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Approximately Quasi Inner Generalized Dynamics on Modules

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Abstract

We investigate some properties of approximately quasi inner generalized dynamics and quasi approximately inner generalized derivations on modules. In particular, we prove that if A is a C^*-algebra, δ is the generator of a generalized dynamics $\{T_t\}_{t\in R}$ on an A-bimodule M satisfying $\|T_t\| \le 1$ and there exist two sequences $\{b_n\}$, $\{c_n\}$ of self adjoint elements in A such that for all x in a core D_0 for δ, $\delta(x) = \lim_{n\to\infty} i\,(c_n x - x b_n)$, then $\{T_t\}_{t\in R}$ is approximately quasi inner.

Keywords: (quasi approximately inner) Generalized derivation; (inner) Generalized isomorphism; (approximately quasi inner) Generalized dynamics

Introduction

Throughout the paper A and B are Banach algebras, M and N are A and $B-$bimodule, respectively and $\boldsymbol{B}(A)$ is the set of all bounded linear operators on A.

A *one parameter group* $\{\varphi_t\}_{t\in R}$ of bounded linear operators on A is a mapping $\varphi : \boldsymbol{R} \to \boldsymbol{B}(A)$ satisfying $\varphi_0 = I$ and $\varphi_{t+s} = \varphi_t \varphi_s$.

A one parameter group $\{\varphi_t\}_{t\in R}$ is called *uniformly (strongly) continuous* if $\varphi : \boldsymbol{R} \to \boldsymbol{B}(A)$ is continuous with respect to the norm (strong) operator topology. We define the *infinitesimal generator* d of φ as a mapping $d : D(d) \subseteq A \to A$ such that

$$d(a) = \lim_{t\to 0} \frac{\varphi_t(a) - a}{t},$$

where

$$D(d) = \left\{ a \in A : \quad \lim_{t\to 0} \frac{\varphi_t(a) - a}{t} \quad \text{exists} \right\}.$$

Also we define the *resolvent set* $\rho(d)$ to be the set of all complex numbers λ for which $\lambda I - d$ is invertible, cf. [13].

A $*-$ *automorphism* on C^*-algebra A is an invertible linear operator $\varphi : A \to A$ such that $\varphi(ab) = \varphi(a)\varphi(b)$ and $\varphi(a^*) = \varphi(a)^*$. An automorphism φ is called *inner* if there exists a unitary element $u \in A$ such that $\varphi(a) = u\,a u^*$. It is easy to check that if $\{\varphi_t\}_{t\in R}$ is a one parameter group of $*-$automorphism on A with the generator d, then d is a $*-$derivation and conversely if d is a bounded $*-$derivation on A, then d induces a uniformly continuous group of $*-$

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automorphisms $\{e^{td}\}_{t\in R}$.

A one parameter group of $*-$automorphisms $\{\varphi_t\}_{t\in R}$ on A is said to be *approximately inner* if there exists a sequence $\{c_n\}$ of self adjoint elements in A such that for each $t\in R$ and $a\in A$, $e^{ic_nt}ae^{-ic_nt}-\varphi(t)a\to 0$ uniformly on every compact subset of **R**, cf. [15]

A *Hilbert C^*-module* over C^*-algebra A is an algebraic left $A-$module M with an $A-$valued inner product, which is $A-$linear in the first and conjugate linear in the second variable such that M is a Banach space with respect to the norm $\|x\|=\|\langle x,x\rangle\|^{\frac{1}{2}}$. The Hilbert Module M is called full if the closed linear span $\langle M,M\rangle$ of all elements of the form $\langle x,y\rangle\ (x,y\in M)$ is equal to A, cf. [8]

We investigate some properties of approximately quasi inner generalized dynamics and quasi approximately inner generalized derivations on modules. In particular, we prove that if A is a C^*-algebra, δ is the generator of a generalized dynamics $\{\mathrm{T_t}\}_{t\in R}$ on an $\mathrm{A}-$bimodule M satisfying $\|\mathrm{T_t}\|\le 1$ and there exist two sequences $\{\mathrm{b_n}\}$, $\{\mathrm{c_n}\}$ of self adjoint elements in A such that for all x in a core $\mathrm{D_0}$ for δ, $\delta(\mathrm{x})=\lim_{n\to\infty}\mathrm{i}(\mathrm{c_n x}-\mathrm{x b_n})$, then $\{\mathrm{T_t}\}_{t\in R}$ is approximately quasi inner.

The reader is referred to [4], [6], [8] and [12] for more details on C^*-algebras and Hilbert C^*-module and to [5] and [15] for more information on dynamical systems.

Generalized Isomorphism and Generalized Dynamics on Modules

In this section, we define the generalized dynamics on modules and mention the relation between generalized dynamics and generalized derivations. For this aim, we need the following definitions:

Definition 1 A linear mapping $\delta: M\to M$ is said to be a *generalized derivation* if there exists a derivation $d:A\to A$ such that $\delta(ax)=a\delta(x)+d(a)x$ $(x\in M, a\in A)$. We call δ a *$d-$derivation.* As an example, let $c,b\in A$ and define $\delta_{c,b}:M\to M$ by $\delta_{c,b}(x)=cx-xb$. Then $\delta_{c,b}$ is a d_c -derivation, where $d_c(a)=ca-ac$. This d_c-derivation is called an *inner generalized derivation*.

We recall that a linear mapping $T:M\to N$ is said to be a *generalized module map* if there exists a linear homomorphism $\varphi:A\to B$ such that $T(ax)=\varphi(a)T(x)$, for all $a\in A, x\in M$. This map is called a $\varphi-$module map. In the case that φ is an automorphism and T is a bijective linear mapping, T is said to be a *$\varphi-$isomorphism* or *generalized isomorphism*, cf. [2] Let $\varphi:A\to A$ be a linear endomorphism. It is easy to see that a linear map $T:M\to M$ is a $\varphi-$module map if and only if

$$T(ax)-ax=a(T(x)-x)+(\varphi(a)-a)x$$
$$+(\varphi(a)-a)(T(x)-x).(*)$$

Example 1 If d is a bounded derivation on A and δ is a bounded $d-$derivation on M, then $e^{t\delta}$ is a $e^{td}-$isomorphism, cf. [2]

Definition 2 Let A be a C^*-algebra. A generalized isomorphism $T:M\to M$ is called *quasi inner* if there exist unitary elements $u,v\in A$ such that for each $x\in M$, $T(x)=uxv^*$.

Definition 3 Suppose that $\{T_t\}_{t\in R}$ is a one parameter group of bounded linear operators on M such that for each $t\in R$, T_t is a φ_t-isomorphism. If moreover $\{T_t\}_{t\in R}$ is uniformly continuous, then it is called φ_t-*dynamics or generalized dynamics* on M. We define the *infinitesimal generator* δ of T as a mapping $\delta: D(\delta)\subseteq M\to M$ such that

$$\delta(x)=\lim_{t\to 0}\frac{T_t(x)-x}{t},$$

where

$$D(\delta)=\left\{x\in M:\lim_{t\to 0}\frac{T_t(x)-x}{t}\ exists\right\}.$$

From now on, A is considered as a C^*-algebra.

Remark 1 Using the relation $(*)$, it can be proved that if $\{\varphi_t\}_{t\in R}$ is a uniformly continuous C^*-dynamics on A with the infinitesimal generator d and $\{T_t\}_{t\in R}$ is a

φ_t – dynamics on M with the infinitesimal generator δ, then δ is an everywhere defined d –derivation. Conversely, if δ is a bounded d – derivation, then it induces the uniformly continuous one parameter group $\{e^{t\delta}\}_{t\in R}$ of e^{td} – isomorphisms, by Example 1.

Theorem 1 Let b, c be two self adjoint elements in A and $\delta_{c,b}$ $M \to M$ be the inner generalized derivation $\delta_{c,b}(x) = i(cx - xb)$. Then there exist a uniformly continuous one parameter group $\{\varphi_t\}_{t\in R}$ of inner $*$– automorphisms on A and a uniformly continuous one parameter group $\{T_t\}_{t\in R}$ of quasi inner φ_t – isomorphisms on M such that d_c is the generator of $\{\varphi_t\}_{t\in R}$ and $\delta_{c,b}$ is the generator of $\{T_t\}_{t\in R}$.

proof. Take $T_t(x) = e^{itc}xe^{-itb}$ and $\varphi_t(a) = e^{itc}ae^{-itc}$. Then trivially $T_t(ax) = \varphi_t(a)T_t(x)$ and $T_0 = I$ and

$$T_t(T_s(x)) = e^{itc}(e^{isc}xe^{-isb})e^{-itb}$$

$$= e^{itc}.e^{isc}xe^{-isb}.e^{-itb}$$

$$= e^{i(t+s)c}xe^{-i(t+s)b}$$

$$= T_{t+s}(x).$$

Also taking $u_t = e^{itc}$ and $v_t = e^{itb}$, we have

$$\|T_t(x) - x\| = \|u_t x v_t^* - x\|$$

$$= \|(u_t x - x v_t)v_t^*\|$$

$$\le \|u_t x - x v_t\|$$

$$\le \|u_t x - x\| + \|x - x v_t\|$$

$$\le (\|u_t - I\| + \|I - v_t\|)\|x\|.$$

Thus

$$\|T_t - I\| \le \|u_t - I\| + \|I - v_t\| \to 0 (as\, t \to 0).$$

Therefore $\{T_t\}_{t\in R}$ is a uniformly continuous one parameter group of quasi inner φ_t – isomorphisms.

Moreover,

$$\lim_{t\to 0}\frac{T_t(x) - x}{t} = \lim_{t\to 0}\frac{e^{itc}xe^{-itb} - x}{t}$$

$$= \lim_{t\to 0}(ice^{itc}xe^{-itb} - ie^{itc}xbe^{-itb})$$

$$= i(cx - xb)$$

$$= \delta_{c,b}(x).$$

The second equality follows from the L'Hopital rule. A similar argument shows that $\{\varphi_t\}_{t\in R}$ is a uniformly continuous one parameter group of inner $*$– automorphisms on A with the generator d_c. ■

We end this section with the following useful lemma which can be found in [3] and [11].

Lemma 1 Let M be a full Hilbert A – module and let $a \in A$ .Then $a = 0$ if and only if $ax = 0$ for all $x \in M$.

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Definition 4 A generalized dynamics $\{T_t\}_{t\in R}$ on A – bimodule M is called *approximately quasi inner* if there exists two sequences $\{c_n\}$ and $\{b_n\}$ of self adjoint elements in A such that for each $t \in \boldsymbol{R}$, $T(t) = s - \lim_{t\to 0} Tn(t)$, where $T_n(t)x = e^{ic_n t}xe^{-ib_n t}$ which, in turn, means that for each $t \in \boldsymbol{R}$ and $x \in M$, $T(t)x = \lim_{t\to 0} e^{ic_n t}xe^{-ib_n t}$.

Theorem 2 Let $\{T_t\}_{t\in R}$ be a generalized dynamics satisfying $T \le 1$ and δ be its generator.

If there exist two sequences $\{b_n\}$ and $\{c_n\}$ of self adjoint elements in A such that

$$(1-\delta)^{-1} = s - \lim_{n\to\infty}(1-\delta_{c_n,b_n})^{-1},$$

where $\delta_{c_n,b_n}(x) = i(c_n x - x b_n)$, then $\{T_t\}_{t\in R}$ is approximately quasi inner.

proof. δ_{c_n,b_n} induces the uniformly continuous one parameter group $T_n(t)x = e^{ic_n t}xe^{-ib_n t}$, by Theorem 1. Now by assumption the range of $(1-\delta)^{-1}$ is dense in M and by Trotter-Kato approximation theorem [13] for each $t \in R$, $T(t) = s - \lim_{n\to\infty} T_n(t)$. ■

Theorem 3 Let $\{\varphi_t\}_{t\in R}$ be a C^*– dynamics on the C^*– algebra A and $\{T_t\}_{t\in R}$ be a generalized dynamics on a full Hilbert C^*– module M. If $\{T_t\}_{t\in R}$ is approximately quasi inner, then $\{\varphi_t\}_{t\in R}$ is approximately inner.

proof. Since $\{T_t\}_{t\in R}$ is approximately quasi inner, then there exist two sequences $\{c_n\}$ and $\{b_n\}$ of self adjoint elements in A such that for each $t\in R$, $x\in M$, $T(t)x=\lim_{n\to\infty}T_n(t)x$, where $T_n(t)x=e^{ic_nt}xe^{-ib_nt}$. Take $\varphi_n(t)a=e^{ic_nt}ae^{-ic_nt}$ and let $z\in M$. Then there exists $x\in M$ such that $z=T_n(t)x$. Thus for all $t\in R$ and $a\in A$ we have

$$
\begin{aligned}
&\varphi_n(t)a.z-\varphi(t)a.z\\
&=\varphi_n(t)aT_n(t)x-\varphi(t)aT_n(t)x\\
&=\varphi_n(t)aT_n(t)x-\varphi(t)aT(t)x\\
&\quad+\varphi(t)aT(t)x-\varphi(t)aT_n(t)x\\
&\le T_n(t)ax-T(t)ax+T(t)x-T_n(t)x\to 0\,.
\end{aligned}
$$

Thus $\lim_{n\to\infty}\varphi_n(t)a=\varphi(t)a$ (By Lemma 1). ■

Remark 2 (i) In the sense of Theorem 2, a generalized dynamics $\{T_t\}_{t\in R}$ on $A-$bimodule M satisfying $T\le 1$ with the generator δ is approximately quasi inner if there exist two sequences $\{b_n\}$ and $\{c_n\}$ of self adjoint elements in A such that $(1-\delta)^{-1}=s-\lim_{n\to\infty}(1-\delta_{c_n,b_n})^{-1}$.

(ii) Following the method as stated in the proof of Theorem 3, it can be shown that if δ is a quasi approximately inner generalized $d-$derivation and $D(\delta)$ is a full Hilbert $D(d)-$module, then d is approximately inner.

Definition 5 A generalized derivation δ is called:

(i) *quasi approximately inner* if there exist two sequences $\{b_n\}$ and $\{c_n\}$ of self adjoint elements in A such that for each $x\in D(\delta)$, $\delta(x)=\lim_{n\to\infty}\delta_{c_n,b_n}(x)$.

(ii) *approximately bounded* if there exist a sequence $\{\delta_n\}$ of bounded generalized derivations on M such that $\{\delta_n\}$ converges strongly to δ on $D(\delta)$.

Because of boundedness of inner generalized derivation, each quasi approximately inner generalized derivation is also approximately bounded. On the other hand every generalized derivation on a unital commutative semi simple Banach algebra, a unital simple C^*-algebra or a Von-Neumann algebra is generalized inner, cf. [2]. Therefore each approximately bounded generalized derivation on the mentioned spaces is approximately inner.

Theorem 4 Let δ be the generator of generalized dynamics $\{T_t\}_{t\in R}$ on $A-$bimodule M satisfying $T_t\le 1$. If δ is a quasi approximately inner generalized $d-$derivation and $(1-\delta)(D(\delta))$ is dense in M, then $\{T_t\}_{t\in R}$ is approximately quasi inner.

proof. Since δ is a quasi approximately inner generalized $d-$derivation, so there exist two sequences $\{b_n\}$ and $\{c_n\}$ of self adjoint elements in A such that $\delta(x)=\lim_{n\to\infty}\delta_{c_n,b_n}(x)$. Also δ_{c_n,b_n} induces the uniformly continuous one parameter group $T_n(t)x=e^{ic_nt}xe^{-ib_nt}$, by Theorem 1. By Remark 2, it is enough to show that $(1-\delta)^{-1}=s-\lim_{n\to\infty}(1-\delta_{c_n,b_n})^{-1}$. For this aim we have

$$
\begin{aligned}
&\left(1-\delta_{c_n,b_n}\right)^{-1}(1-\delta)(x)-(1-\delta)^{-1}(1-\delta)(x)\\
&=\left(1-\delta_{c_n,b_n}\right)^{-1}(1-\delta)(x)-\left(1-\delta_{c_n,b_n}\right)^{-1}\left(1-\delta_{c_n,b_n}\right)(x))\\
&\le\left(1-\delta_{c_n,b_n}\right)^{-1}(1-\delta)(x)-\left(1-\delta_{c_n,b_n}\right)(x)\\
&\le(1-\delta)(x)-\left(1-\delta_{c_n,b_n}\right)(x)\to 0
\end{aligned}
$$

Since $\left(1-\delta_{c_n,b_n}\right)^{-1}\le 1$ (By Hille-Yosida theorem [11]).

Now the density of $(1-\delta)(D(\delta))$ in M implies that $\left(1-\delta_{c_n,b_n}\right)^{-1}\to(1-\delta)^{-1}$ (strongly). ■

Definition 6 A subset D of domain $D(S)$ of a closed linear operator S on a Banach space X is called a *core* for S, if S is the closure of its restriction on D.

Theorem 5 Let δ be the generator of generalized dynamics $\{T_t\}_{t\in R}$ on $A-$bimodule M Satisfying $T_t\le 1$. If there exist two sequences $\{b_n\}$ and $\{c_n\}$ of self adjoint elements in A such that for all x in a core D_0 for δ, $\delta(x)=\lim_{n\to\infty}\delta_{c_n,b_n}(x)$, then $\{T_t\}_{t\in R}$ is approximately quasi inner.

proof. First note that δ_{c_n,b_n} induces the uniformly continuous one parameter group $T_n(t)x = e^{ic_nt}xe^{-ib_nt}$. Also by Hille-Yosida Theorem $\lambda = 1 \in \rho(\delta) \cap \rho(\delta_{c_n,b_n})$. Further $(1-\delta)^{-1} \leq 1$ and $(1-\delta_{c_n,b_n})^{-1} \leq 1$ and the range $R(1-\delta)$ of $1-\delta$ is M.

We are going to show that $(1-\delta_{c_n,b_n})^{-1} \to (1-\delta)^{-1}$ (strongly). For this aim, let $\tilde{M} := \{(1-\delta)(y) : y \in D_0\} = R(1-\delta|_{D_0})$. First we show that $\tilde{M}$ is dense in M. Let $x \in M$. Since $R(1-\delta) = M$, so there exists $z_0 \in D(\delta)$ such that $x = z_0 - \delta(z_0)$. But D_0 is a core for δ. Thus there exists a sequence $\{y_n\}$ in D_0 such that $y_n \to z_0$ and $y_n - \delta(y_n) \to z_0 - \delta(z_0) = x$. Hence $\tilde{M}$ is dense in M.

Now we show that $(1-\delta_{c_n,b_n})^{-1}$ converges strongly on $\tilde{M}$ to $(1-\delta)^{-1}$. For, let $z \in B$.

There exists $y_0 \in D_0$ such that $z = y_0 - \delta(y_0)$ and by assumption $\delta_{c_n,b_n}(y_0) \to \delta(y_0)$. Therefore

$$(1-\delta_{c_n,b_n})^{-1}(z) - (1-\delta)^{-1}(z)$$

$$= (1-\delta_{c_n,b_n})^{-1}(\delta_{c_n,b_n} - \delta)(1-\delta)^{-1}(z)$$

$$\leq (\delta_{c_n,b_n} - \delta)(1-\delta)^{-1}(z)$$

$$= (\delta_{c_n,b_n} - \delta)(y_0) \to 0$$

Finally, given $x \in M$ and $\in > 0$. Since $\tilde{M}$ is dense in M, so there exist $z \in B$ and $N_\epsilon \in \mathbb{N}$ such that $z - x < \frac{\epsilon}{3}$ and for each $n \geq N_\epsilon$, $(1-\delta_{c_n,b_n})^{-1}(z) - (1-\delta)^{-1}(z) < \frac{\epsilon}{3}$. Therefore

$$(1-\delta_{c_n,b_n})^{-1}(x) - (1-\delta)^{-1}(x)$$

$$\leq (1-\delta_{c_n,b_n})^{-1}(z) - (1-\delta_{c_n,b_n})^{-1}(x)$$

$$+ (1-\delta)^{-1}(z) - (1-\delta)^{-1}(x)$$

$$+ (1-\delta_{c_n,b_n})^{-1}(z) - (1-\delta)^{-1}(z)$$

$$\leq (1-\delta_{c_n,b_n})^{-1} z - x + (1-\delta)^{-1} z - x + \frac{\epsilon}{3}$$

$$< \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

Consequently, $(1-\delta_{c_n,b_n})^{-1} \to (1-\delta)^{-1}$ strongly on M. ■

Results and Discussion

Let $\{\varphi_t\}_{t\in R}$ be a C^*-dynamics on a C^*-algebra A with the generator d, $\{T_t\}_{t\in R}$ be a generalized dynamics on a full Hilbert C^*-module M such that $T_t \leq 1$ and let δ be the generator of $\{T_t\}_{t\in R}$. It has been proved that each of the following implies that $\{\varphi_t\}_{t\in R}$ is approximately inner and $\{T_t\}_{t\in R}$ is approximately quasi inner:

(i) There exist two sequences $\{b_n\}$ and $\{c_n\}$ of self adjoint elements in A such that $(1-\delta)^{-1} = s - \lim_{n\to\infty}(1-\delta_{c_n,b_n})^{-1}$, where $\delta_{c_n,b_n}(x) = i(c_nx - xb_n)$.

(ii) δ is a quasi approximately inner generalized $d-$derivation and $(1-\delta)(D(\delta))$ is dense in M.

(iii) There exist two sequences $\{b_n\}$ and $\{c_n\}$ are two sequences of self adjoint elements in A such that $\delta(x) = \lim_{n\to\infty} \delta_{c_n,b_n}(x)$, for all x in a core D_0 for δ.

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