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Estimating a Bounded Normal Mean Relative to Squared Error Loss Function

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Abstract

Let $X_1,...,X_n$ be a random sample from a normal distribution with unknown mean θ and known variance σ^2. The usual estimator of the mean, i.e., sample mean $\bar{X}$, is the maximum likelihood estimator which under squared error loss function is minimax and admissible estimator. In many practical situations, θ is known in advance to lie in an interval, say $[-m, m]$ for some $m > 0$. In this case, the maximum likelihood estimator changes and dominates $\bar{X}$ but it is no longer admissible. Minimax and some other estimators for this problem have been studied by some researchers. In this paper, a new estimator is proposed and the risk function of it is compared with some other competitors. According to our findings, the use of $\bar{X}$ and the maximum likelihood estimator is not recommended when some information are accessible about the finite bounds on $[-m, m]$ in advance. Based on the values taken by θ in $[-m, m]$, the appropriate estimator is suggested.

Keywords: Admissibility; Bounded normal mean; Maximum likelihood estimator; Rao-Blackwellization; Squared error loss

Introduction

In the statistical literature it is often assumed that the parameter space is unbounded which seems to be never fulfilled in practice. In various physical, industrial and biological experiments, the experimenter has often some prior knowledge about the parameter(s) of the underlying population. The average per capita income of a developing country is likely to lie between those of an underdeveloped and a developed country. The average fuel efficiency of a new model of passenger car will lie between those of an old model and a formula one racing car. Examples of similar nature where mean of a real phenomena lies in a bounded interval abound in practice (e.g., physical attributes such as height or weight of people, average life of animals). Therefore there is practical interest to include such additional information into statistical procedures.

Surprisingly, while the assumption of boundedness can be useful in practice, it introduces some challenging problems in theory. Such problems first arose with the practical problem in 1950 in which two probabilities θ_1

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and θ_2 known to satisfy the restriction $\theta_1 \le \theta_2$, needed to be estimated. Maximum likelihood estimation was used for this purpose. Later Maximum Likelihood Estimators (MLEs) were shown to be inadmissible under Squared Error Loss (SEL) function

$$L(\theta,\delta) = (\delta - \theta)^2. \qquad (1)$$

That is, it was shown that there exist estimators which are better than the MLE in the sense that their expected loss, i.e., $R(\theta,\delta) = E[L(\theta,\delta)]$, as a function of the parameter to be estimated, is nowhere larger and somewhere smaller than that of the MLE. This then led to the search for dominators for these inadmissible estimators as well as for admissible estimators with "good properties". One such property is that of minimaxity where an estimator is minimax when there does not exist an estimator with a smaller maximum expected loss. Examples of problems addressed in the restricted parameter spaces, can be found in [1], [21], [29] and the recent treatise by ven Eeden [30] for detailed discussion.

Let $X \sim N(\theta,\sigma^2)$ denote a random variable having normal distribution with the probability density function

$$f(x \mid \theta,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\theta)^2}, x \in \mathbb{R},$$

where, it is supposed that variance σ^2 is known and the unknown mean θ, lies in an interval of the form $[-m,m]$, for some known $m > 0$. The first study in estimating the bounded normal mean under SEL function dates back to 1981. Casella and Strawderman [5] showed that, when $0 < m \le m_0 \simeq 1.05$, there exists a unique admissible and minimax estimator of θ, associated with a symmetric two-point prior on $\{-m,m\}$ and proved that it dominates the MLE of θ, when $0 < m \le 1$. They also gave a class of admissible and minimax estimators for the case when $1.4 \le m \le 1.6$. These estimators are minimax with respect to (w.r.t.) a symmetric three-point prior on $\{-m,0,m\}$. Bickel [3] presented an estimator which is asymptotically minimax as $m \to \infty$, and showed that the weak limit of the least favourable prior (rescaled to $[-1,1]$) has the density $cos^2\left(\frac{\pi\theta}{2}\right)$, $|\theta| \le 1$ and the minimax risk is $1 - \pi^2 m^{-2} + o(m^{-2})$. After these initial works, several authors considered the estimation problem in restricted parameter spaces under SEL function. Moors [23, 24] assumed a bounded estimation problem is invariant w.r.t. a finite group of transformations and constructed dominators of a boundary estimator. He then applied his results into the estimation problem of a bounded normal mean. Gatsonis et al. [10], considered the Bayes estimator w.r.t. the uniform prior on the interval $[-m,m]$, as a competitor for the sample mean $\bar{X}$, and showed that it dominates $\bar{X}$. They further numerically compared risk performance of their Bayes estimator, the MLE, the minimax estimator, and the Bayes estimator w.r.t. the Bickel's prior, and finally recommended the use of their proposed estimator. In addition to [23, 24], the estimation in restricted parameter spaces under SEL function, in a very general setting, was considered in [7] and [6]. DasGupta [7] in estimating a vector $h(\theta)$ when θ is restricted to a small bounded convex subset Θ of $\mathbb{R}^k$ and derived sufficient conditions under which the Bayes estimator w.r.t. a least favourable prior on the boundary of Θ is minimax. He then applied his results in some distributions including the normal distribution and showed that the Bayes estimator w.r.t. two-point prior considered in [5] is minimax when $m \le 0.643$. Conditions for inadmissibility or either methods of constructing dominators within a given class of estimators were given in [6]. Kumar and Tripathi [14], on the basis of MLE, proposed another estimator and compared the risk performance of it with the above-mentioned estimators. Dou and van Eeden [8], showed the inadmissibility conditions in [6] are satisfied for the bounded normal mean problem and hence, by giving an explicit form of a dominating estimator, derived inadmissibility of MLE of the mean θ. Lately, the general theory of estimating parameters of a symmetric distribution which is subject to an interval constraint, under SEL function developed in [20]. See also [16] for a similar development done under SEL function.

It is worth mentioning that the problem of estimating a normal mean θ in the case where θ is bounded below, i.e., $\theta \ge a$, for some constant a, also received considerable attention in the literature. Estimation of a positive normal mean was first considered in Katz [12]. Katz proposed the generalized Bayes estimator of θ w.r.t. the uniform prior on $[0,\infty)$ and proved its admissibility and minimaxity under SEL function. He also proved that the restricted MLE, is minimax. The results of Katz were independently proved in [25] and generalized in [9] to a general location parameter family under certain conditions. Thereafter, the problem of estimating a positive normal mean has developed in the literature, see for example, [28], [26] and references there in.

The concept of Bayesianity, admissibility and minimaxity highly depends on the choose of loss function. It is worth noting that there exist some other works related to restricted parameter estimation problem, considering other losses. Zeytinoglu and Mintz [31] obtained an admissible minimax estimator of $\theta \in [-m, m]$ for the zero-one loss function

$$L(\theta,\delta)=\begin{cases}1 & |\delta-\theta|>e\\ 0 & |\delta-\theta|\le e,\end{cases}$$

where $e>0$ is known and $m>e$. For the case where $e<m\le 2e$, their admissible minimax estimator is given by

$$d_Z(X)=\begin{cases}-m+e & X<-m+e\\ X & |X|\le m-e\\ m-e & X>m-e.\end{cases}$$

Bischoff et al. [4] under the LINear Exponential (LINEX) loss function

$$L(\theta,\delta)=e^{c(\delta-\theta)}-c(\delta-\theta)-1,\ c\ne 0,$$

where c is a known constant, showed that the Bayes estimator w.r.t. a two-point prior on $\{-m, m\}$, when $m\le \min\{c(\sqrt{3}+1)/2, \log 3/(2c)\}$ and $c>0$, is minimax. Towhidi and Behboodian [27] considered the so-called reflected normal loss function

$$L(\theta,\delta)=1-e^{(\delta-\theta)^2/2\gamma^2},$$

where γ is a known positive value. They proved that the Bayes estimator w.r.t. a symmetric two-point prior on $\{-m, m\}$, when $\gamma>2m$, is minimax but their result needs correction (see [30], p. 48). Iwasa and Moritani [11] under Absolute Error Loss (AEL) function

$$L(\theta,\delta)=|\delta-\theta|,$$

showed that MLE of the bounded mean θ is the unique Bayes estimator associated with a specific prior and hence, achieved the admissibility of the MLE.

Recently, Kucerovsky et al. [13] investigated Bayesianity of MLEs under AEL, for estimating the location parameter of symmetric and unimodal density functions in the presence of a lower (or upper) bounded and interval constraints. Their results in the bounded normal mean case, extends the results obtained in [11]. Marchand and Strawderman [22] studied the problem of estimating a location parameter θ for loss functions of the form $L(\theta,\delta)=\rho(\delta-\theta)$, under the restriction $\theta\ge a$ (known a). They showed that the Bayes estimator with respect to a uniform prior on $[a,\infty)$ is minimax. Then extending some previous dominance results due to [12] and [9], they obtained classes of dominators and extended their results to the case when $\theta\in[a,b]$.

In a historical view, it is worth noting that Marchand and Perron [17, 19] showed, in the multivariate version of the estimation problem of a bounded normal mean, that the Bayes estimator w.r.t. the uniform prior on $\{\theta : \|\theta-\theta_0\|=m\}$ (known θ_0) dominates MLE of θ on the restricted parameter space $\Theta_m=\{\theta : \|\theta-\theta_0\|\le m\}$ whenever $m\le\sqrt{p}$ for the model $X \sim N_p(\theta, I_p)$. For the case $p=1$, their result yields the result obtained in [5] (with $\theta_0=0$). Findings concerning the minimaxity of the Bayes estimator for small enough m were given in [2], [18] and [21].

In this paper we considered estimation of mean of a normal distribution under the additional assumption that the mean lies in the interval $\theta\in[-m,m], m>0$. Based on a random sample of size n, a new estimator is proposed and assuming SEL function, it is compared with the other estimators derived by some researches until 2009. To this end, first, all competitors for MLE of the mean θ are extended to the case when the sample size is n and then, a new estimator is derived. Note that the distribution of the obtained estimators is unknown and impossible to obtain. Risk function of the estimators is quite complicated, as well. Hence, using a simulation study, risk performance of estimators is compared to reach the appropriate one. It will be seen that the new estimator takes the minimum risk estimated values among the estimators for moderate values of θ in the interval $[-m, m]$.

Estimators for the Normal Mean When $\theta\in[-m, m]$

Let $X_1,\ldots,X_n$ be a random sample from $N(\theta,\sigma^2)$ where $\theta\in[a,b]$ and σ^2 is known. Without loss of generality, we assume $\sigma=1$ and $\theta\in[-m,m], m>0$. Under SEL function, the sample mean, $\bar{X}$, is not admissible and dominated by MLE of θ, which is given by

$$d_{MLE}(\bar{X})=\begin{cases}-m & \bar{X}<-m\\ \bar{X} & |\bar{X}|\le m\\ m & \bar{X}>m.\end{cases}$$

Hence, when θ belongs to the bounded parameter space $\Theta = [-m, m], m > 0, \bar{X}$ is no longer minimax and d_{MLE} is preferred to $\bar{X}$. Now, considering risk value as a comparative criterion, finding competitors for d_{MLE} becomes interesting.

In the rest of this section, we extend the result of the previous works in estimating a bounded normal mean to the case when the sample size is n. The idea is based on the fact that $\sqrt{n}\bar{X} \sim N(\theta^*, 1)$, where $\theta^* = \sqrt{n}\theta$ and $\theta^* \in [-m\sqrt{n}, m\sqrt{n}]$.

The following lemma plays a pivotal role in pursuing the theories.

Lemma 1 (Lehmann and Casella [15], P. 228) Let given θ, X have distribution P_θ. Then in problem of estimating θ with non-negative SEL function (1), the Bayes estimator is given by $\delta_\pi(X) = E[\theta \mid X]$.

I. Bayes Estimator w.r.t. a Symmetric Two-point Prior

Casella and Strawderman [5] considered the following symmetric two-point prior

$$\tau_m^0(\theta) = \begin{cases} \frac{1}{2} & \theta = -m \\ \frac{1}{2} & \theta = m. \end{cases}$$

It can be easily verified that the posterior distribution of θ given $\bar{X} = \bar{x}$ is

$$\tau_m^0(\theta \mid \bar{x}) = \frac{\phi\{\sqrt{n}(\bar{x} - \theta)\}}{\phi\{\sqrt{n}(\bar{x} - m)\} + \phi\{\sqrt{n}(\bar{x} + m)\}}, \quad \theta = -m, m,$$

where, $\phi(.)$ denotes the density functions of a standard normal random variable. Using Lemma 1, the Bayes estimator w.r.t. SEL function is the mean of the posterior distribution and can be obtained as

$$d_m^0(\bar{X}) = m \tanh(mn\bar{X}),$$

where $\tanh(.)$ is the tangent hyperbolic function.

The result of Casella and Strawderman [5] implies that d_m^0 is the unique minimax and admissible estimator for $m\sqrt{n} \le 1.0567$. It also can be deduced that d_m^0 dominates d_{MLE} when $m\sqrt{n} \le 1$.

II. Bayes Estimator w.r.t. a Symmetric Three-point Prior

For large values of m, Casella and Strawderman [5] considered the following symmetric three-point prior

$$\tau_m^\alpha(\theta) = \begin{cases} \alpha & \theta = 0 \\ \frac{1-\alpha}{2} & \theta = -m, m. \end{cases}$$

Hence, the posterior distribution of θ given $\bar{X} = \bar{x}$ is

$$\tau_m^\alpha(\theta \mid \bar{x}) = \begin{cases} \frac{2\alpha\,\phi\{\sqrt{n}(\bar{x} - \theta)\}}{(1-\alpha)\phi\{\sqrt{n}(\bar{x} - m)\} + (1-\alpha)\phi\{\sqrt{n}(\bar{x} + m)\} + 2\alpha\phi\{\sqrt{n}(\bar{x})\}}, & \theta = 0, \\ \frac{2(1-\alpha)\phi\{\sqrt{n}(\bar{x} - \theta)\}}{(1-\alpha)\phi\{\sqrt{n}(\bar{x} - m)\} + (1-\alpha)\phi\{\sqrt{n}(\bar{x} + m)\} + 2\alpha\phi\{\sqrt{n}(\bar{x})\}}, & \theta = -m, m. \end{cases}$$

Hence, the Bayes estimator of θ under SEL function is obtained as

$$d_m^\alpha(\bar{X}) = \frac{m(1-\alpha)\tanh(mn\bar{X})}{(1-\alpha) + \alpha e^{\frac{nm^2}{2}} \operatorname{sech}(mn\bar{X})},$$

where $\operatorname{sech}(.)$ is the secant hyperbolic function.

The result of Casella and Strawderman [5] implies that when $1.40 \le m\sqrt{n} \le 1.60$, there exists a unique α for which d_m^α is the unique minimax and admissible estimator of θ.

III. Bayes Estimator w.r.t.Bickel's Prior

Bickel [3] introduced the following prior in estimation of θ,

$$g_m(\theta) = \frac{1}{m}\cos^2\left(\frac{\pi\theta}{2m}\right), \quad |\theta| \le m.$$

The posterior density of θ is then

$$g(\theta \mid \bar{x}) = \frac{\phi\{\sqrt{n}(\theta - \bar{x})\} g_m(\theta)}{\int_{-m}^{m} \phi\{\sqrt{n}(\theta - \bar{x})\} g_m(\theta)\, d\theta}, \quad -m \le \theta \le m.$$

As before, the Bayes estimator is the mean of posterior distribution and can be expressed as

$$d_B(\bar{X}) = \bar{X} + \frac{1}{\sqrt{n}} \frac{\int_{-\sqrt{n}(m+\bar{X})}^{\sqrt{n}(m-\bar{X})} u\,\phi(u) \cos^2\{\frac{\pi}{2m}(\frac{u}{\sqrt{n}} + \bar{X})\}\, du}{\int_{-\sqrt{n}(m+\bar{X})}^{\sqrt{n}(m-\bar{X})} \phi(u) \cos^2\{\frac{\pi}{2m}(\frac{u}{\sqrt{n}} + \bar{X})\}\, du}.$$

IV. Bayes Estimation w.r.t. a Uniform Prior

Gatsonis et al. [10] considered the uniform prior

$$u_m(\theta) = \frac{1}{2m}, \quad |\theta| \le m.$$

The posterior distribution of θ given $\bar{X} = \bar{x}$ is

$$u_m(\theta \mid \bar{x}) = \frac{\sqrt{n}\phi\{\sqrt{n}(\bar{x}-\theta)\}}{\Phi\{\sqrt{n}(m-\bar{x})\} - \Phi\{-\sqrt{n}(m+\bar{x})\}}, \quad -m \le \theta \le m.$$

where $\Phi(.)$ and $\phi(.)$ denote the distribution and density functions of a standard normal random variable, respectively. The Bayes estimator under SEL function is obtained as

$$d_m(\bar{X}) = \bar{X} + \frac{1}{n}\frac{\Phi\left(\sqrt{n}(\bar{X}+m)\right) - \Phi\left(\sqrt{n}(\bar{X}-m)\right)}{\phi\left(\sqrt{n}(\bar{X}+m)\right) - \phi\left(\sqrt{n}(\bar{X}-m)\right)},$$

The results of Gatsonis et al. [10] imply that d_m dominates the sample mean $\bar{X}$ (see also [14]).

V. A Linearly Invariant Estimator

Another competitor for d_{MLE} is obtained considering the works done by Moors [23, 24]. He considered a linearly invariant estimator based on a given boundary estimator, i.e., an estimator which takes boundary values of the parameter space with positive probability, and proved inadmissibility of the boundary estimator under SEL function. Obviously, in the estimation problem of a bounded normal mean, d_{MLE} is such a boundary estimator and applying his results, it is inferred that d_{MLE} is inadmissible and dominated by its projection onto

$$\Theta_{\bar{x}} = [-m\tanh(mn|\bar{x}|), m\tanh(mn|\bar{x}|)].$$

(see also Kumar and Tripathi [14]). So, the dominating estimator of d_{MLE} is given by

$$d_{Mr}(\bar{X}) = \begin{cases} -m\tanh(mn|\bar{X}|) & \bar{X} < -m\tanh(mn|\bar{X}|) \\ \bar{X} & |\bar{X}| \le m\tanh(mn|\bar{X}|) \\ m\tanh(mn|\bar{X}|) & \bar{X} > m\tanh(mn|\bar{X}|) \end{cases}$$

VI. Kumar and Tripathi's Estimator

The last but not the least competitor for d_{MLE} presented by Kumar and Tripathi [14]. They considered the average of MLE of θ based on each random variable X_i alone, i.e.,

$$d_{Km}(\mathbf{X}) = \frac{1}{n}\sum_{i=1}^{n} d_{MLE}(X_i).$$

They applied Rao-Blakwell theorem (see Lehmann and Casella [15], p. 47) and derived the following improved estimator of θ

$$\begin{aligned}\hat{d}_{Km}(\bar{X}) = \bar{X} &+ (m-\bar{X})\Phi\{-\sqrt{\frac{n}{n-1}}(m-\bar{X})\} \\ &-(m+\bar{X})\Phi\{-\sqrt{\frac{n}{n-1}}(m+\bar{X})\} \\ &+\sqrt{\frac{n+1}{n}}[\phi\{\sqrt{\frac{n}{n-1}}(m+\bar{X})\} \\ &-\phi\{\sqrt{\frac{n}{n-1}}(m-\bar{X})\}].\end{aligned}$$

Kumar and Tripthi [14], introduced $\hat{d}_{Km}$ as a competitor for d_{MLE} and numerically compared risk performance of all the estimators mentioned above with each other.

A New Smooth Estimator

In this section, we introduce a new competitor for d_{MLE}. This new and smooth estimator is based on the estimator suggested by Dou and van Eeden [8] and using the idea of Kumar and Tripathi [14]. For a single normal random variable, they suggested the use of the shrinkage estimator

$$d_{Ch}(X) = \begin{cases} -m+\epsilon & X < \epsilon - m \\ X & |X| \le m-\epsilon \\ m-\epsilon & X > m-\epsilon, \end{cases}$$

where $0 < \epsilon < m$, and obtained sufficient conditions on ϵ for which d_{Ch} dominates MLE of θ. Their main result is as follows.

Theorem 1. (**Dou and van Eeden [8]**) Let $X \sim N(\theta,1)$ when $|\theta| \le m$, $m > 0$. Then $\{d_{Ch} : 0 < \epsilon \le \epsilon'\}$ is a class of dominating estimators of d_{MLE}, where ϵ' is the unique root of $\psi(x) = 0$, where

$$\psi(x) = g(2m-x) + g(x) - 2x \qquad (2)$$

and $g(x) = 2x\,\Phi(-x)$.

In connection with Theorem 1, the unique root of eq. (2) for some selected values of m have been calculated. These values are given in Table 1.

Now, let d_{Av} be the average of d_{Ch} based on each random variable alone, i.e.

$$d_{Av}(\mathbf{X}) = \frac{1}{n}\sum_{i=1}^{n} d_{Ch}(X_i).$$

Obviously, d_{Av} does not depend on the sufficient statistic $\bar{X}$, and it can be improved using Rao-Blakwell theorem. Hence, a new estimator of θ is derived by conditioning d_{Av} on the sufficient statistic $\bar{X}$. This estimator is given in the following theorem.

Theorem 2 d_{Av} is an inadmissible estimator and its improvement is given by the following estimator

$$\begin{aligned}\hat{d}_{Av}(\bar{X}) &= E\left[d_{Av}(\mathbf{X}) \mid \bar{X}\right]\\ &= \bar{X} + (m-\epsilon' - \bar{X})\Phi\{-\sqrt{\tfrac{n}{n-1}}(m-\epsilon'-\bar{X})\}\\ &\quad -(m-\epsilon'+\bar{X})\Phi\{-\sqrt{\tfrac{n}{n-1}}(m-\epsilon'+\bar{X})\}\\ &\quad +\sqrt{\tfrac{n-1}{n}}\big[\phi\{\sqrt{\tfrac{n}{n-1}}(m-\epsilon'+\bar{X})\}\\ &\quad -\phi\{\sqrt{\tfrac{n}{n-1}}(m-\epsilon'-\bar{X})\}\big],\end{aligned}$$

where ϵ' is the unique root of eq. (2). Moreover $\hat{d}_{Av}$ has following properties

(a) $|\hat{d}_{Av}(\bar{x})| < m - \epsilon' < m$ for all $\bar{x} \in \Re$.

(b) $\lim\limits_{\bar{x}\to-\infty}\hat{d}_{Av}(\bar{x}) = \epsilon' - m$, $\quad \lim\limits_{\bar{x}\to+\infty}\hat{d}_{Av}(\bar{x}) = m - \epsilon'$.

(c) $\lim\limits_{m\to\infty}\hat{d}_{Av}(\bar{x}) = \bar{x}$.

Proof. Notice that

$$\begin{aligned}\hat{d}_{Av}(\bar{X}) &= E\left[d_{Av}(\mathbf{X}) \mid \bar{X}\right]\\ &= \frac{1}{n}\sum_{i=1}^{n} E\left[d_{Ch}(X_i)\middle|\bar{X}\right]\\ &= E\left[d_{Ch}(X_1)\middle|\bar{X}\right],\end{aligned}$$

where the last equality holds because X_i's are identically distributed. Now using the fact that $X_1 \middle| \bar{X} = \bar{x} \sim N(\bar{x}, 1-1/n)$, the above equality can be written as

$$\begin{aligned}\hat{d}_{Av}(\bar{x}) &= -(m-\varepsilon)P\left(X_1 \le -m+\varepsilon \mid \bar{X} = \bar{x}\right)\\ &\quad +\int_{-m+\varepsilon}^{m-\varepsilon} x_1\sqrt{\frac{n}{n-1}}\,\phi\left\{\sqrt{\frac{n}{n-1}}(x_1-\bar{x})\right\}dx_1 \qquad (3)\\ &\quad +(m-\varepsilon)P\left(X_1 \ge m-\varepsilon \mid \bar{X} = \bar{x}\right).\end{aligned}$$

The terms in (3) can be written respectively as

$$\begin{aligned}P\left(X_1 \le -m+\varepsilon \mid \bar{X} = \bar{x}\right) &= P\left(\frac{\sqrt{n}(X_1-\bar{X})}{\sqrt{n-1}} \le \frac{\sqrt{n}(-m+\varepsilon-\bar{x})}{\sqrt{n-1}}\right)\\ &= \Phi\left\{-\sqrt{\frac{n}{n-1}}(m-\varepsilon+\bar{x})\right\},\end{aligned}$$

$$\begin{aligned}&\int_{-m+\varepsilon}^{m-\varepsilon} x_1\sqrt{\frac{n}{n-1}}\phi\left\{\sqrt{\frac{n}{n-1}}(x_1-\bar{x})\right\}dx_1\\ &= \int_{-m+\varepsilon}^{m-\varepsilon}(x_1-\bar{x})\sqrt{\frac{n}{n-1}}\phi\left\{\sqrt{\frac{n}{n-1}}(x_1-\bar{x})\right\}dx_1\\ &\quad +\bar{x}\int_{-m+\varepsilon}^{m-\varepsilon}\sqrt{\frac{n}{n-1}}\phi\left\{\sqrt{\frac{n}{n-1}}(x_1-\bar{x})\right\}dx_1\\ &= \sqrt{\frac{n-1}{n}}\left[\phi\left\{\sqrt{\frac{n}{n-1}}(m-\varepsilon+\bar{x})\right\} - \phi\left\{\sqrt{\frac{n}{n-1}}(m-\varepsilon-\bar{x})\right\}\right]\\ &\quad +\bar{x}\left[1-\Phi\left\{-\sqrt{\frac{n}{n-1}}(m-\varepsilon+\bar{x})\right\} - \Phi\left\{-\sqrt{\frac{n}{n-1}}(m-\varepsilon-\bar{x})\right\}\right],\end{aligned}$$

$$\begin{aligned}P\left(X_1 \ge m-\varepsilon \mid \bar{X} = \bar{x}\right) &= P\left(\frac{\sqrt{n}(X_1-\bar{X})}{\sqrt{n-1}} \ge \frac{\sqrt{n}(m-\varepsilon-\bar{x})}{\sqrt{n-1}}\right)\\ &= \Phi\left\{-\sqrt{\frac{n}{n-1}}(m-\varepsilon-\bar{x})\right\}.\end{aligned}$$

Now, substituting the above relations in (3) and letting $\bar{x} = \bar{X}$, the estimator $\hat{d}_{Av}$ is obtained. To complete the proof, note that d_{Av} takes its value between $-m+\varepsilon$ and $m-\varepsilon$. The properties (a), (b) and (c) are easily observed by using the properties of ϕ and Φ.

Results and Discussion

In the previous sections, estimation of the normal mean θ when it is known to lie in an interval $[-m, m]$, has been considered. In this case, As noted, $\bar{X}$ is dominated by d_{MLE}. Hence in finding competitors for

Table 1. Values of the unique root ϵ' given by Eq. (2)

m	0.25000	0.50000	0.75000	1.00000	1.25000	1.50000
ϵ'	0.19867	0.27846	0.21852	0.10119	0.03273	0.00824

Table 2. Values taken by estimators when $n=4,\ m=0.5$ and $\epsilon^{'}=0.27846$

θ	$\overline{X}$	d_{MLE}	d_m^0	d_m^α	d_B	d_m	d_{Mr}	$\hat{d}_{Km}$	$\hat{d}_{Av}$
0.00	-0.108026	-0.108026	-0.106376	-0.062929	-0.012896	-0.031374	-0.106376	-0.047022	-0.021756
0.05	0.404061	0.404061	0.334277	0.219130	0.047934	0.113870	0.334277	0.170737	0.078776
0.10	-0.100459	-0.100459	-0.099129	-0.058567	-0.011993	-0.029186	-0.099129	-0.043742	-0.020239
0.15	0.273371	0.273371	0.249037	0.154421	0.032549	0.078390	0.249037	0.117524	0.054311
0.20	0.349158	0.349158	0.301649	0.192929	0.041492	0.099192	0.301649	0.148727	0.068672
0.25	0.099063	0.099063	0.097787	0.057761	0.011827	0.028782	0.097787	0.043137	0.019959
0.30	0.655338	0.500000	0.432223	0.318931	0.076910	0.175863	0.432223	0.263289	0.120933
0.35	-0.017732	-0.017732	-0.017724	-0.010388	-0.002118	-0.005162	-0.017724	-0.007736	-0.003580
0.40	0.073097	0.073097	0.072581	0.042717	0.008728	0.021257	0.072581	0.031858	0.014742
0.45	1.324101	0.500000	0.495015	0.450209	0.147837	0.293735	0.495015	0.426451	0.192650
0.50	0.620883	0.500000	0.422979	0.307221	0.072996	0.167915	0.422979	0.251497	0.115599

Table 3. Values taken by estimators when $n=4,\ m=1$ and $\epsilon^{'}=0.10119$

θ	$\overline{X}$	d_{MLE}	d_m^0	d_m^α	d_B	d_m	d_{Mr}	$\hat{d}_{Km}$	$\hat{d}_{Av}$
0.0	0.109818	0.109818	0.413042	0.106345	0.040913	0.084637	0.109818	0.082421	0.076804
0.1	-0.422565	-0.422565	-0.934180	-0.438604	-0.155198	-0.309361	-0.422565	-0.309903	-0.288155
0.2	0.633724	0.633724	0.987511	0.658815	0.228490	0.436249	0.633724	0.450835	0.418053
0.3	0.344863	0.344863	0.880828	0.352437	0.127302	0.257012	0.344863	0.255008	0.237294
0.4	0.549323	0.549323	0.975613	0.575551	0.199686	0.388357	0.549323	0.396122	0.367759
0.5	1.102342	1.000000	0.999704	0.928212	0.373597	0.636100	0.999704	0.705435	0.648657
0.6	0.702436	0.702436	0.992773	0.719387	0.251397	0.472408	0.702436	0.493583	0.457204
0.7	0.692585	0.692585	0.992183	0.711149	0.248144	0.467379	0.692585	0.487558	0.451695
0.8	1.010425	1.000000	0.999383	0.899349	0.347305	0.604924	0.999383	0.662671	0.610435
0.9	0.088560	0.088560	0.340129	0.085504	0.033006	0.068347	0.088560	0.066505	0.061976
1.0	1.203685	1.000000	0.999868	0.951027	0.401315	0.666698	0.999868	0.748347	0.686729

Table 4. Estimated Risk values when $n=4,\ m=0.5$ and $\epsilon^{'}=0.27846$

θ	$R(\theta,d_{MLE})$	$R(\theta,d_m^0)$	$R(\theta,d_m^\alpha)$	$R(\theta,d_B)$	$R(\theta,d_m)$	$R(\theta,d_{Mr})$	$R(\theta,\hat{d}_{Km})$	$R(\theta,\hat{d}_{Av})$
0.00	0.129643	0.099390	0.053518	0.003422	0.016953	0.099390	0.037568	0.007885
0.05	0.128461	0.098245	0.053018	0.005252	0.017831	0.098245	0.037469	0.009355
0.10	0.127330	0.098524	0.055481	0.011195	0.022232	0.098524	0.040851	0.014544
0.15	0.123908	0.096847	0.057329	0.020779	0.028482	0.096847	0.044339	0.022686
0.20	0.123873	0.098594	0.062638	0.034618	0.038446	0.098594	0.051343	0.034825
0.25	0.119793	0.097408	0.067461	0.052111	0.050403	0.097408	0.058885	0.049918
0.30	0.119413	0.097056	0.076369	0.074072	0.066509	0.100554	0.070425	0.069355
0.35	0.115743	0.100554	0.081094	0.098394	0.082007	0.097056	0.079004	0.089871
0.40	0.113317	0.103939	0.094672	0.128577	0.104664	0.103939	0.095700	0.116953
0.45	0.111153	0.104499	0.104697	0.161470	0.127352	0.104499	0.110151	0.145622
0.50	0.113898	0.111489	0.129830	0.199333	0.155200	0.111489	0.129965	0.179412

d_{MLE}, several alternatives have been pointed out. All the proposed estimators with the exception of $\bar{X}$, are range-preserving, i.e., satisfy

$$P_\theta\left(\delta(\bar{X}) \in [-m, m]\right) = 1,$$

and all are invariant with respect to the finite group of transformations $G = \{e, g\}$ where $e(\bar{x}) = \bar{x}$ is an identity transformation and $g(\bar{x}) = -\bar{x}$ for all $\bar{x} \in \chi$. Values taken by the estimators for some selected values of m and n, have been computed based on a numerical simulation. These values are shown in Tables 2 and 3. Note that the values taken by $\bar{X}$ for some θ, are outside the interval $[-m, m]$, while the other estimates take values in the interval $[-m, m]$. Especially $\hat{d}_{Av}$ takes values in the interval $[-m+\varepsilon, m-\varepsilon]$, where ε is the unique root of eq. (2). This property can be seen from Theorem 2. It is worth mentioning that the risk of $\bar{X}$ is the constant $1/n$ but the risk functions of other estimators, cannot be evaluated in a closed form. Estimated risk values of all estimators have been presented for some selected values of n and m. Note that α has been chosen equal to 0.30, when computing the risk function of d_m^α. This is due to the minimaxity condition (see [5] and [14]). Note that in the interval $(0, \epsilon']$, as ϵ tends to ϵ', the difference between the risk function of d_{Ch} and d_{MLE} becomes larger. So, ϵ has been chosen equal to ϵ' (this property can be seen from the results in [8]). Since the risk function of all estimators are symmetric about $\theta = 0$, the estimated risks have been tabulated for $\theta \in [0, m]$. The estimated risks have been evaluated using a simulation study based on 10,000 generations of sample size n from $N(\theta, 1)$ population using a MATLAB package. The following conclusions can be drawn from the results:

(a) The performance of d_{MLE} is very good w.r.t. $\bar{X}$. As mentioned in the proceeding section, d_{MLE} dominates the usual estimator $\bar{X}$. The risk function of d_{MLE} decreases, as θ increases in the interval $[0, m]$ and takes its maximum when θ is close to zero. But in these parts of the interval $[0, m]$, $\hat{d}_{Av}$ has a sensible improvement. These properties are obviously observed from Tables 4-7.

(b) The performance of d_m^0 w.r.t. d_{MLE} is noticeable especially when $m\sqrt{n} \le 1$. As mentioned earlier, extending the results of Casella and Straderman [5], it is inferred that the Bayes estimator w.r.t. the symmetric two-point prior, i.e., d_m^0 dominates d_{MLE}, when $m\sqrt{n} \le 1$. Obviously in Tables 4-7, the minimaxity condition holds true only in Table 4 and this confirms the mentioned claim. However, for small and moderate values of θ in the interval $[0, m]$, $\hat{d}_{Av}$ has better performance than d_m^0. These properties are obviously observed from Tables 4-7.

(c) The performance of d_m^α when $1.40 \le m\sqrt{n} \le 1.60$, is satisfactory. As mentioned in the previous section, when $1.40 \le m\sqrt{n} \le 1.60$, d_m^α is a unique minimax and admissible estimator. Obviously the values of n and m in Table 6 satisfy this condition. Under minimaxity condition, estimated risk of d_m^α varies slowly but among the estimators better alternatives there exist. Tables 4-7 show that for small and moderate values of θ in the interval $[0, m]$, $\hat{d}_{Av}$ performs better than d_m^α.

(d) The performance of d_B when θ is close to zero is quite good. However, its risk increases rapidly as θ closes to m. Numerical calculations yield that for some values of m and n, when θ closes to m, the risk of d_B exceeds that of $\bar{X}$ (see Tables 5-7). It is worth noting that for moderate and large values of θ in $[0, m]$, d_m performs better than d_B. As mentioned in the previous section, d_m dominates $\bar{X}$. This desired property can be seen from Tables 4-7.

(e) d_{Mr} has better risk performance than d_{MLE} for small values of m and n. But its improvement for large values of m and n is insignificant. However as mentioned earlier, d_{Mr} dominates d_{MLE}. But for small and moderate values of θ in the interval $[0, m]$, $\hat{d}_{Av}$ has better performance than d_{Mr} and d_{MLE}. These properties can be seen from Tables 4-7.

(f) The risk performance of $\hat{d}_{Km}$ for small and moderate values of θ in the interval $[0, m]$ is satisfactory. But in these parts $\hat{d}_{Av}$ improves upon $\hat{d}_{Km}$. This can be seen from Tables 4-7. Further, for large values of θ, d_{MLE} and d_{Mr} perform better than $\hat{d}_{Av}$ and $\hat{d}_{Km}$.

(g) Similar observations are made for other various values of m and n.

Remark 1. In a carefully view on the estimated risk values in Table 2, one will find that $R(\theta, d_{Mr})$ and

Table 5. Estimated Risk values when $n=4,\ m=1$ and $\epsilon^{'}=0.10119$

θ	$R(\theta,d_{MLE})$	$R(\theta,d_m^0)$	$R(\theta,d_m^\alpha)$	$R(\theta,d_B)$	$R(\theta,d_m)$	$R(\theta,d_{Mr})$	$R(\theta,\hat{d}_{Km})$	$R(\theta,\hat{d}_{Av})$
0.0	0.234271	0.636863	0.234920	0.031686	0.108105	0.234245	0.119478	0.102194
0.1	0.230517	0.622033	0.226725	0.035227	0.107050	0.230492	0.118090	0.101479
0.2	0.220742	0.579401	0.215745	0.046915	0.106005	0.220715	0.116909	0.101830
0.3	0.210912	0.522805	0.204360	0.066768	0.106884	0.210885	0.117278	0.104474
0.4	0.203971	0.453772	0.194873	0.097063	0.113111	0.203938	0.123088	0.113036
0.5	0.186759	0.366342	0.176310	0.134576	0.118479	0.186726	0.126502	0.120688
0.6	0.173964	0.294625	0.162964	0.183324	0.132167	0.173931	0.136698	0.135886
0.7	0.152328	0.208765	0.141759	0.235838	0.143513	0.152301	0.143074	0.148326
0.8	0.143431	0.152593	0.134932	0.306316	0.172474	0.143410	0.164624	0.176990
0.9	0.128981	0.089191	0.118735	0.380658	0.198472	0.123969	0.180380	0.201682
1.0	0.123922	0.072575	0.127634	0.472322	0.246623	0.128322	0.215759	0.246360

Table 6. Estimated Risk values when $n=10,\ m=0.5$ and $\epsilon^{'}=0.27846$

θ	$R(\theta,d_{MLE})$	$R(\theta,d_m^0)$	$R(\theta,d_m^\alpha)$	$R(\theta,d_B)$	$R(\theta,d_m)$	$R(\theta,d_{Mr})$	$R(\theta,\hat{d}_{Km})$	$R(\theta,\hat{d}_{Av})$
0.00	0.081933	0.139832	0.064293	0.006356	0.025192	0.081598	0.014944	0.003133
0.05	0.080139	0.138263	0.063120	0.007551	0.025226	0.079821	0.015474	0.004710
0.10	0.079422	0.132712	0.062500	0.011842	0.026980	0.079080	0.018431	0.009859
0.15	0.075482	0.122624	0.059983	0.018705	0.029229	0.075161	0.022864	0.018252
0.20	0.072447	0.112194	0.058379	0.028875	0.033519	0.072126	0.029852	0.030420
0.25	0.066252	0.095970	0.054127	0.040688	0.037156	0.065946	0.037150	0.045117
0.30	0.063131	0.085030	0.053718	0.056980	0.045077	0.062872	0.048623	0.064462
0.35	0.057105	0.069576	0.050877	0.075093	0.052664	0.056895	0.060534	0.086503
0.40	0.054159	0.058925	0.051309	0.097173	0.063801	0.054011	0.075896	0.112826
0.45	0.051133	0.049586	0.051665	0.121218	0.075511	0.051048	0.091976	0.141935
0.50	0.050800	0.043818	0.055322	0.149730	0.091424	0.050797	0.112142	0.175733

Table 7. Estimated Risk values when $n=10,\ m=1$ and $\epsilon^{'}=0.10119$

θ	$R(\theta,d_{MLE})$	$R(\theta,d_m^0)$	$R(\theta,d_m^\alpha)$	$R(\theta,d_B)$	$R(\theta,d_m)$	$R(\theta,d_{Mr})$	$R(\theta,\hat{d}_{Km})$	$R(\theta,\hat{d}_{Av})$
0.0	0.100955	0.751881	0.089058	0.034653	0.081306	0.097097	0.045465	0.038878
0.1	0.097097	0.729511	0.100594	0.037220	0.083360	0.100955	0.047864	0.041374
0.2	0.098702	0.661062	0.110639	0.041836	0.079380	0.098702	0.050234	0.045079
0.3	0.097987	0.545296	0.131363	0.048541	0.074610	0.097987	0.053753	0.050455
0.4	0.093752	0.418567	0.142655	0.059108	0.067915	0.093752	0.059180	0.058826
0.5	0.090639	0.301427	0.144453	0.073677	0.062510	0.090639	0.067366	0.070566
0.6	0.081839	0.195776	0.130346	0.091955	0.056890	0.081839	0.077039	0.084916
0.7	0.074422	0.112011	0.109441	0.120561	0.058858	0.074422	0.095065	0.108589
0.8	0.063924	0.052567	0.080792	0.154443	0.064243	0.063924	0.115903	0.136247
0.9	0.056685	0.016776	0.057413	0.197403	0.079105	0.056685	0.143851	0.171881
1.0	0.051855	0.002080	0.041255	0.248287	0.101354	0.051855	0.176775	0.213971

$R(\theta, d_m^0)$ are the same. For a single observation, the reason can be found in [8]. They showed that for a single normal random variable, when $m \le 1$, $d_{Mr}(x) = d_m^0(x)$, for all $x \in \mathcal{X}$. Extending their result to a random sample of size n, one can deduce that when $m\sqrt{n} \le 1$, $d_{Mr}(\bar{x}) = d_m^0(\bar{x})$, for all $\bar{x} \in \mathcal{X}$. Clearly, in Table 4, $m\sqrt{n} \le 1$ and $R(\theta, d_{Mr})$ = $R(\theta, d_m^0)$. This relation also can be seen in Table 1 of [14], but they did not mention anything about it.

Conclusion

On the basis of our numerical results, using $\bar{X}$ and d_{MLE} is not recommended when some information are accessible about the finite bounds on $[0, m]$. In this case, d_{Mr}, d_B or $\hat{d}_{Av}$ can be used instead. When the prior information indicates θ to be close to zero or moderate values of the interval $[0, m]$, $\hat{d}_{Av}$ or d_B can be used. Otherwise, it is recommended to use d_{Mr}.

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