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A Quadratically Convergent $O(\sqrt{n})$ Interior-Point Algorithm for the $P_*(\kappa)$-Matrix Horizontal Linear Complementarity Problem

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Abstract

In this paper, we present a new path-following interior-point algorithm for $P_*(\kappa)$-horizontal linear complementarity problems (HLCPs). The algorithm uses only full-Newton steps which has the advantage that no line searchs are needed. Moreover, we obtain the currently best known iteration bound for the algorithm with small-update method, namely, $O\left(\sqrt{n}(1+\kappa)\log\frac{n}{\varepsilon}\right)$, which is as good as the linear analogue.

Keywords: Horizontal linear complementarity problem (HLCP); Interior point method (IPM); Central path

Introduction

Given $Q, R \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$, the horizontal linear complementarity problem (HLCP) is to find a pair $(x, s) \in \mathbb{R}^{2n}$ such that

$$Qx + Rs = b,\ (x, s) \geq 0,\ x^T s = 0.\ (P)$$

The standard (monotone) linear complementarity problem (SLCP or simply LCP) corresponds to the case where $R = -I$ and Q is positive semidefinite.

We say that (P) is a $P_*(\kappa)$-HLCP if

$$Qu + Rv = 0 \Rightarrow (1+4\kappa)\sum_{i\in I^+} u_i v_i + \sum_{i\in I^-} u_i v_i \geq 0, \forall u, v \in \mathbb{R}^n, \quad (1)$$

where κ is nonnegative constant and $I^+ = \{i : u_i v_i > 0\}$ and $I^- = \{i : u_i v_i < 0\}$. If the above condition satisfied, then we say that the pair (Q, R) is a $P_*(\kappa)$-pair and write $(Q, R) \in P_*(\kappa)$. For $\kappa = 0$, $P_*(\kappa)$ −HLCP is called the monotone HLCP.

There exist many approaches for solving the $P_*(\kappa)$-HLCPs. Among them, the interior-point methods (IPMs) gained more attention than other methods. IPMs that were initiated by Karmakar for linear optimization (LO) problems, extended by many researchers for convex quadratic optimization (CQO) and the standard (monotone) linear complementarity problem (SLCP) and achieved plentiful and beautiful results [1-4]. Theoretically, HLCP can be solved by using any algorithm for SLCP [5], but directly solving HLCP is a

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better choice than using any algorithm for SLCP to solve the HLCP. Close connection between LO, CQO, SLCP and HLCP cause the extension of some IPMs from LO, CQO and SLCP to HLCP. For instance, Gonzaga et al. [6, 7] studied the largest step path following algorithm for monotone HLCP and showed the fast convergence of the simplified path following algorithm. Mizuno et al. [8] proposed the (MTY) predictor-corrector method that was the first polynomial-time and superlinear convergent IPM for general LO, with precisely $O\left(\sqrt{n}L\right)$ iteration complexity.(cf. [8]). The MTY method generalized to SLCP in [9] and the resulting algorithm has $O\left(\sqrt{n}L\right)$ iterations. Also in [10] the MTY method has been generalized to HLCP and a class of corrector-predictor IPMs for solving $P_*(\kappa)$-HLCP has been proposed therein. Huang [11] proposed a high-order feasible interior-point method for HLCP with $O\left(\sqrt{n}\log\frac{\varepsilon_0}{\varepsilon}\right)$ iterations. Monteiro et al. [12] studied the limiting behavior of the derivatives of certain trajectories associated with the monotone HLCP. Some other relevant references can be found in [13, 14]. It should be noted that all most known polynomial various of IPMs used the so-called central path as a guideline to the optimal set, and some various of the Newton method to follow the central path approximately. However there is still a gap between the practical behavior of these algorithms and the theoretical performance results with respect to the update strategies of the duality gap parameter in the algorithm. The so-called large-update IPMs have superior practical performance but with relatively weak theoretical results. While the so-called small-update IPMs enjoy the best known worst-case iteration bound but their performance in computational practice is poor. This gap was reduced by Peng et al. [15] who introduced the so-called self-regular barrier functions based on IPMs for LO and semidefinite optimization (SDO). See also Salahi et al. [16]. Bai et al. [17] and Amini et al. [18] who presented IPMs based on a new class of non-self-regular kernel functions for LO and $P_*(\kappa)$-linear complementarity problems and also obtained the same best known iteration bounds for the algorithms with large- and small-update methods as they are in [15]. In very recently, Mansouri et al. [19-21] presented the first full-Newton step IPM for Linear Complementarity problems (LCPs) and $P_*(\kappa)$-HLCPs, which are an extension of the work for linear optimization [22-24].

In this paper we present a new feasible primal-dual IPM with full-Newton steps for HLCP problems. We prove that the complexity of our algorithm is $O\left(\sqrt{n}(1+\kappa)\log\frac{n}{\varepsilon}\right)$ iteration, which coincides with the best known iteration bound for feasible IPMs .

The notations used throughout the paper is rather standard: capital letters denote matrices, lower case letters denote vectors, script capital letters denote sets, and Greek letters denote scalars. All vectors are considered to be column vectors. The components of a vector $u \in \mathbb{R}^n$ will be denoted by u_i, $i = 1,\cdots,n$. The relation $u > 0$ is equivalent to $u_i > 0,\ i = 1,\cdots,n$, while $u \geq 0$ means $u_i \geq 0,\ i = 1,\cdots,n$. We denote $\mathbb{R}^n_+ = \{u \in \mathbb{R}^n : u \geq 0\}$ and $\mathbb{R}^n_{++} = \{u \in \mathbb{R}^n : u > 0\}$. If $u \in \mathbb{R}^n$, then $U = diag(u)$ denotes the diagonal matrix having the components of u as its diagonal entries. If $x, s \in \mathbb{R}^n$, then xs denotes the componentwise (Hadamard) product of the vectors x and s . Furthermore, e denotes the all-one vector of length n . The 2-norm and the infinity norm for vectors are denoted by $\|\bullet\|$ and $\|\bullet\|_\infty$, respectively. We denote the set of feasible points of the HLCP by

$$F = \{(x, s) \in \mathbb{R}^{2n}_+ : Qx + Rs = b\}, \tag{2}$$

and the set of strictly feasible (or interior) points by

$$F^0 = \{(x, s) \in \mathbb{R}^{2n}_{++} : Qx + Rs = b\}, \tag{3}$$

and the solution set of HLCP by

$$F^* = \{(x^*, s^*) \in F : x^* s^* = 0\}. \tag{4}$$

Throughout this paper it is assumed that F^* is not empty, i.e. (P) has at least one solution.

Materials and Methods

1. Feasible Full Newton Step IPMs and Central Path

Solving HLCP is equivalent with finding a solution of the following system of equations:

$$\begin{aligned} Qx + Rs &= b, \quad x \geq 0, \\ xs &= 0, \quad s \geq 0, \end{aligned} \tag{5}$$

where the first constraint represents feasibility and the second is the so-called complementarity condition.

The basic idea of IPMs is to relax the complementarity condition by the so-called centering condition $xs = \mu e$, where μ may be any positive number. This yields the following parameterized system:

$$\begin{aligned} Qx + Rs &= b, \quad x \geq 0, \\ xs &= \mu e, \quad s \geq 0, \end{aligned} \tag{6}$$

In [25] it has been shown that if HLCP satisfies the interior-point condition i.e. there exists $(x, s) > 0$ such that $Qx + Rs = b$, then the above system has a unique solution for each $\mu > 0$. Denote this unique solution by $(x(\mu), s(\mu))$, for every $\mu > 0$. Then we call $(x(\mu), s(\mu))$ the $\mu-$center of HLCP. The set of $\mu-$centers (with μ running through all positive real numbers) gives a homotype path, which is called the central path of HLCP. If $\mu \to 0$, then the limit of the central path exists [25] and it yields the optimal solution for HLCP.

2. Definition and Properties of the Newton Step

IPMs follow the central path approximately. Let us describe how this proceeds. A direct application of Newton's method to solve the system (6) with fixed μ, and assuming $(x, s) > 0$, produces the following system for the displacement Δx and Δs :

$$\begin{aligned} Q(x + \Delta x) + R(s + \Delta s) &= b, \\ (x + \Delta x)(s + \Delta s) &= \mu e. \end{aligned}$$

By omitting the quadratic term $\Delta x \Delta s$ in the second equation, we have the following linear system of equations:

$$\begin{aligned} Q\Delta x + R\Delta s &= b - (Qx + Rs), \\ s\Delta x + x\Delta s &= \mu e - xs. \end{aligned}$$

Note that if (x, s) is a feasible solution of HLCP, then $Qx + Rs = b$. Hence, the above system reduces to

$$\begin{aligned} Q\Delta x + R\Delta s &= 0, \\ s\Delta x + x\Delta s &= \mu e - xs. \end{aligned} \tag{7}$$

The new iterates are given by

$$\begin{aligned} x^+ &= x + \Delta x, \\ s^+ &= s + \Delta s. \end{aligned}$$

3. Proximity Measure

In the case of a feasible method we call the (x, s) an $\varepsilon-$solution of HLCP if $x^T s \leq \varepsilon$. To measure the quality of any approximation (x, s) of $(x(\mu), s(\mu))$, we introduce $\delta(x, s; \mu)$ that vanishes if $(x, s) = (x(\mu), s(\mu))$ and is positive otherwise. To this end we introduce the variance vector of (x, s) with respect to μ as follows

$$v = \sqrt{\frac{xs}{\mu}},$$

where all operations are componentwise. Note that

$$xs = \mu e \Leftrightarrow v = e.$$

The proximity meature $\delta(x, s; \mu)$ is now defined by

$$\delta(x, s; \mu) = \frac{1}{\sqrt{2}} \left\| v - v^{-1} \right\|, \tag{8}$$

Note that if $(x, s) = (x(\mu), s(\mu))$, then $v = e$ and hence $\delta(x, s; \mu) = 0$ and otherwise $\delta(x, s; \mu) > 0$.

Results

1. Feasibility and Quadratic Convergence of the Feasible Full-Newton Step

In this section we find a condition for feasibility of full Newton steps. We also prove that the value of $x^T s$ after one step is less than or equal to $\left(n + \delta^2\right)\mu$. We also prove that the full Newton steps are quadratically convergent to the target point $(x(\mu), s(\mu))$. Define

$$d_x = \frac{v \Delta x}{x}, d_s = \frac{v \Delta s}{s}, \bar{Q} = QV^{-1}X, \bar{R} = RV^{-1}S, \tag{9}$$

where $X = diag(x), S = diag(s)$ and $V = diag(v)$. Now we can easily check that the system (7), which defines the search directions Δx and Δs, can be expressed in terms of the scaled search directions d_x and d_s as follows:

$$\begin{aligned} \bar{Q}d_x + \bar{R}d_s &= 0, \\ d_x + d_s &= v^{-1} - v. \end{aligned} \tag{10}$$

Now, using (9) and the second equation in (7), we have

$$x^+ s^+ = (x + \Delta x)(s + \Delta s)$$

$$= xs + (x\Delta s + s\Delta x) + \Delta x\Delta s = \mu e + \Delta x\Delta s$$

$$= \mu e + \frac{xs}{v^2} d_x d_s = \mu(e + d_x d_s).$$

Lemma 1.1 (Cf. Lemma II.45 in [1]) The new iterates (x^+, s^+) are strictly feasible if and only if $e + d_x d_s > 0$.

Proof: Note that if x^+ and s^+ are positive, then the above equality makes clear that $e + d_x d_s > 0$, proving the 'only if' part of the statement in the lemma. For the proof of the converse implication, we introduce a step length $\alpha \in [0,1]$, and we define

$$x^\alpha = x + \alpha\Delta x, \qquad s^\alpha = s + \alpha\Delta s.$$

We then have $x^0 = x$, $x^1 = x^+$ and similar relations for s. Hence we have $x^0 s^0 = xs > 0$. We may write

$$x^\alpha s^\alpha = (x + \alpha\Delta x)(s + \alpha\Delta s)$$
$$= xs + \alpha(s\Delta x + x\Delta s) + \alpha^2\Delta x\Delta s.$$

Using $s\Delta x + x\Delta s = \mu e - xs$ gives

$$x^\alpha s^\alpha = xs + \alpha(\mu e - xs) + \alpha^2\Delta x\Delta s.$$

Now suppose that $e + d_x d_s > 0$. From the definitions of d_x and d_s in (9) we deduce that $\mu d_x d_s = \Delta x\Delta s$. Hence $\mu e + \Delta x\Delta s > 0$, or, equivalently, $\Delta x\Delta s > -\mu e$. Substitution gives

$$x^\alpha s^\alpha > xs + \alpha(\mu e - xs) - \alpha^2\mu e = (1-\alpha)(xs + \alpha\mu e),$$
$\alpha \in [0,1]$.

Since $(1-\alpha)(xs + \alpha\mu e) \geq 0$, it follows that $x^\alpha s^\alpha > 0$, for all $0 \leq \alpha \leq 1$. Hence, none of the entries of x^α and s^α vanishes for $0 \leq \alpha \leq 1$. Since x^0 and s^0 are positive, and x^α and s^α depend linearly on α, this implies that $x^\alpha > 0$ and $s^\alpha > 0$, for all $0 \leq \alpha \leq 1$. Hence, x^1 and s^1 must be positive which proves that x^+ and s^+ are positive. □

Corollary 1.2 The iterates (x^+, s^+) are strictly feasible if $\|d_x d_s\|_\infty < 1$.

Proof: By Lemma 1.1, x^+ and s^+ are strictly feasible if and only if $e + d_x d_s > 0$. Since the last inequality holds if $\|d_x d_s\|_\infty < 1$, then the corollary follows. □

The following lemma gives some bounds for the solution of a linear system of the form:

$$\begin{aligned} su + xv &= a, \\ Qu + Rv &= 0. \end{aligned} \tag{11}$$

Using the notations

$$D = X^{\frac{-1}{2}} S^{\frac{1}{2}}, \qquad \tilde{a} = (xs)^{\frac{-1}{2}} a,$$

where $X = diag(x)$ and $S = diag(s)$, we have the following result.

Lemma 1.3 Let (Q, R) in the HLCP be a $P_*(\kappa)$-pair. Then for any $(x, s) \in \mathbb{R}^{2n}_{++}$ and $a \in \mathbb{R}^n$, the linear system (11) has a unique solution (u, v), for which the following estimates hold

$$-\kappa\|\tilde{a}\|^2 \leq u^T v \leq \frac{1}{4}\|\tilde{a}\|^2, \qquad \|uv\| \leq \left(\frac{1}{\sqrt{8}} + \kappa\right)\|\tilde{a}\|^2.$$

Proof: We consider the index sets:

$$I^+ = \{i : u_i v_i > 0\}, \qquad I^- = \{i : u_i v_i < 0\}.$$

Using the relations

$$0 < 4u_i v_i \leq (Du - D^{-1}v)_i^2 = \tilde{a}_i^2, \ \ \forall i \in I^+,$$

$$\sum_{i\in I^-} |u_i v_i| \leq (1+4\kappa)\sum_{i\in I^+} |u_i v_i| \leq \left(\frac{1}{4} + \kappa\right)\|\tilde{a}\|^2,$$

where we use this fact that (Q, R) is a $P_*(\kappa)$-pair, we deduce that

$$u^T v = \sum_{i\in I^+} u_i v_i + \sum_{i\in I^-} u_i v_i \leq \sum_{i\in I^+} u_i v_i \leq \frac{1}{4}\|\tilde{a}\|^2.$$

Also we have

$$u^T v = \sum_{i\in I^+} u_i v_i + \sum_{i\in I^-} u_i v_i$$
$$= (1+4\kappa)\sum_{i\in I^+} u_i v_i + \sum_{i\in I^-} u_i v_i - 4\kappa\sum_{i\in I^+} u_i v_i$$
$$\geq -4\kappa\sum_{i\in I^+} u_i v_i \geq -\kappa\|\tilde{a}\|^2.$$

This proves the first inequality in the lemma. For the second inequality we have

$$\|uv\|^2 = \sum_{i\in I^+} u_i^2 v_i^2 + \sum_{i\in I^-} u_i^2 v_i^2 \leq \frac{1}{16}\sum_{i\in I^+} \tilde{a}_i^4$$
$$+\left(\sum_{i\in I^-} u_i v_i\right)^2 \leq \frac{1}{16}\|\tilde{a}\|^4 + \left(\frac{1}{4} + \kappa\right)^2 \|\tilde{a}\|^4$$
$$= \left(\frac{1}{8} + \frac{\kappa}{2} + \kappa^2\right)\|\tilde{a}\|^4 \leq \left(\frac{1}{\sqrt{8}} + \kappa\right)^2 \|\tilde{a}\|^4.$$

See [26] for the uniqueness of the solution. □

Corollary 1.4 Let (Q, R) in the HLCP be a $P_*(\kappa)$-

pair. Then the unique solution $(\Delta x, \Delta s)$ of the system (7) satisfies the following inequalities:

$$-2\mu\kappa\delta^2 \le (\Delta x)^T \Delta s \le \frac{\mu\delta^2}{2}, \tag{12}$$

$$\|\Delta x \Delta s\| \le 2\left(\frac{1}{\sqrt{8}}+\kappa\right)\mu\delta^2. \tag{13}$$

Proof: It suffices that we apply Lemma 1.3 with $a = \mu e - xs$ and $(\Delta x, \Delta s)$ instead of (u, v) and note that

$$\tilde{a} = \frac{1}{\sqrt{xs}}(\mu e - xs) = \frac{\mu e}{\sqrt{xs}} - \sqrt{xs}$$
$$= \sqrt{\mu}\left(\sqrt{\frac{\mu}{xs}} - \sqrt{\frac{xs}{\mu}}\right) = \sqrt{\mu}(v^{-1} - v),$$

which implies

$$\|\tilde{a}\|^2 = \mu\left\|v^{-1} - v\right\|^2 = 2\mu\delta^2,$$

that completes the proof. □

Lemma 1.5 After a Newton step one has $(x^+)^T s^+ \le (n+\delta^2)\mu$.

Proof: By using $x^+ = x + \Delta x$ and $s^+ = s + \Delta s$, after a Newton step one has

$$(x^+)^T s^+ = e^T(x^+ s^+) = e^T((x+\Delta x)(s+\Delta s))$$
$$= e^T(xs + x\Delta s + s\Delta x + \Delta x \Delta s)$$
$$= e^T(\mu e + \Delta x \Delta s) \le n\mu + \mu\delta^2 = (n+\delta^2)\mu,$$

where the inequality follows because of (12). This completes the proof. □

Lemma 1.6 Let $\delta^+ = \delta(x^+, s^+; \mu)$. Then

$$\delta^+ \le \frac{\left(\frac{1+2\sqrt{2}\kappa}{2}\right)\delta^2}{\sqrt{1-\left(\frac{1+2\sqrt{2}\kappa}{\sqrt{2}}\right)\delta^2}}.$$

Proof: Let v^+ be the variance vector of (x^+, s^+) with respect to μ, i.e. $v^+ = \sqrt{\frac{x^+ s^+}{\mu}}$, then we have

$$\sqrt{2}\delta^+ = \left\|(v^+)^{-1} - v^+\right\| = \left\|(v^+)^{-1}\left(e - (v^+)^2\right)\right\|.$$

Since $x^+ s^+ = \mu e + \Delta x \Delta s$, we obtain $(v^+)^2 = e + \frac{\Delta x \Delta s}{\mu}$.

Then

$$\sqrt{2}\delta^+ = \left\|\frac{-\frac{\Delta x \Delta s}{\mu}}{\sqrt{e + \frac{\Delta x \Delta s}{\mu}}}\right\| \le \frac{\left\|\frac{\Delta x \Delta s}{\mu}\right\|}{\sqrt{1 - \left\|\frac{\Delta x \Delta s}{\mu}\right\|_\infty}}.$$

Using (13) and the fact that $\|\Delta x \Delta s\|_\infty \le \|\Delta x \Delta s\|$, we have

$$\sqrt{2}\delta^+ \le \frac{2\left(\frac{1}{\sqrt{8}}+\kappa\right)\delta^2}{\sqrt{1-2\left(\frac{1}{\sqrt{8}}+\kappa\right)\delta^2}},$$

which completes the proof. □

Corollary 1.7 If $\delta = \delta(x, s; \mu) \le \frac{1}{\sqrt{2\left(1+2\sqrt{2}\kappa\right)}}$ then we have

$$\delta^+ = \delta(x^+, s^+; \mu) \le \left(\sqrt{1+2\sqrt{2}\kappa}\,\delta\right)^2,$$

i.e. quadratic convergence to the μ-center is obtained.

2. Updating the Barrier Parameter μ

In this section, we obtain a simple relation for our proximity measure just before and after a μ-update.

Lemma 2.1 Let (x, s) be a positive pair and $\mu > 0$ is such that $x^T s \le (n+\delta^2)\mu$. Moreover, let $\delta = \delta(x, s; \mu)$ and $\mu^+ = (1-\theta)\mu$. Then, one has

$$\delta(x, s; \mu^+)^2 \le (1-\theta)\delta^2 + \frac{n\theta^2}{2(1-\theta)} + \frac{\delta^2}{2(1-\theta)}.$$

Proof: Assume that $\delta^+ = \delta(x, s; \mu^+)$, then we have

$$2(\delta^+)^2 = \left\|\sqrt{1-\theta}\,v^{-1} - \frac{v}{\sqrt{1-\theta}}\right\|^2$$
$$= \left\|\sqrt{1-\theta}(v^{-1} - v) - \frac{\theta v}{\sqrt{1-\theta}}\right\|^2$$
$$= (1-\theta)\left\|v^{-1} - v\right\|^2 + \frac{\theta^2}{1-\theta}\|v\|^2 - 2\theta v^T(v^{-1} - v).$$

Since $x^T s \le (n+\delta^2)\mu$ we obtain that $\|v\|^2 \le n + \delta^2$. This implies

$$2(\delta^+)^2 \le 2(1-\theta)\delta^2 + \frac{\theta^2}{1-\theta}\left(n+\delta^2\right) - 2n\theta + 2\theta\left(n+\delta^2\right)$$

$$= 2(1-\theta)\delta^2 + \frac{n\theta^2}{1-\theta} + \left(\frac{\theta^2}{1-\theta} + 2\theta\right)\delta^2$$

$$= 2(1-\theta)\delta^2 + \frac{n\theta^2}{1-\theta} + \left(\frac{2\theta-\theta^2}{1-\theta}\right)\delta^2$$

$$= 2(1-\theta)\delta^2 + \frac{n\theta^2}{1-\theta} + \left(\frac{1-(\theta-1)^2}{1-\theta}\right)\delta^2$$

$$\le 2(1-\theta)\delta^2 + \frac{n\theta^2}{1-\theta} + \frac{1}{1-\theta}\delta^2.$$

It completes the proof. □

3. Complexity Analysis

In this subsection we present a lemma that gives the complexity of the algorithm. At the start of the algorithm, we have a point (x, s) that is strictly feasible for (P) and a $\mu > 0$ such that $\delta(x, s; \mu) \le \tau = \frac{1}{2\left(1+2\sqrt{2}\kappa\right)}$. Then, after the barrier parameter is updated to $\mu^+ = (1-\theta)\mu$, with $\theta = \frac{1}{\left(1+2\sqrt{2}\kappa\right)\sqrt{8n}}$, Lemma 2.1, yields the following upper bound for $\delta(x, s; \mu^+)$:

$$\delta(x, s; \mu^+)^2 \le \frac{1-\theta}{4\left(1+2\sqrt{2}\kappa\right)^2} + \frac{1}{16(1-\theta)\left(1+2\sqrt{2}\kappa\right)^2} + \frac{1}{8(1-\theta)\left(1+2\sqrt{2}\kappa\right)^2}$$

$$= \frac{1-\theta}{4\left(1+2\sqrt{2}\kappa\right)^2} + \frac{3}{16(1-\theta)\left(1+2\sqrt{2}\kappa\right)^2}$$

$$\le \frac{1}{2\left(1+2\sqrt{2}\kappa\right)^2}.$$

Assuming $n \ge 2$, The last inequality follows since its left hand side is a convex function of θ, whose value is $\frac{7}{16\left(1+2\sqrt{2}\kappa\right)^2}$ both in $\theta = 0$ and $\theta = \frac{1}{4\left(1+2\sqrt{2}\kappa\right)}$. Since $\theta \in \left[0, \frac{1}{4\left(1+2\sqrt{2}\kappa\right)}\right]$, the left hand side does not exceed $\frac{7}{16\left(1+2\sqrt{2}\kappa\right)^2}$. Since $\frac{7}{16\left(1+2\sqrt{2}\kappa\right)^2} < \frac{1}{2\left(1+2\sqrt{2}\kappa\right)^2}$, it follows that after the μ-update we have $\delta^2 = \delta(x, s; \mu^+)^2 \le \frac{1}{2\left(1+2\sqrt{2}\kappa\right)^2}$. Thus, by Corollary 1.7, after performing the Newton step we certainly have

$$\delta(x^+, s^+; \mu^+) \le \left(1+2\sqrt{2}\kappa\right)\delta^2$$

$$\le \left(1+2\sqrt{2}\kappa\right)\frac{1}{2\left(1+2\sqrt{2}\kappa\right)^2} \le \frac{1}{2\left(1+2\sqrt{2}\kappa\right)} = \tau,$$

therefore the algorithm is well defined. The above explanation implies the following result which establishes the polynomial iteration complexity of the algorithm.

Theorem 3.1 If $\theta = \frac{1}{\left(1+2\sqrt{2}\kappa\right)\sqrt{8n}}$, the number of iterations of the feasible primal-dual path-following algorithm with full-Newton steps does not exceed

$$\sqrt{8n}\left(1+2\sqrt{2}\kappa\right)\log\frac{n\mu^0}{\varepsilon}.$$

4. Numerical Results

In this section we present some numerical results. We solve the following $P_*(0)$ (monotone) linear complementarity problems, so $\mathrm{R} = -\mathrm{I}$, using the algorithm in Figure 1. The initialization parameters are assumed as described in Section 3, and the accuracy parameter ε is set to 10^{-4} and $\tau = 0.5$. Tables 1-4 show the number of iterations to obtain ε-solutions of the problems with the algorithm.

Problem 4.1

$$Q = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 1 & 2 \\ -1 & -1 & -2 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 8 \\ 6 \\ 4 \\ -3 \end{bmatrix}.$$

Problem 4.2

$$Q=\begin{bmatrix} 1 & 0 & -0.5 & 0 & 1 & 3 & 0 \\ 0 & 0.5 & 0 & 0 & 2 & 1 & -1 \\ -0.5 & 0 & 1 & 0.5 & 1 & 2 & -4 \\ 0 & 0 & 0.5 & 0.5 & 1 & -1 & 0 \\ -1 & -2 & -1 & -1 & 0 & 0 & 0 \\ -3 & -1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad b=\begin{bmatrix} 1 \\ 3 \\ -1 \\ 1 \\ -5 \\ -4 \\ 1.5 \end{bmatrix}.$$

Problem 4.3

$$Q=\begin{bmatrix} 1 & 2 & 2 & \cdots & 2 \\ 0 & 1 & 2 & \cdots & 2 \\ 0 & 0 & 1 & \cdots & 2 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}, \quad b=\begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

Problem 4.4

$$Q=\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 2 & 1 & 0 & \cdots & 0 \\ 2 & 2 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 2 & 2 & 2 & \cdots & 1 \end{bmatrix}, \quad b_i=\sum_{j=i}^{n} 2^i .$$

Problems 4.3 and 4.4 are all known to have exponential complexity for pivoting methods, but our results show slow growth as n increases, which is precisely what is hoped for interior-point methods.

5. *Concluding Remarks and Further Research*

We have presented an interior-point algorithm for HLCPs. At each iteration, we use only full-Newton steps. The favorable polynomial complexity bound for the algorithm with the small-update method is deserved, namely, $O\left(\sqrt{n}(1+\kappa)\log\frac{n}{\varepsilon}\right)$. Moreover, the resulting analysis is relatively simple and straightforward to the LO analogue. It may be clear that this full-Newton step method, may not be efficient in practice, Just as almost all feasible IPMs with the best theoretical performance. But this gap between the practical and theoretical performance can be reduced with changing the search direction by using methods that are based on kernel functions, as presented in [3, 16, 18, 27]. We leave it to the future to analyze a full-Newton step method based on kernel functions.

Feasible IPM for $P_*(\kappa)$-HLCP

Input:
  Accuracy parameter $\varepsilon > 0$;
  threshold parameter $\tau < 1$;
  barrier update parameter θ, $0<\theta<1$;
  feasible pair (x^0, s^0) with $(x^0)^T s^0 = n\mu^0$ and
    $\mu^0 > 0$ such that $\delta(x^0, s^0; \mu^0) \le \tau$.
begin
    $x := x^0$; $s := s^0$; $\mu := \mu^0$;
    while $n\mu \ge \varepsilon$ do
  begin
    update of μ:
    $\mu := (1-\theta)\mu$;
    $(x, s) := (x, s) + (\Delta x, \Delta s)$;
  end
end

Figure 1. Feasible full-Newton-step algorithm.

Table 1. The number of iterations for problem 4.1

θ	Iterations	$(x^*)^T$
0.17	56	[0.09,0.99,1.72,0.38]

Table 2. The number of iterations for problem 4.2

θ	Iterations	$(x^*)^T$
0.13	79	[0.1,0,0.3,0,0.7,0.16,0.33]

Table 3. The number of iterations for problem 4.3

n	θ	Iterations	$(S^*)^T$
10	0.11	99	[0,0,…,0.09]
20	0.08	150	[0,0,…,0.05]
30	0.06	191	[0,0,…,0.03]

Table 4. The number of iterations for problem 4.4

n	θ	Iterations	$(S^*)^T$
10	0.11	89	[0,0,…,0.23]
20	0.08	136	[0,0,…,0.23]
30	0.06	170	[0,0,…,0.23]

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