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On Special Generalized Douglas-Weyl Metrics

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Abstract

In this paper, we study a special class of generalized Douglas-Weyl metrics whose Douglas curvature is constant along any Finslerian geodesic. We prove that for every Landsberg metric in this class of Finsler metrics, $\bar{\mathrm{E}} = 0$ if and only if $\mathrm{H} = 0$. Then we show that every Finsler metric of non-zero isotropic flag curvature in this class of metrics is a Riemannian if and only if $\bar{\mathrm{E}} = 0$.

Keywords: Douglas space; Landsberg metric; The non-Riemannian quantity H

Introduction

For a Finsler metric $F = F(x,y)$ on a manifold M, its geodesics curves are characterized by the system of differential equations $\ddot{c}^i + 2G^i(\dot{c}^i) = 0$, where the local functions $G^i = G^i(x,y)$ are called the spray coefficients and given by following

$$G^i := \frac{1}{4} g^{il} \left\{ \frac{\partial^2 \left[F^2\right]}{\partial x^k \partial y^l} y^k - \frac{\partial \left[F^2\right]}{\partial x^l} \right\}, \quad y \in T_x M.$$

Thus F induced a spray $\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$ which determines the geodesics [9,15].

Two Finsler metrics F and $\bar{F}$ on a manifold M are called projectively related if any geodesic of the first is also geodesic for the second and the other way around. Hereby, there is a scalar function $P = P(x,y)$ defined on TM_0 such that

$$G^i = \bar{G}^i + P y^i,$$

where G^i and $\bar{G}^i$ are the geodesic spray coefficients of F and $\bar{F}$, respectively and P is positively y-homogeneous of degree one [6,8].

Let

$$D^i_{j\ kl} := \frac{\partial^3}{\partial y^j \partial y^k \partial y^l} \left[G^i - \frac{1}{n+1} \frac{\partial G^m}{\partial y^m} y^i \right].$$

It is easy to verify that, $D := D^i_{j\ kl} dx^j \otimes \partial_i \otimes dx^k \otimes dx^l$ is a well-defined tensor on slit tangent bundle TM_0. We call D the Douglas tensor. The Douglas tensor D is a non-Riemannian projective invariant, namely, if two Finsler metrics F and $\bar{F}$ are projectively equivalent, $G^i = \bar{G}^i + Py^i$, where $P = P(x,y)$ is positively y-homogeneous of degree one, then the Douglas tensor of F is same as that of $\bar{F}$ [8]. Finsler metrics with vanishing Douglas tensor are called Douglas metrics. The notion of Douglas curvature was proposed by Bácsó-Matsumoto as a generalization of Berwald curvature [3]. There is another projective invariant in Finsler geometry, namely $D^i_{j\ kl|m} y^m = T_{jkl} y^i$, that is hold for some tensor T_{jkl}, where $D^i_{j\ kl|m}$ denotes the

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horizontal covariant derivatives of $D^i_{j\ kl}$ whit respect to the Berwald connection of Finsler metric F. This equation implies that the rate of change of the Douglas curvature along a geodesic is tangent to the geodesic [6].

In this paper, we study on aclass of Finsler metrics whose Douglas curvature satisfies

$$D^i_{j\ kl|s}y^s = 0 \tag{1}$$

The geometric mining of this equation is that on these new spaces, the Douglas tensor is constant along a geodesics.

Other than Douglas curvature, there are several important non-Riemannian quantities: the Cartan torsoin **C**, the Berwald curvature **B**, the mean Berwald curvature **E**, and the Landsberg curvature **L**, etc. [12-15]. The study show that the above mentioned non-Riemannian quantities are closely related to the Douglas metrics, namely Bácsó-Matsumoto proved that every Douglas metric with vanishing Landsberg curvature is a Berwald metric [1,2]. Is there any other interesting non-Riemannian quantity with such property? In [10], Shen find a new non-Riemannian quantity for Finsler metrics that is closely related to the E-curvature and call it $\bar{E}$ – curvature. Recall $\bar{E}$ is obtained from the mean Berwald curvature by the covariant horizontal differentiation along geodesics.

In this paper, we prove that for every Landsberg metric satisfies (1), $\bar{\mathrm{E}} = 0$ if and only if $\mathrm{H} = 0$. More precisely, we prove the following.

Theorem 1. *Let (M, F) be a Finsler space satisfies (1). Suppose that F is a Landsberg metric. Then* $\bar{\mathrm{E}} = 0$ if and only if $\mathrm{H} = 0$.

For a non-zero vector $y \in T_xM_0$, the Riemann curvature $R_y : T_xM \to T_xM$ is defined by $R_y(u) := R^i_{\ k}(y)u^k \frac{\partial}{\partial x^i}$, where $R^i_{\ k}(y) = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k}y^j + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}$. The family $R := \{R_y\}_{y \in TM_0}$ is called the Riemann curvature [5]. Suppose $P \subset T_xM$ (flag) is an arbitrary plane and $y \in P$ (flag pole). The flag curvature $K(P, y)$ is defined by

$$K(P, y) = \frac{g_y(R_y(u), v)}{g_y(y, y)g_y(v, v) - g_y(v, y)g_y(v, y)}$$

where v is an arbitrary vector in P such that $P = span\{y, v\}$. A Finsler metric F is said to be of isotopic flag curvature if $K = K(x)$. In this paper, we show that every metrics in this class of Finsler metrics with non-zero isotropic flag curvature is a Riemannian metric if and only if $\bar{\mathrm{E}} = 0$.

Theorem 2. Let F be a Finsler metric satisfies (1) of non-zero isotropic flag curvature K = K(x). Then F is a Riemannian metric if and only if $\bar{\mathrm{E}} = 0$.

There are many connections in Finsler geometry [11]. In this paper we set the Berwald connection on Finsler manifolds. The h- and v- covariant derivatives of a Finsler tensor field are denoted by " | " and ", " respectively.

Preliminaries

Let M be a n-dimensional C^∞ manifold. Dnote by T_xM the tangent space at $x \in M$ by $TM = \bigcup_{x \in M} T_xM$ the tangent bundle of M, and by $TM_0 = TM \setminus \{0\}$ the slit tangent bundle on M. A Finsler metric on M is a function $F : TM \to [0, \infty)$ which has the following properties:

(i) F is C^∞ on TM_0; (ii) F is positively 1-homogeneous on the fibers of tangent bundle TM, (iii) for each $y \in T_xM$, the following quadratic form g_y on T_xM is positive-definite,

$$g_y(u, v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\left[F^2(y + su + tv)\right]\Big|_{s,t=0}, u, v \in T_xM.$$

Let $x \in M$ and $F_x := F\big|_{T_xM}$. To measure the non-Euclidean feature of F_x define $C_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ *by*

$$\mathrm{C}_y(u, v, w) := \frac{1}{2}\frac{d}{dt}\left[\mathrm{g}_{y+tw}(u, v)\right]\Big|_{t=0}, u, v, w \in T_xM.$$

The family $\mathrm{C} := \{\mathrm{C}_y\}_{y \in TM_0}$ is called the Cartan torsion. It is well known that $C = 0$ if and only if F is Riemannian.

For $y \in T_xM_0$, define $L_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by $L_y(u, v, w) := L_{ijk}(y)u^iv^jw^k$, where $L_{ijk} := C_{ijk|s}y^s$. The family $L := \{L_y\}_{y \in TM_0}$ is called the

Landsberg curvature. A Finsler metric F is called a Landsberg metric if **L**=0 [4].

Given a Finsler manifold (M, F), then a global vector field G is induced by F on TM_0, which in a standard coordinate (x^i, y^i) for TM_0 is given by $G = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i}$, where G^i are local function on TM given by

$$G^i := \frac{1}{4} g^{il}\left\{\frac{\partial^2[F^2]}{\partial x^k \partial y^l} y^k - \frac{\partial[F^2]}{\partial x^l}\right\}, \quad y \in T_x M$$

G is called the associated spray to (M, F). The projection of an integral curve of G is called a geodesic in M. In local coordinates, a curve $c(t)$ is a geodesic if and only if its coordinates $(c^i(t))$ satisfy $\ddot{c}^i + 2G^i(\dot{c}) = 0$.

For a non-zero vector $y \in T_x M_0$, we can define $\mathrm{B}_y : T_x M \otimes T_x M \otimes T_x M \to T_x M$ and $\mathrm{E}_y : T_x M \otimes T_x M \to \mathbb{R}$ by $\mathrm{B}_y(u, v, w) := B^i_{j\ kl}(y) u^j v^k w^l \frac{\partial}{\partial x^i}|_x$ and $\mathrm{E}_y(u, v) := E_{jk}(y) u^j v^k$ where

$$B^i_{j\ kl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}, \qquad E_{jk} := \frac{1}{2} B^m_{j\ km}$$

$u = u^i \frac{\partial}{\partial x^i}|_x$, $v = v^i \frac{\partial}{\partial x^i}|_x$ and $w = w^i \frac{\partial}{\partial x^i}|_x$. The B and E are called the Berwald curvature and mean Berwald curvature respectively. A Finsler metric is called a Berwald metric and weakly Berwald metric if $\mathrm{B} = 0$ and $\mathrm{E} = 0$, respectively [11].

The quantity $\mathrm{H}_y = H_{ij} dx^i \otimes dx^j$ is defined as the covariant derivative of E along geodesics [7]. More precisely $H_{ij} := E_{ij|m} y^m$.

For a flag $P = span\{y, u\} \subset T_x M$ flagpole y, the flag curvature $K = K(P, y)$ is defined by

$$K(P, y) := \frac{g_y(u, R_y(u))}{g_y(y, y) g_y(u, u) - g_y(y, u)^2},$$

We say that a Finsler metric F is of scalar curvature if for any $y \in T_x M$, the flag curvature $K = K(x, y)$ is a scaler function on the slit tangent bundle TM_0.

By means of E-curvature, we can define $\bar{E}_y : T_x M \otimes T_x M \otimes T_x M \to \mathbb{R}$ by

$$\bar{E}_y(u, v, w) := \bar{E}_{jkl}(y) u^i v^j w^k,$$

where $\bar{E}_{ijk} := E_{ij|k}$. We call it $\bar{E}$-curvature. From a Bianchi identity, we have

$$B^i_{j\ ml|k} - B^i_{j\ km|l} = R^i_{j\ kl,m}$$

where R^i_{jkl} is the Riemannian curvature of Berwald connection [11]. This implies that $\bar{E}_{jlk} - \bar{E}_{jkl} = 2R^m_{j\ kl,m}$. Then $\bar{E}_{ijk}$ is not totally symmetric in all three of its indices.

Results and Discusion

Sakaguchi Theorem

In this section, we are going to prove the well-known theorem of Sakaguchi. Our method is different from the Sakaguchi.

Theorem 3. Every Finsler metric of scalar flag curvature is a generalized Douglas-Weyl metric.

Proof. Let F be a Finsler metric of scalar flag curvature K. The following holds

$$\begin{aligned} B^i{}_{jml|k} y^k &= 2K C_{jlm} y^i - \frac{1}{3} K_{.j.m} F^2 h^i{}_l \\ &- \frac{1}{3} K_{.j.l} F^2 h^i{}_m - \frac{1}{3} K_{.l.m} F^2 h^i{}_j \\ &- \frac{1}{3} K_{.l} \{F F_{.j} \delta^i{}_m + F F_{.m} \delta^i{}_j - 2g_{jm} y^i\} \\ &- \frac{1}{3} K_{.m} \{F F_{.j} \delta^i{}_m + F F_{.l} \delta^i{}_j - 2g_{jl} y^i\} \\ &- \frac{1}{3} K_{.j} \{F F_{.l} \delta^i{}_m + F F_{.m} \delta^i{}_l - 2g_{lm} y^i\} \end{aligned} \tag{2}$$

It follows from (2) that

$$H_{jl} = -\frac{n+1}{6}\{y_l K_{.j} + y_j K_{.l} + K_{.j.l} F^2\}. \tag{3}$$

We obtain

$$D^{i}{}_{jkl|m}\, y^{m} = 2K\,C_{jkl}\, y^{i} - \frac{2}{3}\{K_{.j}\, g_{kl} + K_{.l}\, g_{jk} + K_{.k}\, g_{jl}\} y^{i} - \frac{1}{3}\{K_{.j.l}\, y_{k} + K_{.j.k}\, y_{l} + K_{.k.l}\, y_{j}\} y^{i} - \frac{2}{n+1} E_{jk.l|m}\, y^{m} y^{i} \tag{4}$$

Thus, we can conclude that every Finsler metric of scalar flag curvature a generalized Douglas-Weyl metric. □

Proof of Theorem 1

To prove the Theorem 1, we need the following.

Lemma 2. Let (M,F) be a Finsler manifold. Then the following holds

$$B_{j\ kl|m}^{\ i} y^{m} = \frac{2}{n+1}\left\{H_{jk}\delta^{i}_{l} + H_{kl}\delta^{i}_{j} + H_{lj}\delta^{i}_{k} + H_{jk,l}\, y^{i} - \bar{E}_{jkl}\, y^{i}\right\}. \tag{5}$$

Proof. By definition, we have

$$D^{i}_{jkl} = B^{\ i}_{jkl} - \frac{2}{n+1}\left\{E_{jk}\delta^{i}_{l} + E_{kl}\delta^{i}_{j} + E_{lj}\delta^{i}_{k} + E_{jk,l}\, y^{i}\right\}. \tag{6}$$

Thus

$$D_{j\ kl|m}^{\ i} y^{m} = B_{j\ kl|m}^{\ i} y^{m} - \frac{2}{n+1}\left\{E_{jk|m}\, y^{m}\delta^{i}_{l} + E_{kl|m}\, y^{m}\delta^{i}_{j} + E_{lj|m}\, y^{m}\delta^{i}_{k}\right\} - \frac{2}{n+1} E_{jk,l|m}\, y^{m} y^{i}. \tag{7}$$

On the other hand, the following Ricci identity for E_{ij} hold

$$E_{jk,l|k} - E_{ij|k,l} = E_{pj} B_{i\ kl}^{\ p} + E_{ip} B_{j\ kl}^{\ p}. \tag{8}$$

It follows from (5) that

$$E_{jk,l|m}\, y^{m} = E_{jk|m,l}\, y^{m} = \left[E_{jk|m}\, y^{m}\right]_{,l} - E_{jkl}, \tag{9}$$

This yields that

$$E_{jk,l|m}\, y^{m} = E_{jk,l|m}\, y^{m} = H_{jk,l} - \bar{E}_{jkl}. \tag{10}$$

By (7) and (10), we get (5). □

Lemma 2. Let (M,F) be a Finsler manifold. Then the following hold

$$R_{j\ kl|m}^{\ i} + R_{j\ lm|k}^{\ i} + R_{j\ mk|l}^{\ i} = B_{j\ ku}^{\ i} R^{u}{}_{lm} + B_{j\ lu}^{\ i} R^{u}{}_{mk} + B_{j\ mu}^{\ i} R^{u}{}_{kl} \tag{11}$$

$$B_{j\ kl|m}^{\ i} - B_{j\ ml|k}^{\ i} = R_{j\ ml,k}^{\ i} \tag{12}$$

$$B_{j\ kl,m}^{\ i} = B_{j\ km,l}^{\ i} \tag{13}$$

Proof. The curvature form of Berwald connection is

$$\Omega^{i}{}_{j} = d\,\omega^{i}_{j} - \omega^{k}_{j}\wedge\omega^{i}_{k} = \frac{1}{2} R^{\ i}_{jkl}\,\omega^{k}\wedge\omega^{l} - B^{\ h}_{jkl}\,\omega^{k}\wedge\omega^{n+1}. \tag{14}$$

For the Berwald connection, we have the following structure equation

$$dg_{ij} - g_{jk}\Omega^{k}_{i} - g_{ik}\Omega^{k}_{j} = -2L_{ijk}\,\omega^{k} + 2C_{ijk}\,\omega^{n+1}. \tag{15}$$

Differentiating (15) yields the following Ricci identity

$$g_{pj}\Omega^{p}_{i} - g_{pi}\Omega^{p}_{j} = -2L_{ijk}\,\omega^{k}\wedge\omega^{l} - 2L_{ijk}\,\omega^{k}\wedge\omega^{n+1} - 2C_{ijl|k}\,\omega^{k}\wedge\omega^{n+1} - 2C_{ijl,k}\,\omega^{n+k}\wedge\omega^{n+1} - 2C_{ijp}\Omega^{p}_{l}\, y^{l}. \tag{16}$$

Differentiating of (14) yields

$$d\,\Omega^{j}_{i} - \omega^{k}_{i}\wedge\Omega^{k}_{j} + \omega^{j}_{k}\wedge\Omega^{k}_{i} = 0. \tag{17}$$

Define $B^{\ i}_{jkl|m}$ and $B^{\ i}_{jkl,m}$ by

$$dB^{\ i}_{jkl} - B^{\ i}_{mkl}\,\omega^{m}_{i} - B^{\ i}_{jml}\,\omega^{m}_{k} - B^{\ i}_{jkm}\,\omega^{m}_{l} + B^{\ i}_{jkl}\,\omega^{i}_{m} = B^{\ i}_{jkl|m}\,\omega^{m} + B^{\ i}_{jkl,m}\,\omega^{n+m}. \tag{18}$$

Similary, we define $R^{\ i}_{jkl|m}$ and $R^{\ i}_{jkl,m}$ by

$$dR^{\ i}_{jkl} - R^{\ i}_{mkl}\,\omega^{m}_{i} - B^{\ i}_{jml}\,\omega^{m}_{k} - R^{\ i}_{jkm}\,\omega^{m}_{l} + R^{\ i}_{jkl}\,\omega^{i}_{m} = R^{\ i}_{jkl|m}\,\omega^{m} + R^{\ i}_{jkl,m}\,\omega^{n+m}. \tag{19}$$

From (16), (17), (18) and (19), we get the proof. □

Proof of Theorem 1: From (16), it follows that

$$C_{ijl|k} - L_{ijk.l} = \frac{1}{2} g_{pj} B^{p}{}_{ikl} + \frac{1}{2} g_{ip} B^{p}{}_{jkl} \tag{20}$$

Contracting (20) with y^j and using $y^i{}_{.j} = \delta^i_j$ and $y_{i.j} = 0$ yields

$$L_{jkl} = -\frac{1}{2} g_{im} y^m B^i{}_{jkl}. \tag{21}$$

By assumption, we have

$$B^i_{j\,kl|m} y^m = \frac{2}{n+1}\left\{H_{jk}\delta^i_l + H_{kl}\delta^i_j + H_{lj}\delta^i_k + H_{jk,l} y^i - \bar{E}_{jkl} y^i\right\}. \tag{22}$$

Multiplying (22) with y_i and using (21), we get

$$\bar{E}_{jkl} = \left\{H_{jk} y_l + H_{kl} y_j + H_{lj} y_k\right\} F^{-2} + H_{jk.l}. \tag{23}$$

By (23), we get the proof. □

Proof of Theorem 2: Let $R^i{}_{kl} := y^j R^i_{j\,kl}$. Then we have

$$R^i_{j\,kl} = \frac{1}{3}\left\{\frac{\partial^2 R^i_k}{\partial y^j \partial y^l} - \frac{\partial^2 R^i_l}{\partial y^j \partial y^k}\right\}. \tag{24}$$

Here, we assume that a Finsler metric F is of scalar curvature $K = K(x, y)$. In local coordinates,

$$R^i{}_k = K F^2 h^i_k. \tag{25}$$

Plugging (25) into (24) gives

$$\begin{aligned} R^i_{j\,kl} = {} & \frac{K_{.j.l}}{3} F^2 h^i{}_k - \frac{K_{.j.k}}{3} F^2 h^i{}_l \\ & + K_{.j}\{F F_{.l} h^i{}_k - F F_{.k} h^i{}_l\} \\ & + \frac{1}{3} K_{.k}\{2F F_{.j}\delta^i{}_l - g_{jl} y^i - F F_{.l}\delta^i{}_j\} \\ & + K\{g_{jl}\delta^i{}_k - g_{jk}\delta^i{}_l\} \\ & + \frac{1}{3} K_{.l}\{2F F_{.j}\delta^i{}_k - g_{jk} y^i - F F_{.k}\delta^i{}_j\} \end{aligned} \tag{26}$$

Differentiating (26) with respect to y^m gives a formula for $R^i_{j\,kl.m}$ expressed in terms of K and its derivatives. Contracting (12) with y^k, we obtain

$$\begin{aligned} B^i{}_{jml|k} y^k = {} & 2K C_{jlm} y^i - \frac{1}{3} K_{.j.m} F^2 h^i{}_l \\ & - \frac{1}{3} K_{.j.l} F^2 h^i{}_m - \frac{1}{3} K_{.l.m} F^2 h^i{}_j \\ & - \frac{1}{3} K_{.l}\{F F_{.j}\delta^i{}_m + F F_{.m}\delta^i{}_j - 2g_{jm} y^i\} \\ & - \frac{1}{3} K_{.m}\{F F_{.j}\delta^i{}_m + F F_{.l}\delta^i{}_j - 2g_{jl} y^i\} \\ & - \frac{1}{3} K_{.j}\{F F_{.l}\delta^i{}_m + F F_{.m}\delta^i{}_l - 2g_{lm} y^i\} \end{aligned} \tag{27}$$

Since $K = K(x)$, then by (27) we get

$$B^i_{j\,ml|k} y^k = 2K C_{jlm} y^i \tag{28}$$

Since F be a weakly Douglas Finsler metric, then we have

$$B^i_{j\,kl|m} y^m = \frac{2}{n+1}\left\{H_{jk}\delta^i_l + H_{kl}\delta^i_j + H_{lj}\delta^i_k + H_{jk,l} y^i - \bar{E}_{jkl} y^i\right\}. \tag{29}$$

From the assumptions, one can obtains

$$B^i_{j\,kl|m} y^m = 0.$$

By (28), we can conclude that $C_{ijk} = 0$ and then F is Riemannian. □

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References

1. Bácsó, S. Ilosvay, F. and Kis, B. Landsberg spaces with common geodesics. Publ. Math. Debrecen. **42**: 139-144 (1993).
2. Bácsó, S. and Matsumoto, M. Reduction theorems of certain Landsberg spaces to Berwald spaces, Publ. Math. Debrecen. **48**: 357-366 (1996).
3. Bácsó, S. and Matsumoto, M. On Finsler spaces of Douglas type, A generalization of notion of Berwald space, Publ. Math. Debrecen. **51**: 385-406 (1997).
4. Hashiguchi, M. and Ichijyo, Y. On Some special (α,β)-

metrics, Rep. Fac. Sci., Kagoshima Univ. **8**: 39-46 (1975).
5. Matsumoto, M. On Finsler spaces with Randers metric and special forms of important tensors, J. Math. Kyoto Univ. **14**: 477-498 (1974).
6. Najafi, B. Shen, Z. and Tayebi, A. On a projective class of Finsler metrics, Publ. Math. Debrecen. **70**: 211-219 (2007).
7. Najafi, B. Shen, Z. and Tayebi, A. Finsler metrics of scalar flag curvature with special non-Riemannian curvature properties, Geom. Dedicata. **131**: 87-97 (2008).
8. Najafi, B. and Tayebi, A. Finsler Metrics of scalar flag curvature and projective invariants, Balkan. J. Geom. Appl. **15**: 90-99 (2010).
9. Peyghan, E. and Tayebi, A., Generalized Berwald metrics, Turkish Journal Math, **35**: 1-10 (2011).
10. Shen, Z. Differential Geometry of Spray and Finsler Spaces, Kluwer Academic Publishers, Dordrecht, (2001).
11. Tayebi, A. Azizpour, E. and Esrafilian, E. On a family of connections in Finsler geometry, Publ. Math. Debrecen. **72**: 1-15 (2008).
12. Tayebi, A. and Najafi, B. On m-th root Finsler metrics, Journal Geometry and Physics. **61**: 1479-1484 (2011).
13. Tayebi, A. and Najafi, B. On m-th Root Metrics with special curvature properties, C. R. Acad. Sci. Paris, Ser. I, **349**: 691-693 (2011)
14. Tayebi, A. and Peyghan, E. On Ricci tensors of Randers metrics, Journal of Geometry and Physics. **60**: 1665-1670 (2010).
15. Tayebi, A. and Peyghan, E. Special Berwald Metrics, Symmetry, Integrability and Geometry: Methods and its Applications (SIGMA), **6**: 008 (2010).