On Special Generalized Douglas-Weyl Metrics

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Received: 16 August 2011 / Revised: 8 January 2012 / Accepted: 22 January 2012

Abstract

In this paper, we study a special class of generalized Douglas-Weyl metrics whose Douglas curvature is constant along any Finslerian geodesic. We prove that for every Landsberg metric in this class of Finsler metrics, \( \mathcal{E} = 0 \) if and only if \( H = 0 \). Then we show that every Finsler metric of non-zero isotropic flag curvature in this class of metrics is a Riemannian if and only if \( \mathcal{E} = 0 \).

Keywords: Douglas space; Landsberg metric; The non-Riemannian quantity H

Introduction

For a Finsler metric \( F = F(x, y) \) on a manifold \( M \), its geodesics curves are characterized by the system of differential equations \( \ddot{c}^i + 2G^i(c^i) = 0 \), where the local functions \( G^i = G^i(x, y) \) are called the spray coefficients and given by following

\[
G^i = \frac{1}{4} g^{il} \left( \frac{\partial^2 F^2}{\partial x^l \partial y^i} y^k - \frac{\partial F^2}{\partial x^l} \right), \quad y \in T_x M.
\]

Thus \( F \) induced a spray \( G = y^i \frac{\partial}{\partial x^i} - 2G^j \frac{\partial}{\partial y^j} \) which determines the geodesics [9,15].

Two Finsler metrics \( F \) and \( \overline{F} \) on a manifold \( M \) are called projectively related if any geodesic of the first is also geodesic for the second and the other way around. Hereby, there is a scalar function \( P = P(x, y) \) defined on \( TM_y \) such that

\[
G^i = \overline{G}^i + P \ y^i,
\]

where \( G^i \) and \( \overline{G}^i \) are the geodesic spray coefficients of \( F \) and \( \overline{F} \), respectively and \( P \) is positively \( y \)-homogeneous of degree one [6,8].

Let

\[
D^j_{\ i \ \ \ k \ l} := \frac{\partial}{\partial y^j} \left[ G^i - \frac{1}{n+1} \frac{\partial G^m}{\partial y^m} y^i \right].
\]

It is easy to verify that, \( D := D^j_{\ i \ \ \ k \ l} \otimes \frac{\partial}{\partial x^k} \otimes \frac{\partial}{\partial x^l} \) is a well-defined tensor on slit tangent bundle \( TM_y \). We call \( D \) the Douglas tensor. The Douglas tensor \( D \) is a non-Riemannian projective invariant, namely, if two Finsler metrics \( F \) and \( \overline{F} \) are projectively equivalent, \( G^i = \overline{G}^i + P \ y^i \), where \( P = P(x, y) \) is positively \( y \)-homogeneous of degree one, then the Douglas tensor of \( F \) is same as that of \( \overline{F} \) [8]. Finsler metrics with vanishing Douglas tensor are called Douglas metrics. The notion of Douglas curvature was proposed by Bacsó-Matsumoto as a generalization of Berwald curvature [3]. There is another projective invariant in Finsler geometry, namely \( D^j_{\ i \ \ \ k \ l} y^m = T_{\ j \ i \ \ \ k \ l} y^m \), that is hold for some tensor \( T_{\ j \ i \ \ \ k \ l} \), where \( D^j_{\ i \ \ \ k \ l} \) denotes the

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horizontal covariant derivatives of $D^i_{jk}$ with respect to the Berwald connection of Finsler metric $F$. This equation implies that the rate of change of the Douglas curvature along a geodesic is tangent to the geodesic.

In this paper, we study on a class of Finsler metrics whose Douglas curvature satisfies

$$D^i_{jk}y^j = 0 \quad (1)$$

The geometric mining of this equation is that on these new spaces, the Douglas tensor is constant along a geodesics.

Other than Douglas curvature, there are several important non-Riemannian quantities: the Cartan torsion $C$, the Berwald curvature $B$, the mean Berwald curvature $E$, and the Landsberg curvature $L$, etc. [12-15]. The study show that the above mentioned non-Riemannian quantities are closely related to the Douglas metrics, namely Bácsó-Matsumoto proved that every Douglas metric with vanishing Landsberg curvature is a Berwald metric [1,2]. Is there any other interesting non-Riemannian quantity with such property? In [10], Shen find a new non-Riemannian quantity for Finsler metrics that is closely related to the $E$-curvature and call it $E$-curvature. Recall $E$ is obtained from the mean Berwald curvature by the covariant horizontal differentiation along geodesics.

In this paper, we prove that for every Landsberg metric satisfies (1), $E = 0$ if and only if $H = 0$. More precisely, we prove the following.

**Theorem 1.** Let $(M,F)$ be a Finsler space satisfies (1). Suppose that $F$ is a Landsberg metric. Then $E = 0$ if and only if $H = 0$.

For a non-zero vector $y \in T_xM$, the Riemann curvature $R_y : T_xM \to T_xM$ is defined by

$$R_y(u)v = R^i_j(u)v^i = \frac{\partial G^i}{\partial x^j}v^j - \frac{\partial G^i}{\partial x^j}v^j + 2G^i_j \frac{\partial G^j}{\partial x^i}v^k \frac{\partial G^k}{\partial x^i}v^j.$$  

The family $R = \{R_y\}_{y \in TM}$ is called the Riemann curvature [5].

Suppose $P \subset T_xM$ (flag) is an arbitrary plane and $y \in P$ (flag pole). The flag curvature $K(P,y)$ is defined by

$$K(P,y) = \frac{g_y(R_y(u)v)}{g_y(y,y)g_y(v,v) - g_y(v,y)g_y(v,y)}$$

where $v$ is an arbitrary vector in $P$ such that $P = \text{span}\{y,v\}$. A Finsler metric $F$ is said to be of isotropic flag curvature if $K = K(x)$. In this paper, we show that every metrics in this class of Finsler metrics with non-zero isotropic flag curvature is a Riemannian metric if and only if $E = 0$.

**Theorem 2.** Let $F$ be a Finsler metric satisfies (1) of non-zero isotropic flag curvature $K = K(x)$. Then $F$ is a Riemannian metric if and only if $E = 0$.

There are many connections in Finsler geometry [11]. In this paper we set the Berwald connection on Finsler manifolds. The h- and v- covariant derivatives of a Finsler tensor field are denoted by “|” and “,” respectively.

**Preliminaries**

Let $M$ be a $n$-dimensional $C^\infty$ manifold. Dnote by $T_xM$ the tangent space at $x \in M$ by $TM = \bigcup_{x \in M} T_xM$ the tangent bundle of $M$, and by $TM_0 = TM \setminus \{0\}$ the slit tangent bundle on $M$. A Finsler metric on $M$ is a function $F : TM \to [0,\infty)$ which has the following properties:

(i) $F$ is $C^\infty$ on $TM_0$; (ii) $F$ is positively 1-homogeneous on the fibers of tangent bundle $TM$, (iii) for each $y \in T_xM$, the following quadratic form $g_y$ on $T_yM$ is positive-definite,

$$g_y(u, v) = \frac{1}{2} \frac{\partial^2}{\partial \alpha^2} \left( F^2(y + \alpha u + tv) \right)_{u^\alpha v^\alpha = 0} \quad y \in T_xM.$$  

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of $F_x$ define $C_x : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by

$$C_x(u,v,w) := \frac{d}{dt} \left( g_{uvw}(u, v, w) \right)_{u^\alpha v^\alpha w^\alpha = 0} \quad u, v, w \in T_xM.$$  

The family $C := \{C_x\}_{x \in TM_0}$ is called the Cartan torsion. It is well known that $C = 0$ if and only if $F$ is Riemannian.

For $y \in T_xM_0$, define $L_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by $L_y(u,v,w) := L_{y^\alpha}(y)u^\alpha v^\alpha w^\alpha$, where $L_{y^\alpha} := C_{y^\alpha}w^\alpha$. The family $L := \{L_y\}_{y \in TM_0}$ is called the
Landsberg curvature. A Finsler metric $F$ is called a Landsberg metric if $L=0$ [4].

Given a Finsler manifold $(M,F)$, then a global vector field $G$ is induced by $F$ on $TM_0$, which in a standard coordinate $(x^i,y^i)$ for $TM_0$ is given by $G = y^i \frac{\partial}{\partial x^i} - 2G^i (x,y) \frac{\partial}{\partial y^i}$, where $G^i$ are local function on $TM$ given by

$$G^i := \frac{1}{4} g^{ij} \left( \frac{\partial^2 \left[ F^2 \right]}{\partial x^i \partial y^j} y^k - \frac{\partial \left[ F^2 \right]}{\partial x^i} \right), \quad y \in T_x M$$

$G$ is called the associated spray to $(M,F)$. The projection of an integral curve of $G$ is called a geodesic in $M$. In local coordinates, a curve $c(t)$ is a geodesic if and only if its coordinates $(c^i(t))$ satisfy $\dot{c}^i + 2G^i (\dot{c}) = 0$.

For a non-zero vector $y \in T_x M$, we can define $B_j \in T_y M \otimes T_y M \rightarrow \mathbb{R}$ and $E_j : T_y M \otimes T_y M \rightarrow \mathbb{R}$ by

$$B_j (u,v,w) = B_j^k (y) u^k \frac{\partial}{\partial x^j} w^i,$$

and

$$E_j (y) = E_j^k (y) \frac{\partial}{\partial x^j} y^i,$$

where $B_j^k$ is the Berwald curvature and $E_j^k$ is the mean Berwald curvature respectively. A Finsler metric is called a Berwald metric and weakly Berwald metric if $B_0 = E_0 = 0$, respectively [11].

The quantity $H_{ij} = H_{ij}^k dx^k \otimes dx^j$ is defined as the covariant derivative of $E$ along geodesics [7]. More precisely $H_{ij} = E_{ijk} y^k$.

For a flag $P = \text{span} \{y,u\} \subset T_x M$ flagpole $y$, the flag curvature $K = K(P,y)$ is defined by

$$K(P,y) = \frac{g_y(u,R_y(u))}{g_y(y,y)g_y(u,u) - g_y(y,u)^2},$$

We say that a Finsler metric $F$ is of scalar curvature if for any $y \in T_x M$, the flag curvature $K = K(x,y)$ is a scaler function on the slit tangent bundle $TM_0$.

By means of $E$-curvature, we can define $E_j : T_x M \otimes T_x M \otimes T_x M \rightarrow \mathbb{R}$ by

$$E_j (u,v,w) = E_{ijkl} (y) u^i v^j w^k,$$

where $E_{ijkl} = E_{ijk l}$. We call it $E$-curvature. From a Bianchi identity, we have

$$B_{jkl}^i - B_{jlk}^i = R_{ijkl}^m,$$

where $R_{ijkl}$ is the Riemannian curvature of Berwald connection [11]. This implies that $E_{ijk} - E_{ikj} = 2R_{ijkl}^m$. Then $E_{ijkl}$ is not totally symmetric in all three of its indices.

**Results and Discussion**

**Sakaguchi Theorem**

In this section, we are going to prove the well-known theorem of Sakaguchi. Our method is different from the Sakaguchi.

**Theorem 3.** Every Finsler metric of scalar flag curvature is a generalized Douglas-Weyl metric.

**Proof.** Let $F$ be a Finsler metric of scalar flag curvature $K$. The following holds

$$B_{jkm}^i y^k = 2K C_{jlm} y^l - \frac{1}{3} K_{jkm} F^2 h^i_j,$$

where

$$\frac{1}{3} K_{jkm} F^2 h^i_j = \frac{1}{3} K_{jkm} F^2 h^i_j.$$

We obtain

$$H_{ij} = -\frac{n+1}{6} \{y_j K_{ij} + y_j K_{ji} + K_{iji} F^2\}.$$
Thus, we can conclude that every Finsler metric of scalar flag curvature a generalized Douglas-Weyl metric. □

Proof of Theorem 1
To prove the Theorem 1, we need the following.

Lemma 2. Let \((M,F)\) be a Finsler manifold. Then the following holds

\[
D^i_{j,k,l,m} y^m = 2K C^i_{j,k} y^i - \frac{2}{3} K^i_{j,k} g_{kl} y^i + K^i_{j,k} g_{jk} + K_{j,k} g_{j,i} y^i - \frac{1}{3} \left\{ K_{j,i} y_k + K_{j,k} y_i + K_{k,j} y_y \right\} y^i \tag{4}
\]

Thus, we can conclude that every Finsler metric of scalar flag curvature a generalized Douglas-Weyl metric. □

Proof. The curvature form of Berwald connection is

\[
\Omega^i_j = \frac{1}{2} R^{i,j}_k \omega^k \wedge \omega^j - B^h_{j,k} \omega^j \wedge \omega^{j+h}. \tag{14}
\]

For the Berwald connection, we have the following structure equation

\[
dg_j = -g_{j,k} \Omega^k_i - g_{i,k} \Omega^i_j = -2L_{i,k} \omega^j + 2C_{i,k} \omega^{j+1}. \tag{15}
\]

Differentiating (15) yields the following Ricci identity

\[
d\Omega^j_i - \omega^j \wedge \omega^j + \omega^k \wedge \Omega^j_k = 0. \tag{17}
\]

Define \(B^i_{j,k,l,m}\) and \(R^i_{j,k,l,m}\) by

\[
dB^i_{j,k,l,m} = B^i_{j,k,l,m} \wedge \omega^j + \omega^k \wedge \Omega^j_k \wedge \omega^m. \tag{18}
\]

From (16), (17), (18) and (19), we get the proof. □

Proof of Theorem 1: From (16), it follows that
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\[ C_{ijkl} - L_{ijkl} = \frac{1}{2} g_{ij} B^p_{ikl} + \frac{1}{2} g_{pj} B^p_{ikl} \]  

(20)

Contracting (20) with \( y^j \) and using \( y^i_{,jk} + H_{ij,k} y^i + H_{ij,k} y^i = -\overline{E}_{ikl} y^i \) yields

\[ L_{ijkl} = -\frac{1}{2} g_{mn} y^{mn} B_{ijkl}^i. \]  

(21)

By assumption, we have

\[ B_{ijkl}^i y^m = \frac{2}{n+1} \{ H_{ik} \delta_j^i + H_{ik} \delta_j^i + H_{ik} \delta_j^i + H_{ik} \delta_j^i \}. \]  

(22)

Multiplying (22) with \( y^j \) and using (21), we get

\[ E_{ikl} = \{ H_{ik} y^j + H_{ik} y^j + H_{ik} y^j \} F^2 + H_{ikl}. \]  

(23)

By (23), we get the proof.  \[ \square \]

Proof of Theorem 2: Let \( R_{ikl} = y^i R_{ijkl} \). Then we have

\[ R_{ikl} = \frac{1}{3} \left[ \frac{\partial^2 R_{ikl}^i}{\partial y^i \partial y^j} - \frac{\partial^2 R_{ikl}^i}{\partial y^i \partial y^j} \right]. \]  

(24)

Here, we assume that a Finsler metric \( F \) is of scalar curvature \( K = K(x,y) \). In local coordinates,

\[ R_{ikl} = K F^2 h_{ikl}. \]  

(25)

Plugging (25) into (24) gives

\[ R_{ikl} = \frac{K_{ikl}}{3} F^2 h_{ikl} - \frac{K_{ikl}}{3} F^2 h_{ikl} \]
\[ + K \left\{ \frac{\partial F h_{ikl}}{\partial y^i} - \frac{\partial F h_{ikl}}{\partial y^j} \right\} \]
\[ + \frac{1}{3} K \left\{ 2 F F_{ij} \delta^i_k + g_{ij} y^i - F F_{ij} \delta^i_k \right\} \]
\[ + K \left\{ g_{ij} \delta^i_k - g_{ij} \delta^i_k \right\} \]
\[ + \frac{1}{3} K \left\{ 2 F F_{ij} \delta^i_k + g_{ij} y^i - F F_{ij} \delta^i_k \right\} \]

Differentiating (26) with respect to \( y^m \) gives a formula for \( R_{ijkl} \) expressed in terms of \( K \) and its derivatives. Contracting (12) with \( y^k \), we obtain

\[ B_{ijkl}^i y^k = 2 K C_{ijn} y^i - \frac{1}{3} K_{jlm} F^2 h_{ij} \]
\[ - \frac{1}{3} K_{jlm} F^2 h_{ij} - \frac{1}{3} K_{jlm} \{ F F_{ij} \delta^i_m + F F_{ij} \delta^i_m - 2 g_{pm} y^i \} \]
\[ - \frac{1}{3} K_{jlm} \{ F F_{ij} \delta^i_m + F F_{ij} \delta^i_m - 2 g_{pm} y^i \} \]

(27)

Since \( K = K(x) \), then by (27) we get

\[ B_{ijkl}^i y^k = 2 K C_{ijn} y^i \]  

(28)

Since \( F \) be a weakly Douglas Finsler metric, then we have

\[ B_{ijkl}^i y^m = \frac{2}{n+1} \{ H_{ik} \delta_j^i + H_{ik} \delta_j^i + H_{ik} \delta_j^i + H_{ik} \delta_j^i \}. \]  

(29)

From the assumptions, one can obtains

\[ B_{ijkl}^i y^m = 0. \]

By (28), we can conclude that \( C_{ikl} = 0 \) and then \( F \) is Riemannian.  \[ \square \]

Acknowledgements

The authors wish to thank referees and the editor for their useful comments and suggestions.

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