Construction of some Join Spaces from Boolean Algebras

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Abstract
The aim of this paper is to construct an algebraic hyperstructure over a set $G$ corresponding to a Boolean algebra $B$ and a function $s: G \rightarrow B$. In order to accomplish this goal we will need to define a hyperoperation $*$ on the set $G$. We define,

$$a^s b = \{ g \in G | s(g) \leq s(a) \lor s(b) \}$$

and prove that if the image of $G$ is a $\lor$-semilattice or constitute a partition of $1$ in $B$, then $(G, ^s)$ is a hypergroup.

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1. Introduction and Preliminaries
First of all we will recall some algebraic definitions that will be used in the paper. A hyperstructure is a set $H$ together with a function $*: H \times H \rightarrow \mathcal{P}(H)$ called hyperoperation, where $\mathcal{P}(H)$ denotes the set of all non-empty subsets of $H$. Marty (Marty, 1934) defined a hypergroup as a hyperstructure $(H,*)$ such that the following axioms hold: (i) $(xy).z = z(x.y)$ for all $x, y, z$ in $H$, (ii) $x.H = H.x = H$ for all $x$ in $H$. (ii) is called the reproduction axiom. A commutative hypergroup $(H,.)$ is called a join space if for all $a, b, c, d \in H$, the implication $a/b \neq 0 \Rightarrow c/d \neq 0 \Rightarrow aod \neq boc \neq 0$ is valid, in which $a/b = \{ x | a \in x \lor b \}$.

The concept of an $H_\lor$-group is introduced by Vougiouklis in (Vougiouklis, 1994) and it is a hyperstructure $(H,\lor)$ such that (i) $(xy).z = x(y.z)$ for all $x, y, z$ in $H$, (ii) $x.H = H.x = H$ for all $x$ in $H$. The first axiom is called weak associativity.
The partition function \( p(n) \) is defined as the number of sequences \( (a_1, a_2, ..., a_r) \), with \( 0 < a_1 \leq a_2 \leq ... \leq a_r \), that the positive integer \( n \) can be written as a sum of positive integers, \( a_i \), as in \( n = a_1 + a_2 + ... + a_r \). The summands \( a_i \) are called the parts of the partition. Also, \( \mathcal{P}(n) \) will denote the set of all integer partitions of \( n \) and for every \( \mathcal{P}(n) \) we denote \( \text{Part}(\mathcal{P}(n)) \) the set of positive integers \( a_j \) such that \( \sum_{j=1}^{r} a_j = n \).

In this paper we construct some join space from Boolean algebras. Our notations are standard and taken mainly from (Corsini, 1993) and (Vougiouklis, 1994).

2. Construction of some Join Spaces

Let \( G \) be a set, \( B \) a Boolean algebra and \( s \) be a function from \( G \) into \( B \).

We define the hyperoperation \( * \) as follows:

\[
(a * b) = \{ x \in G | s(x) \in \text{Part}(s(a)) \}
\]

Since for all \( x, y \in G \), \( \{x, y\} \) \( * s \) \( y \), hence \((G, * s)\) is an \( H_v \)-group.

Also, it is obvious that the hyperoperation \( * s \) is commutative.

**Example 1.**

Suppose \( G = \mathcal{P}(6) \), \( I(6) = \) \( \{1, 2, ..., 6\} \) and \( s : \mathcal{P}(6) \to \mathcal{P}(I(6)) \) is defined by \( s(\mathcal{P}(6)) = \text{Part}(\mathcal{P}(6)) \). In Table 1, we compute all of integer partitions of six. From this table we can see that:

\[
(a * d) \neq (d * a)
\]

Therefore, \((G, * s)\) is not a hypergroup.

Some special cases where \((G, * s)\) is a hypergroup are discussed in the following results.

**Proposition 2.**

If the image \( G \) is a \( \vee \)-sub-semilattice of \( B \) then \((G, * s)\) is a commutative hypergroup.
Table I - Integer partitions of 6

<table>
<thead>
<tr>
<th></th>
<th>6=1+1+1+1+1+1</th>
<th>b</th>
<th>6=1+1+1+1+1+2</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>6=1+1+2+2</td>
<td></td>
<td>6=1+2+2+2</td>
</tr>
<tr>
<td>c</td>
<td>6=1+1+1+3</td>
<td></td>
<td>6=1+2+3</td>
</tr>
<tr>
<td>e</td>
<td>6=1+1+4</td>
<td></td>
<td>6=1+5</td>
</tr>
<tr>
<td>g</td>
<td>6=3+3</td>
<td></td>
<td>6=2+4</td>
</tr>
<tr>
<td>i</td>
<td>6=6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>k</td>
<td>6=6</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Proof.

Suppose \( y \in (a \ast b) \ast c \), then there exists \( g \in G \) such that \( s(g) \in s(a) \ast s(b) \ast s(c) \). Therefore, \( s(y) \in (s(a) \ast s(b) \ast s(c)) \). Since the image \( G \) is a \( \ast \)-sub-semilattice of \( B \), there exists \( \ast \)-sub-semilattice of \( B \), there exists \( t \in G \) such that \( s(t) \ast s(c) = s(a) \ast s(b) \ast s(c) \). Therefore, the associative law is valid.

Lemma 3.

If the image \( G \) is a \( \ast \)-sub-semilattice then we have,

\[ a_1 \ast a_2 \ast \ldots \ast a_n = \{ g \in G | s(g) \leq s(a_1) \ast \ldots \ast s(a_n) \} \]

Proof.

Suppose \( U = a_1 \ast a_2 \ast \ldots \ast a_n \) and \( V = g \ast s(a_1) \ast \ldots \ast s(a_n) \), then we must prove \( U = V \). It is easy to see that \( U \neq V \). Suppose \( y \in V \), then \( s(y) \in s(a_1) \ast \ldots \ast s(a_n) \). Since the image \( G \) is a \( \ast \)-sub-semilattice of \( B \), hence there exists an element \( g \in G \) such that \( s(g) = s(a_1) \ast \ldots \ast s(a_n) \). Using an inductive proof, we have \( g \ast a_1 \ast a_2 \ast \ldots \ast a_n \) and \( y \in u \ast \ldots \ast u \ast a_n \). Therefore, \( u \in \{ u \ast \ldots \ast u \ast a_n \} \). This completes the proof.
Definition 4.
Let \( B=(B, \vee, \wedge, 0, 1) \) be a Boolean algebra. A subset \( X \subseteq B \) is called a partition of 1 if and only if,
1) For all \( x \in X \), \( x \neq 0 \),
2) \( 1=\vee X \),
3) For all \( x, y \in X \), \( x \neq y \), we have \( x \wedge y = 0 \).

For a Boolean algebra \( B \), suppose \( A=\text{Atom}(B) \) is the set of all atoms of \( B \). By 25. 1 and 2 of (Sikorski, 1964), if \( B \) is a complete Boolean algebra, \( B \) is a atomic if and only if it is completely distributive if and only if it is the field of all subsets of the set of all atoms of \( B \). Therefore, the following lemma is true:

Lemma 5.
Let \( B=(B, \vee, \wedge, 0, 1) \) be an atomic complete Boolean algebra and \( A=\text{Atom}(B) \). An equivalence relation in the set \( A \) determines a partition of 1 and conversely, a partition of 1 defines an equivalence relation in \( A \).

In fact the above lemma defines a one-to-one correspondence between the set of all partitions of 1 and the set of all equivalence relations on the set \( A \).

Proposition 6.
If the image \( G \) is a partition of 1 then \( (G, \ast) \) is a commutative hypergroup.

Proof.
It is enough to show the associativity. Suppose \( a, b, c \in G \),
\[
(a \ast b) \ast c = \{ g \in G | s(g) \leq s(a) \vee s(b) \} \ast c
\]
\[
= \bigcup_{s(g) \leq s(a) \vee s(b)} g \ast c
\]
Set, \( T=\{ x \in G | s(x) \leq s(a) \vee s(b) \} \). We now show that \( T=(a \ast b) \ast c \). It is easy to see that \( a \ast b \ast c \subseteq T \). Suppose \( y \in T \), then \( s(y) \leq s(a) \vee s(b) \) and so \( s(y)=(s(y) \wedge s(a)) \vee (s(y) \wedge s(b)) \vee (s(y) \wedge s(c)) \). Now the hypothesis \( \{ s(g) | g \in G \} \) is a partition of 1 and our main proof will consider a number of cases.
Case 1) \( s(y) = s(a) \) or \( s(y) \neq s(a) \) and \( s(y) = s(b) \). In this case we choose \( g = y \) and we have,

\[
y \leq g \land s(g) = s(y) \lor s(a).
\]

Therefore, \( y \leq (a \lor b) \land c \).

Case 2) \( s(y) \neq s(a) \) and \( s(y) \neq s(b) \) and \( s(y) = s(c) \). In this case we choose \( g = a \) and we have,

\[
y \leq g \land s(g) = s(a) \lor s(b).
\]

Thus, \( y \leq (a \lor b) \land c \).

Case 3) \( s(y) \neq s(a) \) and \( s(y) \neq s(b) \) and \( s(y) \neq s(c) \). In this case we have \( s(y) = 0 \) and so \( y \leq (a \lor b) \land c \). Similarly, \( T = a \lor (b \land c) \) and so \( (a \lor b) \land c = a \lor (b \land c) \).

Lemma 7.

If the image \( G \) is a partition of 1 then we have,

\[
a_1 \lor a_2 \lor \ldots \lor a_n = \{ g \in s(g) \leq s(a_1) \lor \ldots \lor s(a_n) \}
\]

Proof.

Suppose \( a_1 \lor \ldots \lor a_n = \{ g \in s(g) \leq s(a_1) \lor \ldots \lor s(a_n) \} \), then we have,

\[
a_1 \lor \ldots \lor a_n = \{ g \in s(g) \leq s(a_1) \lor \ldots \lor s(a_n) \} \lor a_n
\]

\[
= \bigcup_{s(g) \leq s(a_1) \lor \ldots \lor s(a_n)} s(g) \uparrow a_n
\]

\[
= \bigcup_{s(g) \leq s(a_1) \lor \ldots \lor s(a_n)} \{ x \in G \mid s(x) \leq s(g) \lor s(a_n) \}
\]

Set, \( R = a_1 \lor \ldots \lor a_n \) and \( S = \{ g \in G \mid s(g) \leq s(a_1) \lor \ldots \lor s(a_n) \} \). It is obvious that \( R \subseteq S \), so it is enough to show that \( S \subseteq R \). Suppose \( x \) is an arbitrary element of \( S \), then \( s(x) \leq s(a_1) \lor \ldots \lor s(a_n) \), and we have

\[
s(x) = s(x) \land (s(a_1) \lor \ldots \lor s(a_n))
\]

\[
= s(x) \land [s(a_1) \lor \ldots \lor s(a_n)] \lor [s(x) \land s(a_1)]
\]
If \( s(x) = s(a_n) \) then we choose \( g = a_1 \) and we have, 
\[ s(g) \vee \ldots \vee s(a_n) = s(x) \vee \ldots \vee s(a_n). \]
We now assume that \( s(x) \wedge s(a_n) = 0 \), therefore \( s(x) = s(x) \wedge (s(a_1) \vee \ldots \vee s(a_n)) \).
Choose \( g = x \) and we have, \( s(x) \vee s(a_n) \), so \( s(g) = s(x) \vee s(a_n) \).
This completes the proof.

**Proposition 8.**

If \( s \) is on to then \((G, \ast)\) is a join space.

**Proof.**

Suppose \( s \) is onto, then by proposition 2, \((G, \ast)\) is a hypergroup. Choose an element \( t \) such that \( s(t) = 0 \), then \( t \ast a \ast d \neq b \ast c \), for all \( a, b, c, d \in H \). Therefore, \((G, \ast)\) is a join space.

**Lemma 9.**

There exists a function \( s \) such that \((G, \ast)\) is a hypergroup but it is not a join space.

**Proof.**

Suppose \( G \) is a Boolean algebra such that \( |\text{Atom}(G)| \neq 0 \) and \( s:G \rightarrow \mathcal{P} \) defined by \( s(0) = 1 \) and \( s(x) = x \), for all \( x \neq 0 \). Since the image \( G \) is a \( \vee \)-sub-semilattice of \( \mathcal{P} \) then by proposition 2, \((G, \ast)\) is a hypergroup. We now assume that \( a, b, c, d \) are distinct atoms of \( G \). It is clear that \( 1 \ast a/b \neq c/d \) and so \( a/b \neq c/d \).

**Proposition 10.**

If the image \( G \) is a partition of \( 1 \) then \((G, \ast)\) is a join space.
Proof.

Suppose the image of $G$ is a partition of 1 and $a, b, c, d \in G$, such that $a/b \neq 0$ if $s(a)=s(b)$ then $a \neq a^s d$ and if $s(c)=s(d)$ then $c \neq a^s d$. Therefore, we can assume that $s(a), s(b)$ and $s(c), s(d)$. Now since the image $G$ is a partition of 1, hence $a/b=s^{-1}(s(a))$ and $c/d=s^{-1}(s(c))$. By assumption $s^{-1}(s(a)) = s^{-1}(s(c)) \neq 0$ and so $s(a) = s(c)$, i.e., $a \neq a^s d$, $b^s c$, as required.

References


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