partial f}{\partial t}$. Therefore

$$h=\dot{x}^i\frac{\partial f}{\partial x^i}\circ\pi+\frac{\partial f}{\partial t}\circ\pi$$

This equation means that if f be a function on $I\!R\times M$, then the restriction of $1-$form df on $J^1(I\!R\times M)$ is equal to h. So, $L-L'$ is a closed $1-$form locally and consequently $L-L'$ is a closed $1-$form on E.

Conversely, let $L-L'$ be a closed $1-$form on E. There exists a function f on $I\!R\times M$ such that $L'=L+df$ locally.

The above calculations show the condition $\omega_L=\omega_{L'}$ is equivalent to this fact that three above equations must hold for $h_t=\dot{x}^i\frac{\partial f}{\partial x^i}\circ\pi+\frac{\partial f}{\partial t}\circ\pi$.

An easy computation shows that above three equations hold. ■

Results and Discussion

Symmetries of Time-Dependent Lagrangian Systems

Definition 12. If $\pi:E\to I\!R$ be a Galilean space-time, then a bundle map $f:E\to E$ is called a Galilean transformation if f be a diffeomorphism and it's induced map on $I\!R$ be a translation.

If f be a Galilean transformation on E, then J^1E

is invariant under f_* and it's restriction to J^1E is denoted by J^1f.

Definition 13. For a Lagrangian system (E,L), a Galilean transformation $f:E\to E$ is called a symmetry if L and $L\circ J^1f$ be equivalent.

Theorem 8. A Galilean transformation $f:E\to E$ is a symmetry of (E,L) if and only if $(J^1f)^*\omega_L=\omega_L$.

Proof. Without loss of generality we can assume $E=IR\times M$. In this case $f:IR\times M\to IR\times M$ has the form $f(t,p)=(t+c,g_t(p))$ and $J^1f:IR\times TM\to IR\times TM$ has the form $J^1f(t,v)=(t,g_{t*}(v))$. So $(J^1f)^*(d\bar{t})=d\bar{t}$ and $(J^1f)^*(\Theta_L)=(J^1f)^*(dL_t\circ J-H_{L_t}d\bar{t})$

$$=dL_t\circ J\circ(J^1f)_*-(H_{L_t}\circ J^1f)d\bar{t}$$

Since $J^1f=g_{t*}$ on TM, then
$(J^1f)^*(\Theta_L)=dL_t\circ J\circ g_{t**}-H_{L_t}\circ g_{t*}d\bar{t}$

The same computations as in the case time-independent Lagrangian systems yield the follwing:

$$(J^1f)^*(\Theta_L)=d(L_t\circ g_{t*})\circ J-H_{L_t\circ g_{t*}}d\bar{t}$$
$$=\Theta_{L\circ J^1f}$$

To prove the theorem, first assume that f is a symmetry of the the Lagrangin system. So $\omega_L=\omega_{L\circ J^1f}$ and consequently

$$(J^1f)^*(\omega_L)=(J^1f)^*(-d\Theta_L)$$
$$=-d(J^1f)^*(\Theta_L)$$
$$=-d(\Theta_{L\circ J^1f})$$
$$=\omega_{L\circ J^1f}=\omega_L$$

Conversely assume $(J^1f)^*(\omega_L)=\omega_L$. The last computation shows

$$\omega_L=(J^1f)^*(\omega_L)=\omega_{L\circ J^1f}$$

So L and $L\circ J^1f$ are equivalent ■

For a Galilean space-time $\pi:E\to IR$, if $\{\phi_t\}$ is the flow of a vector field $Y\in \mathrm{X}(E)$ then each ϕ_t is a local bundle map on E, if and only if Y is $\pi-$related to some vector field on IR. Each ϕ_t is a local Galilean transformation if and only if for some $\lambda\in IR$, Y is $\pi-$related to $\lambda\frac{d}{dt}$. This kind of vector fields are called infinitesimal Galilean transformations. For example, the vector fields of all observers are infinitesimal Galilean transformations, because all of them are $\pi-$related to $\frac{d}{dt}$. If $\{\phi_t\}$ be the flow of an infinitesimal Galilean transformation Y, then $\{\phi_{t*}\}$ is a flow on J^1E and it's induced vector field is denoted by J^1Y. Actually, J^1Y is the restriction of Y^c to J^1E.

Symmetries and Constants of Motion

Definition 14. Let Y be an infinitesimal Galilean transformation of a Galilean space-time $\pi:E\to IR$. If the flow of Y be $\{\phi_t\}$, then Y is called an infinitesimal symmetry of (E,L) if ϕ_t be a local symmetry of (E,L) for every $t\in IR$.

Corollary 3. Y is an infinitesimal symmetry of a Lagrangian system (E,L) if and only if J^1Y is an infinitesimal symmetry of (J^1E,ω_L), i.e. $L_{J^1Y}\omega_L=0$.

Theorem 9. If Y be an infinitesimal symmetry of the lagrangian system (E,L) then $[J^1Y,X_L]=0$.

Proof. Since J^1Y is an infinitesimal symmetry of (J^1E,ω_L), then

$$0=L_{J^1Y}(i_{X_L}\omega_L)=i_{[J^1Y,X_L]}\omega_L+i_{X_L}L_{J^1Y}\omega_L$$
$$=i_{[J^1Y,X_L]}\omega_L$$

So $[J^1Y,X_L]$ must be a multiple of X_L. Since X_L is $\bar{t}$-related to $\frac{d}{dt}$ and for some λ, J^1Y is $\bar{t}$-related to $\lambda\frac{d}{dt}$, then $[J^1Y,X_L]$ is $\bar{t}$-related to zero. Since $[J^1Y,X_L]=fX_L$, then we conclude that $f=0$ and $[J^1Y,X_L]=0$. ■

Theorem 10. Let Y be an infinitesimal Galilean transformation of E. Y is an infinitesimal symmetry of a Lagrangian system (E,L) if and only if $(J^1Y)L$ is a closed 1-form on E.

Proof. The proof is completely the same as the time-independent case. ■

Theorem 11. Let Y be an infinitesimal Galilean transformation of Lagrangian system (E,L) and set $f_Y=\Theta_L(J^1Y)$, then $X_L(f_Y)=J^1Y(L)$.

Proof. Note that in this case, similar to the case of time-independent Lagrangian system, by the following computation we have $L_{X_L}(\Theta_L) = dL$.

$$L_{X_L}(\Theta_L) = di_{X_L}\Theta_L + i_{X_L}d\Theta_L = d(\Theta_L(X_L)) - i_{X_L}\omega_L = d(dL_t \circ J(X_L) - H_{L_t}d\bar{t}(X_L)) = d(dL_t(\Delta) - H_{L_t}) = d(\Delta(L_t) - \Delta(L_t) + L) = dL$$

Since the vector field $[J^1Y, X_L]$ is π_1-vertical[5],then

$$X_L(f_Y) = X_L(\Theta_L(J^1Y)) = (L_{X_L}\Theta_L)(J^1Y) + \Theta_L([J^1Y, X_L]) = dL(J^1Y) + 0 = J^1Y(L).$$

■

Theorem 12. If Y be an infinitesimal symmetry of a Lagrangian system (E, L) and $(J^1Y)L = dg$, then the function $C_Y = g \circ \pi_1 - f_Y$ is a constant of motion.

Proof. Since X_L is a semi-spray of J^1E , then $X_L(g \circ \pi_1) = dg$. Now

$$X_L(C_Y) = X_L(g \circ \pi_1) - X_L(f_Y) = dg - J^1Y(L) = dg - dg = 0 \blacksquare$$

If X_ϕ be the vector field of an observer ϕ, then C_{X_ϕ} is energy of the system relative to that observer. Moreover, if X_ϕ be an infinitesimal symmetry of (M, L) , then the energy is constant and we can choose an time-independent equivalent Lagrangian L relative to observer ϕ .

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References

1. Abraham R. and Marsden J.E. *Foundation of Mechanics*. Addison-Wesley, Englewood Cliffs,791 p. (1978).
2. Trautman A. M. Noether Equations and Conservation Laws. *Commun. Math. Phy.*, **6**: 248-261 (1967).
3. Arnold V.I. *Mathematical Methods of Classical Mechanics*. Vol. I. Publish Berkeley, 552 p. (1979).
4. Broojerdian N. On the Lifts of Semi-Riemannian Metrics. *J. Sci . I R. of Iran*, **5**(4): 181-191 (1994).
5. Echeverria-Enriquez A., Muñoz-Lecanda M.C. and Román-Roy N. Non-standard connections in classical mechanics. *J. Phys. A: Math. Gen.* **28**: 553-5567 (1995).
6. Giachetta G., Mangiarotti L. and Sardanashvily G. Differential Geometry of Time-Dependent Mechanics. arXiv:dg-ga/9702020 v1. 25 Feb (1997).
7. MUÑOZ-LECANDA M.C., ROMÁN-ROY N. and YÁNIZ-FERNÁNDEZ F.J. Time-Dependent Lagrangians Invariant by a Vector Field. *Letters in Mathematical Physics*, **57**: 107-121 (2001).
8. Poor W. A. *Differential Geometric Structures*. McGraw-Hill (1981).
9. Puta M. *Hamiltonian Mechanical System and Geometric Quantization*. Kluwer Academic publishers. London (1993).