Isotropic Lagrangian Submanifolds in Complex Space Forms

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Received: 21 November 2011 / Revised: 22 September 2012 / Accepted: 25 September 2012

Abstract

In this paper we study isotropic Lagrangian submanifolds $M^n$, in complex space forms $M^n(4c)$. It is shown that they are either totally geodesic or minimal in the complex projective space $\mathbb{C}P^n$, if $n \geq 3$. When $n = 2$, they are either totally geodesic or minimal in $M^2(4c)$. We also give a classification of semi-parallel Lagrangian H-umbilical submanifolds.

Keywords: Lagrangian; Isotropic; Semi-parallel submanifold; H-umbilical; Complex space form

Introduction

The notion of an isotropic submanifold of a Riemannian manifold was introduced by B. O'Neill [14]. These submanifolds which can be considered as generalized totally geodesic submanifolds usually have been studied under some additional hypothesis, [12,13]. Here, we assume that these submanifolds are semi-parallel. On the other hand, Lagrangian submanifolds of complex space forms have been deeply studied since the decade 1970’s. A survey of the main results about Lagrangian submanifolds can be found in [7]. Since there is no complete classification of Lagrangian submanifolds, it is natural to study these submanifolds with some additional constraint.

Recall that, an $n$-dimensional Riemannian submanifold $M$ of an $m$-dimensional Riemannian manifold $\tilde{M}$ is called parallel if its second fundamental form $\mathcal{I}$, satisfies

$$(\nabla^\perp)(X,Y,Z) = \nabla^\perp_z \mathcal{I}(X,Y)$$

$$-\mathcal{I}(\nabla_z X,Y) = \mathcal{I}(X,\nabla_z Y) = 0$$

for all vectors $X,Y,Z$, tangent to $M$ where $\nabla$ is the Van der Waerden–Bortolotti connection [11]. By their definition, semi-parallel submanifolds are generalized parallel submanifolds. The classification of semi-parallel submanifolds in real space forms is still an open problem, although several authors have obtained many important results. We can refer the reader to [10] for a survey. Also a recent good reference for the whole subject of Lagrangian and symplectic manifolds is [1].

Since 2009, it is known that all isotropic Lagrangian submanifolds are parallel, hence semi-parallel [2]. In [9], semi-parallel isotropic Lagrangian submanifolds have been studied. In this paper we follows [2,12,13] to continue the study of isotropic Lagrangian submanifolds in complex space forms.

Our main results are Proposition 1 and Theorems 2,5.
**Preliminaries**

We recall some prerequisites from [2,6,14]. Let $(\tilde{M}^m,J)$ be an $m$ -dimensional Riemannian complex manifold and $M^s$ be an $n$ -dimensional real submanifold of $\tilde{M}$. We denote by $X,Y,W,Z,...$ vectors tangent to $\tilde{M}$ and by $U,V,...$ generic vectors tangent to $\tilde{M}$. $\nabla$ and $\tilde{\nabla}$ denote the Levi-Civita connections of $\tilde{M}$ and $M$, respectively. $\parallel$ is the second fundamental form of $\tilde{M}$, and $A_{\xi}$ is the shape operator of $\tilde{M}$ in the direction of the normal vector field $\xi \in \mathcal{N}(\tilde{M})$.

The curvature $\tilde{R}$ of $\tilde{\nabla}$ is defined by:

$$\tilde{R}(U_1, U_2)W = \tilde{\nabla}_{U_1} \tilde{\nabla}_{U_2} W - \tilde{\nabla}_{\tilde{\nabla}_{U_1} U_2} W - \tilde{\nabla}_{U_2} \tilde{\nabla}_{U_1} W + \tilde{\nabla}_{\tilde{\nabla}_{\tilde{\nabla}_{U_1} U_2} W} W,$$

and the sectional curvature of a plane spanned by $\{U,V\}$ is given by:

$$\langle R(U,V)U,V \rangle / \langle U,U \rangle \langle V,V \rangle - \langle U,V \rangle^2. $$

If $R$ denote the Riemann curvature tensor of $\nabla$, then the Gauss equation is,

$$\langle \tilde{R}(X,Y)Z,W \rangle = \langle R(X,Y)Z,W \rangle + \langle \mathbb{I}(X,Z), \mathbb{I}(Y,W) \rangle - \langle \mathbb{I}(X,W), \mathbb{I}(Y,Z) \rangle.$$

$\nabla = \nabla \oplus \nabla^\perp$ is the Van der Waerden–Bortolotti connection, where $\nabla^\perp$ is normal curvature. The curvature operator $\tilde{R}(X,Y)$ of $\tilde{\nabla}$, can be extended as derivation of tensor fields in the usual way. Its action on $\mathbb{I}$ is as follows,

$$\langle \tilde{R}(X,Y)\cdot \mathbb{I}(Z,W) \rangle = R^\perp(X,Y) \langle \mathbb{I}(Z,W) \rangle$$

$$- \langle \mathbb{I}(R(X,Y)Z,W) \rangle = - \langle \mathbb{I}(Z,R(X,Y)W) \rangle,$$

where $R^\perp$ denote the Riemannian curvature operator of $\nabla^\perp$. The submanifold $M$ of $\tilde{M}$ is called semi-parallel if its second fundamental form $\mathbb{I}$ satisfies

$$\tilde{R}(X,Y) \cdot \mathbb{I} = 0$$

An almost complex structure on $\tilde{M}^m$ is a tensor field $J$ of type (1,1) on $\tilde{M}$, such that $J^2 = -I_{\tilde{M}}$. If $J$ is an isometry, i.e., $\langle U,V \rangle = \langle JU,JV \rangle$, $\tilde{M}^m$ is called an almost Hermitian manifold. The most interesting Hermitian manifolds are, the Kähler manifolds $(\tilde{M},J,(\cdot,\cdot))$ defined by the condition $\tilde{\nabla}J = 0$,

where $\left(\tilde{\nabla}J \right)(U,V) = \tilde{\nabla}_{U}JV - J\tilde{\nabla}_{V}U$.

The holomorphic sectional curvature of an almost Hermitian manifold is the restriction of the sectional curvature to holomorphic planes (the planes that are spanned by $U$ and $JU$) in the tangent spaces. The curvature tensor of a space of constant holomorphic sectional curvature $4c^*$, $\tilde{M}(4c^*)$, is given by:

$$\tilde{R}(U_1,U_2)W = c(\langle U_1 \wedge U_2 \rangle W)$$

$$+ (\langle JU_1 \wedge JU_2 \rangle W + 2\langle JU_1 \wedge U_2 \rangle JW)$$

where $\langle U_1 \wedge U_2 \rangle W = \langle U_1 W \rangle U_2 - \langle U_2 W \rangle U_1$.

A complex space form is a complete, simply connected, Kähler manifold with constant holomorphic sectional curvature. So, a complex space form is isometric to either the complex projective space $\mathbb{C}^n$, if $c > 0$, or the complex Euclidean space $\mathbb{C}^n$, if $c=0$, or the complex projective hyperbolic space $\mathbb{H}^n(4c)$, if $c < 0$.

An $n$ -dimensional submanifold $M^s$ of an almost Hermitian complex manifold $\tilde{M}^m$ is said to be totally real if $J(I_p M) \subset (I_p M)^\perp$ for all $p \in M$. A totally real submanifold $M^s$ of $\tilde{M}^m$ is said to be Lagrangian when $m=n$. For Lagrangian submanifolds of a Kähler manifold the following relations hold, [2]

$$JA_{nX} = (\mathbb{I}(X,Y)) = JA_{nX},$$

$$\langle \mathbb{I}(X,Y), JZ \rangle = \langle \mathbb{I}(Y,Z), JX \rangle = \langle \mathbb{I}(Z,X), JY \rangle,$$

$$R^\perp(X,Y)JZ = JR(X,Y)Z.$$

Moreover, from the Gauss equation one has

$$R(X,Y) = \tilde{R}(X,Y) + A_{\nabla}A_{\nabla} - A_{\nabla}A_{\nabla}. $$

In [4], it is proved that there exists no totally umbilical Lagrangian submanifold in a complex space form $\tilde{M}(4c^*)$ with $n \geq 2$ except the totally geodesic ones. The Lagrangian $H$ -umbilical submanifolds are the simplest Lagrangian submanifolds next to the totally geodesic submanifolds in a complex space form. A Lagrangian $H$ -umbilical submanifold of a Kähler manifold $\tilde{M}(4c^*)$ is a Lagrangian submanifold whose second fundamental form takes the following simple form, [6],

$$A_{\kappa_1}e_1 = \mu e_1, \quad A_{\kappa_2}e_2 = ... = A_{\kappa_n}e_n = \mu e_1,$$

$$A_{\kappa_j}e_j = \mu e_j, \quad A_{\kappa_k}e_k = 0, \quad 2 \leq j \neq k \leq n,$$

where $\langle \tilde{\nabla}J \rangle(U,V) = \tilde{\nabla}_{U}JV - J\tilde{\nabla}_{V}U$. 

with respect to some suitable orthonormal local frame field, and for some suitable functions \( \lambda \) and \( \mu \).

A Lagrangian submanifold \( M \) of \( \tilde{M} \) is said to be \( \lambda \)-isotropic if there exists an smooth function \( \lambda : M \to \mathbb{R} \) such that \( \| (X, X) \| = \lambda^2(p) \), for any unit vector \( X \in T_p M \) and for all \( p \in M \), [14]. In particular, if \( \lambda \) is constant then \( M \) is called constant isotropic.

From [12], it is known that, if \( M^n \) \( (n \geq 3) \) is a minimal totally real and isotropic submanifold of a Kähler manifold, then either \( M \) is totally geodesic or \( n = 5,8,14,26 \). Also, if \( M^n \) is a complete, constant isotropic totally real submanifold of \( \mathbb{C}P^n(4c) \), then either \( M \) is totally geodesic or \( M \) is locally isometric to \( S^1 \times S^{n-1} (n \geq 2) \); \( SU(3)/SO(3) \), \( n = 5; SU(3) \), \( n = 8 \); \( SU(6)/Sp(3) \), \( n = 14 \); \( E_7 \), \( n = 26 \).

### Results

In [2], P. M Chacon and G. A. Lobos give some properties of semi-parallel Lagrangian \( H \)-umbilical submanifold. Here, we give the classification of such submanifolds, by using their result.

**Proposition 1:** If \( n \geq 3 \) and \( M^n \) is a semi-parallel Lagrangian \( H \)-umbilical submanifold of \( \mathbb{C}P^n(4c) \), then \( M^n \) is one of the following submanifolds.

a) A totally geodesic one,

b) A flat submanifold of \( \mathbb{C}^n \),

c) A non-flat and non-totally geodesic minimal submanifold of \( \mathbb{C}P^n(4c) \).

**Proof:** suppose that \( \{e_1, \ldots, e_n\} \) is a suitable orthonormal local frame field, such that with respect to it the shape operators of \( M \) have the form (1.6). From the Gauss equation for \( i, j = 2, \ldots, n \) and \( i \neq j \) we have,

\[
R(e_i, e_j) e_i = (c - \mu^2) e_j,
\]

(2.1)

From (1.6), for Lagrangian \( H \)-umbilical submanifolds, \( H = \sum_{i=1}^{n} A_{e_i} e_i = \frac{\lambda}{n} \mu e_1 \). If \( \mu \neq 0 \) from [2], we have \( \lambda = (1-n) \mu \) and \( c = n \mu^2 > 0 \), so \( H = 0 \), i.e. \( M \) is minimal. Also, from (2.1) we have \( \langle R(e_i, e_j)e_i, e_j \rangle = (n-1)\mu^2 > 0 \), so \( M \) is a non-flat, non-totally geodesic minimal submanifold of \( \mathbb{C}P^n(4c) \).

A semi-parallel Lagrangian submanifold \( M^n \) of constant sectional curvature \( c_1 \) in \( \mathbb{C}P^n(4c) \) is flat or totally geodesic [2]. If \( \mu = 0 \), from (2.1) we get that \( R(e_i, e_j)e_i = ce_j \), and \( R(e_i, e_j)e_i = ce_i \). So \( M \) has constant sectional curvature \( c \), Hence, \( M \) is either totally geodesic or flat. If \( M \) is flat, we get that \( c = 0 \), i.e. \( M^n \) is a flat submanifold of \( \mathbb{C}^n \).

One should see [3] for new results about Lagrangian \( H \)-umbilical submanifolds of para-Kahler manifolds. It is known that for \( n \geq 3 \) any \( \lambda \)-isotropic Lagrangian submanifold \( M^n \) of \( \mathbb{C}P^n(4c) \) is constant isotropic and \( M^n \) is parallel in \( \mathbb{C}P^n(4c) \), [2]. So, any isotropic Lagrangian submanifold of \( \mathbb{C}P^n(4c) \) is semi-parallel. In [12] minimal isotropic Lagrangian submanifolds have been studied. Now, we use the fact that isotropic Lagrangian submanifolds are semi-parallel, and give the classification of such submanifolds.

**Theorem 2:** Let \( M^n \) \( (n \geq 3) \) be a \( \lambda \)-isotropic Lagrangian submanifold of \( \mathbb{C}P^n(4c) \). Then \( M \) is either totally geodesic or minimal in \( \mathbb{C}P^n(4c) \).

**Proof:** From [9], a semi-parallel isotropic Lagrangian submanifold of dimension \( n \geq 3 \) is either totally geodesic or \( c = 2\lambda^2 > 0 \). So, every isotropic Lagrangian submanifold of \( \mathbb{C}P^n(4c) \) with \( c \leq 0 \) is totally geodesic. It follows that non-totally geodesic isotropic Lagrangian submanifolds can only exist in \( \mathbb{C}P^n(4c) \).

If \( X = \sin \theta e_1 + \cos \theta e_n \), we have,

\[
\lambda^2 = \| (X, X) \| = \lambda^2 \cos^4 \theta + \lambda^2 \sin^4 \theta
\]

(2.2)

\[
+ \left( \| (e_i, e_j) \| + \| (e_i, e_j) \| \right) \sin^2 \theta \cos \theta
\]

(2.3)

Since \( \lambda \) is independent of \( \theta \), we obtain from (2.2) that,

\[
0 = \frac{d}{d\theta} \lambda^2 = -2\lambda^2 \sin 2\theta \cos 2\theta + 2\| (e_i, e_j) \|
\]

(2.3)

\[
+ \frac{1}{2} \left( \| (e_i, e_j) \| \right) \sin 4\theta
\]

(2.3)

\[
+ \left( \| (e_i, e_j) \| \right) \cos 4\theta - 3\sin^2 \theta \cos^2 \theta
\]
Choose \( \theta = 0 \) in (2.3) to get
\[
\langle \mathcal{I}(e_i, e_j), \mathcal{I}(e_i, e_j) \rangle = 0
\]
(2.4)

Choose \( \theta = \frac{\pi}{2} \) to obtain
\[
2\|\mathcal{I}(e_i, e_j)\|^2 + \langle \mathcal{I}(e_i, e_j), \mathcal{I}(e_i, e_j) \rangle = \lambda^2
\]
(2.5)

From [12], \( \forall p \in M \) there exists an orthonormal basis \( \{e_1, \ldots, e_n\} \) of \( T_pM \) satisfying \( A_{\lambda_i} e_i = \lambda_i e_i \), and \( \lambda_i = \lambda \), and \( i = 2, \ldots, n \). Let \( \lambda_i \) is either \(-\lambda\) or \( \frac{1}{2}\lambda \). Let \( V_1 \) and \( V_2 \) be the eigenspaces of \( A_{\lambda_i} \) corresponding to the eigenvalues \(-\lambda\) and \( \frac{1}{2}\lambda \) respectively. Then, \( \langle \mathcal{I}(e_i, e_j) \rangle = \mathcal{I}(e_i, e_j) \mathcal{I}(e_i, e_j) = \frac{1}{2} \lambda^2 \). Since \( M \) is \( \lambda \)-isotropic, hence \( \frac{1}{2} \lambda^2 = \lambda^2 \), so \( \lambda = 0 \).

**Case i:** If \( V_1 = \emptyset \), we have \( \mathcal{I}(e_i, e_j) = \frac{1}{2} \lambda e_i \) for \( e_i \in V_2 \), so \( \|\mathcal{I}(e_i, e_j)\|^2 = \frac{1}{4} \lambda^2 \).

**Case ii:** If \( V_2 = \emptyset \), since \( M \) is an \( H \)-umbilical Lagrangian submanifold. From Proposition 2, \( M \) is either totally geodesic or minimal. We have \( H = (1 - \dim V_1) \lambda e_i \) and \( \dim V_1 > 1 \), so if \( M \) is minimal, hence \( \lambda = 0 \) and \( M \) is totally geodesic.

**Case iii:** If \( \dim V_1 = 1 \) and \( \dim V_2 = n - 2 \), from (2.4) and (2.5) we obtain that,
\[
\begin{align*}
A_{\lambda_i} e_1 &= \lambda e_1, \quad A_{\lambda_i} e_2 = -\lambda e_1, \\
A_{\lambda_i} e_3 &= -\lambda e_2, \quad A_{\lambda_i} e_4 = \frac{1}{2} \lambda e_1, \\
A_{\lambda_i} e_5 &= \frac{1}{2} \lambda e_1, \quad A_{\lambda_i} e_6 = \frac{1}{2} \lambda e_2, \\
A_{\lambda_i} e_7 &= e_i, \quad A_{\lambda_i} e_7 = 0,
\end{align*}
\]
(2.6)

where \( e_i \in V_1 \) and \( e_j \in V_2 \), and \( e_i = \pm 1 \). We have from Gauss equation and (2.6) that,
\[
\begin{align*}
R(e_i, e_j) e_k &= c(e_i \wedge e_j) e_k + A_{\lambda_i} A_{\lambda_j} e_k - A_{\lambda_i} A_{\lambda_j} e_k \\
&= ce_j + \frac{1}{2} \lambda A_{\lambda_i} e_i - \lambda A_{\lambda_i} e_j \\
&= ce_j + \frac{1}{2} \lambda^2 e_i - \frac{1}{2} \lambda^2 e_j = (c - \frac{1}{2} \lambda^2) e_j, \\
R(e_i, e_j) e_k &= c(e_i \wedge e_j) e_k + A_{\lambda_i} A_{\lambda_j} e_k - A_{\lambda_i} A_{\lambda_j} e_k \\
&= ce_i + \frac{1}{2} \lambda A_{\lambda_j} e_j - \lambda A_{\lambda_j} e_i \\
&= ce_i + \frac{1}{2} \lambda^2 e_j - \frac{1}{2} \lambda^2 e_i = (c - \frac{1}{2} \lambda^2) e_i.
\end{align*}
\]
(2.7)

For semi-parallel Lagrangian submanifold \( M^a \) of \( M^a (4c) \) we have, [2]
\[
R(X, Y) \sum_{k=1}^{nM} A_{\lambda_k} e_k = 0
\]
(2.9)

Then from (2.7), (2.8) and (2.9) one obtains that,
\[
0 = R(e_2, e_1) e_2 = \frac{nM}{2} \lambda R(e_2, e_1) e_1
\]
(2.10)

and
\[
0 = R(e_1, e_1) e_1 = \frac{nM}{2} \lambda R(e_1, e_1) e_1
\]
(2.11)

If \( \lambda \neq 0 \), from (2.10), \( \sum_{k=1}^{nM} e_k = -e_1 \frac{nM}{2} (n - 2) \), then from (2.11), \( n = 2 \), this is in contrast with the assumption \( n \geq 3 \). So, \( \lambda = 0 \).

**Case iv:** If \( V_2 \neq \emptyset \) and \( \dim V_1 \geq 2 \), from Gauss equation we have,
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Let 
\[ R(\epsilon_i, \epsilon_j)\epsilon_k = c(\epsilon_i \wedge \epsilon_j)\epsilon_k + A_{\alpha_i} A_{\alpha_j} \epsilon_k - A_{\alpha_i} A_{\alpha_j} \epsilon_k = 0, \]

\[ R(\epsilon_i, \epsilon_j)\epsilon_1 = c(\epsilon_i \wedge \epsilon_j)\epsilon_1 + A_{\alpha_i} A_{\alpha_j} \epsilon_1 - A_{\alpha_i} A_{\alpha_j} \epsilon_1 = 0, \]

\[ R(\epsilon_i, \epsilon_j)\epsilon_1 = c(\epsilon_i \wedge \epsilon_j)\epsilon_1 + A_{\alpha_i} A_{\alpha_j} \epsilon_1 - A_{\alpha_i} A_{\alpha_j} \epsilon_1 = 0, \]  \hspace{1cm} (2.12)

for \( \epsilon_i, \epsilon_j, \epsilon_k \in V_1 \). Then, the restriction of \( R(\epsilon_i, \epsilon_j) \) to \( V_1 \cap \text{span}\{\epsilon_i\} \) is equal to \( R(\epsilon_i, \epsilon_j) = (c + \lambda^2)(\epsilon_i \wedge \epsilon_j) \). Using (2.9) gives,

\[ 0 = R(\epsilon_i, \epsilon_j)\sum_{k=1}^{n} A_{\alpha_k} \epsilon_k = \sum_{k=1}^{n} R(\epsilon_i, \epsilon_j) A_{\alpha_k} \epsilon_k \]

\[ = (c + \lambda^2)(\sum_{k=1}^{n} (\epsilon_i \wedge \epsilon_j) A_{\alpha_k} \epsilon_k) \]

\[ = (c + \lambda^2)(\sum_{k=1}^{n} (\epsilon_i, A_{\alpha_k} \epsilon_k) \epsilon_j - \sum_{k=1}^{n} (\epsilon_j, A_{\alpha_k} \epsilon_k) \epsilon_i) \]

\[ = (c + \lambda^2)(\epsilon_i, \sum_{k=1}^{n} A_{\alpha_k} \epsilon_k) \epsilon_j - (\epsilon_j, \sum_{k=1}^{n} A_{\alpha_k} \epsilon_k) \epsilon_i), \]

If \( M \) is not totally geodesic, \( c + \lambda^2 = 3 \lambda^2 \neq 0 \), so for each \( \epsilon_i \in V_1 \), \( \langle \epsilon_i, \sum_{k=1}^{n} A_{\alpha_k} \epsilon_k \rangle = 0 \), therefore

\[ \sum_{k=1}^{n} A_{\alpha_k} \epsilon_k \]

is in the direction of \( \epsilon_i \). Then,

\[ \sum_{k=1}^{n} A_{\alpha_k} \epsilon_k = \sum_{k=1}^{n} (\epsilon_i, A_{\alpha_k} \epsilon_k) \epsilon_i \]

\[ = \lambda(1 - \text{dim} V_1 + \frac{1}{2}\text{dim} V_2)\epsilon_i \]  \hspace{1cm} (2.13)

Let \( \epsilon_i \in V_1 \) and \( \epsilon_i \in V_2 \), from Gauss equation it is seen that

\[ R(\epsilon_i, \epsilon_i)\epsilon_i = (\epsilon_i \wedge \epsilon_i)\epsilon_i + A_{\alpha_i} A_{\alpha_i} \epsilon_i - A_{\alpha_i} A_{\alpha_i} \epsilon_i \]

\[ = -\lambda A_{\alpha_i} \epsilon_i - \lambda A_{\alpha_i} \epsilon_i = -\frac{1}{2}\lambda A_{\alpha_i} \epsilon_i \]  \hspace{1cm} (2.14)

If \( A_{\alpha_i} \epsilon_i = 0 \), (2.5) yields \( \langle I_{\alpha_i}, I_{\alpha_i} \rangle = \lambda^2 \), but since \( \langle I_{\alpha_i}, I_{\alpha_i} \rangle = \frac{1}{2} \lambda^2 \), so \( \lambda = 0 \). From (2.9), (2.13) and (2.14) we have

\[ 0 = R(\epsilon_i, \epsilon_i)\sum_{n=1}^{\infty} A_{\alpha_n} \epsilon_n \]

\[ = \lambda(1 - \text{dim} V_1 + \frac{1}{2}\text{dim} V_2) R(\epsilon_i, \epsilon_i)\epsilon_i \]  \hspace{1cm} (2.15)

\[ = -\frac{1}{2}(1 - \text{dim} V_1 + \frac{1}{2}\text{dim} V_2)\lambda^2 A_{\alpha_i} \epsilon_i \]

If \( A_{\alpha_i} \epsilon_i \neq 0 \) from (2.15) we get that either \( \lambda = 0 \) or \( (1 - \text{dim} V_1 + \frac{1}{2}\text{dim} V_2) = 0 \). Then from (2.13), \( H = 0 \).

So, isotropic Lagrangian submanifolds of \( \tilde{M}^*(4c) \) are either totally geodesic or minimal in \( \mathbb{C}P^n(4c) \). \( \square \)

From the classification of minimal isotropic Lagrangian submanifolds of \( \tilde{M}^*(4c) \) and constant isotropic Lagrangian submanifolds of \( \tilde{M}^*(4c) \) given in [12], we obtain the following corollaries:

**Corollary 3:** Let \( n \geq 3 \), non-totally geodesic isotropic Lagrangian submanifolds \( M^* \) in complex space form \( \tilde{M}^*(4c) \) are minimal and \( n = 5, 8, 14 \) or \( 26 \).

**Corollary 4:** Let \( M^* \) \( (n \geq 3) \) be a complete isotropic Lagrangian submanifold of \( \tilde{M}^*(4c) \). Then \( M^* \) is a totally geodesic or minimal submanifold in \( \mathbb{C}P^n(4c) \), and locally isometric to one of the following spaces:

- a) \( n = 5 \); \( SU(3)/SO(3) \),
- b) \( n = 8 \); \( SU(3) \),
- c) \( n = 14 \); \( SU(6)/Sp(3) \),
- d) \( n = 26 \); \( E_6/F_4 \), where \( E_6 \), \( F_4 \) are exceptional Lie groups.

From [7], we know that all submanifolds in Corollary 4, are minimal.

We also know that isotropic Lagrangian submanifolds, are semi-parallel, for \( n \geq 3 \), [2]. In Theorem 2, we proved that, for \( n \geq 3 \), isotropic Lagrangian submanifolds are either totally geodesic or minimal. To prove the theorem we used the condition that \( M \) is semi-parallel. Now we prove the same result for isotropic Lagrangian surfaces.

**Theorem 5:** If \( M^2 \) is a \( \lambda \)-isotropic Lagrangian surface in the complex space form \( \tilde{M}^2(4c) \), then \( M \) is either totally geodesic or minimal. Moreover, all constant isotropic Lagrangian surfaces in \( \tilde{M}^2(4c) \), are either totally geodesic or flat and minimal surfaces in \( \mathbb{C}P^2(4c) \).

**Proof:** If \( \lambda = 0 \) from (2.5) \( M \) is totally geodesic. If
\( \lambda \neq 0 \), for Lagrangian surfaces, from [12], we have that \( \forall p \in M \) there exist an orthonormal basis \( \{e_1, e_2\} \) of 
\( T_pM \) satisfying \( A_{e_1}e_1 = \lambda e_1 \) and \( A_{e_2}e_2 = \lambda e_2 \), 
\( \lambda_1 \geq 2\lambda_2 \). Since, \( M \) is \( \lambda \)-isotropic, \( \lambda_2^2 = \lambda^2 \). From 
(1.4) we get that \( \langle A_{e_1}e_2, e_2 \rangle = \langle A_{e_2}e_1, e_1 \rangle = \lambda_2 \), and from 
(2.4), 
\[ 0 = \langle A_{e_1}e_2, A_{e_2}e_2 \rangle = \lambda_1 \langle A_{e_1}e_2, e_2 \rangle, \]
so, \( A_{e_1}e_2 = \lambda_2 e_1 \). Using the fact that \( M \) is \( \lambda \)-isotropic gives, \( \lambda_1^2 = \lambda_2^2 \). Also, from \( \lambda_1 \geq 2\lambda_2 \), one gets that 
\( \lambda_1 = \lambda \) and \( \lambda_2 = -\lambda \). So, \( H = JA_{e_1}e_1 + JA_{e_2}e_2 = 0 \), i.e. 
\( M \) is minimal.

Now, suppose that \( M^2 \) is a constant isotropic Lagrangian surface, so \( M^2 \) is either totally geodesic or minimal. From Gauss equation, we obtain that \( M \) has constant Gaussian curvature \( c - 2\lambda^2 \).

In [8], it has been shown that, minimal Lagrangian submanifolds of constant sectional curvature in complex space forms are either totally geodesic or flat. So, if \( M^2 \) is not totally geodesic, we have \( c - 2\lambda^2 = 0 \), so \( c = 2\lambda^2 > 0 \), i.e. \( M^2 \) is a flat, minimal Lagrangian surface in \( \mathbb{CP}^2(4c) \).

From [12], we know that every minimal Lagrangian surface \( M^2 \) in \( \mathbb{CP}^2(4c) \) is isotropic. From Gauss equation we know that the Gaussian curvature of each \( \lambda \)-isotropic Lagrangian surface is \( c - 2\lambda^2 \). So, from Theorem 5 we obtain the following corollary.

**Corollary 6:** A minimal Lagrangian surface \( M^2 \) of constant Gaussian curvature \( c_1 \) in \( \mathbb{CP}^2(4c) \) is either totally geodesic or flat, constant \( \lambda \)-isotropic surface in \( \mathbb{CP}^2(4c) \).

**Acknowledgment**

The author would like to thank gratefully Mr. M.J. Vanaei for his help in the preparation of the final corrected version of the paper.

**References**