A Note on Transformation Semigroups

Masoud Sabbaghan and Fatemah Ayatollah Zadeh Shirazi
Dept. of Math, Faculty of Science, University of Tehran
Enghelab Ave., Tehran, Iran
(sabbagh@khayam.ut.ac.ir)

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Abstract
In this note we study the transformation semigroup \((X,S)\), where \(S\) is a finite union of its subsemigroups.

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Preliminaries:
By a transformation semigroup \((X,S,\cdot)\) (or simply \((X,S)\)) we mean a compact Hausdorff topological space \(X\), a discrete topological semigroup \(S\) with identity \(e\) and a continuous map \(\cdot : X \times S \rightarrow X\) \((\cdot, S) = xs (\forall x \in X, \forall s \in S)\) such that:

- \(\forall x \in X, xe = x\)
- \(\forall x \in X, \forall s, t \in S, x (st) = (xs) t\)

In the transformation semigroup \((X,S)\) we have the following definitions:

1. For each \(s \in S\), define the continuous map \(^s : X \rightarrow X\) by \(^s = xs\) \((\forall x \in X)\), we used to write \(s\) instead of \(^s\). The closure of \(\{s \mid s \in S\}\) in \(X^X\) with pointwise convergence, is called the enveloping semigroup (or Ellis semigroup) of \((X,S)\) and it is written by \(E(X,S)\) or simply \(E(X)\). \(E(X,S)\) has a semigroup structure (Ellis, 1969, Chapter 3), a nonempty subset \(K\) of \(E(X,S)\) is called a right ideal if \(KE(X,S) \subseteq K\), and it is called a minimal right ideal if none of the right ideals of \(E(X,S)\) is a proper subset of \(K\).

2. A nonempty subset \(Z\) of \(X\) is called invariant if \(ZS \subseteq Z\), moreover it is called minimal if it is closed and none of the closed invariant subsets...
of \( X \) is a proper subset of \( Z \). Let \( a \in X \), \( A \) be a nonempty subset of \( X \) and \( C \) be a nonempty subset of \( E(X,S) \), we introduce the following sets:
\[
F(a,C) = \{ p \in C \mid ap = p \}, \quad F(A,C) = \{ p \in C \mid \forall b \in A \quad bp = b \},
\]
and
\[
\overline{F}(A,C) \cup \{ p \mid p \in C \mid Ap \notin A \}, \quad J(C) \cup \{ p \mid p \in C \mid p^* \notin p \}.
\]

3. Let \( a \in X \), \( A \) be a nonempty subset of \( X \) and \( K \) be a closed right ideal of \( E(X,S) \), then (Sabbaghan and Shirazi, 2001, Definition 1):
- We say \( K \) is an \( a \)-minimal set if:
  - \( aK = aE(X,S) \),
  - \( K \) does not have any proper subset like \( L \), such that \( L \) is a closed right ideal of \( E(X,S) \) with \( aL = aE(X,S) \).
- We say \( K \) is an \( A \)-minimal set if:
  - \( \forall b \in A \quad bK = bE(X,S) \),
  - \( K \) does not have any proper subset like \( L \), such that \( L \) is a closed right ideal of \( E(X,S) \) with \( bL = bE(X,S) \) for all \( b \in A \).
- We say \( K \) is an \( A \)-minimal set if:
  - \( AK = AE(X,S) \),
  - \( K \) does not have any proper subset like \( L \), such that \( L \) is a closed right ideal of \( E(X,S) \) with \( AL = AE(X,S) \).

The sets of all \( a \)-minimal (resp. \( A \)-minimal, \( A \)-minimal) sets is written by \( M_{(X,S)}(a) \) (resp. \( \overline{M}_{(X,S)}(A) \), \( M_{(X,S)}(a) \)).

4. Let \( A \) be a nonempty subset of \( X \), we introduce the following sets (Sabbaghan and Shirazi, 2001b, Definition 1):
\[
P(X,S) = \{ (xy) \in X \times X \mid \exists p \in E(X,S) \quad xp = yp \},
\]
\[
P_{(a)}(X,S) = \{ (xy) \in X \times X \mid \exists a \in A \quad \exists l \in M_{(X,S)}(a) \quad \forall p \in l \quad xp = yp \},
\]
\[
\overline{P}_{(a)}(X,S) = \{ (xy) \in X \times X \mid \exists l \in \overline{M}_{(X,S)}(A) \quad \forall p \in l \quad xp = yp \},
\]
\[
\overline{M}_{(X,S)}(D) = \{ ? \mid \exists D \subseteq X \mid \forall K \in \overline{M}_{(X,S)}(D) \quad J(F(D,K)) \neq ? \},
\]
\[
\overline{M}_{(X,S)}(D) = \{ ? \mid \exists D \subseteq X \mid \forall K \in \overline{M}_{(X,S)}(D) \quad J(F(D,K)) \neq ? \},
\]
\[
\Delta x = \{ (xy) \mid x \in X \}, \quad \Delta = \{ (xy) \mid x \in X \}, \quad A \]
be a nonempty subset of \( X \) and \( B \)
be a nonempty subset of $Y$. We say (Sabbaghan and Shirazi, 2001b, Definition 7):

- $(Y,S)$ is a distal (resp. $A$-distal, $A^{[\mathfrak{N}]}$ distal) factor of $(X,S)$ if $R(\varnothing) \cap P(X,S) = \Delta_X$ (resp. $R(\varnothing) \cap P_A(X,S) = \Delta_X$, $R(\varnothing) \cap \overline{P}_A(X,S) = \Delta_X$),

- $(X,S)$ is a distal (resp. $B$-distal, $B^{[\mathfrak{N}]}$ distal) extension of $(Y,S)$ if $R(\varnothing) \cap P(X,S) = \Delta_X$ (resp. $R(\varnothing) \cap P_{\gamma \gamma_{B}}(X,S) = \Delta_X$, $R(\varnothing) \cap \overline{P}_{\gamma \gamma_{B}}(X,S) = \Delta_X$),

- $(Y,S)$ is a proximal (resp. $A$-proximal, $A^{[\mathfrak{M}]}$ proximal) factor of $(X,S)$ if $R(\varnothing) \subseteq P(X,S)$ (resp. $R(\varnothing) \subseteq P_A(X,S)$, $R(\varnothing) \subseteq \overline{P}_A(X,S)$),

- $(X,S)$ is a proximal (resp. $B$-proximal, $B^{[\mathfrak{M}]}$ proximal) extension of $(Y,S)$ if $R(\varnothing) \subseteq P(X,S)$ (resp. $R(\varnothing) \subseteq P_{\gamma \gamma_{B}}(X,S)$, $R(\varnothing) \subseteq \overline{P}_{\gamma \gamma_{B}}(X,S)$).

6. Let $A$ be a nonempty subset of $X$, then (Sabbaghan and Shirazi, 2001a, Definition 13):

- $(X,S)$ is distal if $E(X,S)$ is a minimal right ideal,

- $(X,S)$ is called $A$-distal if for each $a \in A$, $E(X,S) \in M_{X,S}(a)$,

- $(X,S)$ is called, $A^{[\mathfrak{M}]}$ distal if $E(X,S) \in \overline{M}_{X,S}(A)$,

- $(X,S)$ is called, $A^{[\mathfrak{M}]}$ distal if $E(X,S) \in \overline{M}_{X,S}(A)$.

7. Let $Z$ be a closed invariant subset of $X$, define:

$h_{X,S}(Z) = \{ n \in \mathbb{N} \cup \{0\} \mid \exists Z_0, Z_1, \ldots, Z_n \quad \exists((Z_0 \subseteq Z_1 \subseteq \cdots \subseteq Z_n) \wedge (\forall i \in \{0, \ldots, n\} \quad \forall j \in \{0, \ldots, n\} \setminus \{i\} \quad Z_i \neq Z_j) \\
\wedge (\forall i \in \{0, \ldots, n\} \quad Z_i \text{ is a closed invariant subset of } Z))\}.$

**Convention 1.** In what follows $(X,S)$ is a transformation semigroup, $e$ is the identity of $S$ and $S_0, S_1, \ldots, S_n$ are subsemigroups of $S$, such that $e \in \bigcap_{i=0}^{n} S_i$ and $S = \bigcup_{i=1}^{n} S_i$.

**Lemma 2.**

1. $E(X,S) = \bigcap_{i=1}^{n} E(X,S_i)$.

2. $S = S_i \cdots S_1$ and $E(X,S) = E(X,S_i) \cdots E(X,S_1)$. 
Proof.  
1. If $p \in E(X, S)$, then there exists a net $\{s_\gamma\}_{\gamma \in \Gamma} \subseteq S$, such that $\lim_{\gamma \in \Gamma} s_\gamma = p$ (i.e., $\lim_{\gamma \in \Gamma} x s_\gamma = xp$ ($\forall x \in X$)), since $S = \bigcup_{i=1}^n S_i$, so there exists $i \in \{1, \ldots, n\}$ and a subnet $\{s_{\gamma \gamma}\}_{\gamma \in \Lambda}$ of $\{s_\gamma\}_{\gamma \in \Gamma}$, such that $s_{\gamma \gamma} \in S_i$, therefore $\lim_{\gamma \in \Lambda} s_{\gamma \gamma} = p \in E(X, S_i)$, so $E(X, S) \subseteq \bigcup_{i=1}^n E(X, S_i)$.  

2. Use $e \in \bigcap_{i=1}^n S_i$.  

Theorem 3. Let $A$ be a nonempty subset of $X$, then:  
1. $(X, S)$ is distal if and only if for each $i \in \{1, \ldots, n\}$, $(X, S_i)$ is distal.  
2. $(X, S)$ is $A$-distal if and only if for each $i \in \{1, \ldots, n\}$, $(X, S_i)$ is $A$-distal.  
3. Let $A \in \bigcap_{i=1}^n \overline{M} (X, S_i) \cap \overline{M} (X, S)$. Then $(X, S)$ is $A^{[\mathbb{1}]}$ distal if and only if for each $i \in \{1, \ldots, n\}$, $(X, S_i)$ is $A^{[\mathbb{1}]}$ distal.  
4. Let $A \in \bigcap_{i=1}^n \overline{M} (X, S_i) \cap \overline{M} (X, S)$. Then $(X, S)$ is $A^{[\mathbb{1}]}$ distal if and only if for each $i \in \{1, \ldots, n\}$, $(X, S_i)$ is $A^{[\mathbb{1}]}$ distal.  

Proof.  
1. $(X, S)$ is distal if and only if $(X, S)$ is $X$-distal (Sabbaghan and Shirazi, 2001a, Theorem 18), so this is a special case of (2).  
2. We have (by (Sabbaghan and Shirazi, 2001a, Theorem 18)):  
$(X, S)$ is $A$-distal $\iff \forall a \in A \quad J (F(a, E(X, S))) = \{e\}$  
$\iff \forall a \in A \quad J (F(a, \bigcup_{i=1}^n E(X, S_i))) = \{e\}$ (by Lemma 2)  
$\iff \forall a \in A \quad \bigcup_{i=1}^n J (F(a, E(X, S_i))) = \{e\}$  
$\iff \forall a \in A \quad \forall i \in \{1, \ldots, n\} \quad J (F(a, E(X, S))) = \{e\}$
A Note on Transformation Semigroups

3. We have (by Sabbaghan and Shirazi, 2001a, Theorem 18): 
\[(X, S) \text{ is } A^{[\overline{M}]} \text{ distal } \iff J(F(A, E(X, S))) = \{e\}\]
\[\iff \bigcap_{i=1}^{n} J(F(A, E(X, S))) = \{e\}\]
\[\iff \forall i \in \{1, \ldots, n\} \quad J(F(A, E(X, S))) = \{e\}\]
4. Use a similar method described in (3).

Theorem 4. Let \( n = 2 \) in Convention 1 and let \( A \) be a nonempty subset of \( X \).
1. If \((X, S)\) is distal then there exists \( i \in \{1, 2\} \) such that \( E(X, S) = E(X, S_i) \).
2. If \((X, S)\) is \( A \)-distal, then for each \( a \in A \) there exists \( i \in \{1, 2\} \) such that \( F(a, E(X, S)) = F(a, E(X, S_i)) \).
3. If \( A \subseteq \mathcal{M}^{-1}(X, S_1) \cap \mathcal{M}^{-1}(X, S_2) \) and \((X, S)\) is \( A^{[\overline{M}]} \) distal, then there exists \( i \in \{1, 2\} \) such that \( F(A, E(X, S)) = F(A, E(X, S_i)) \).
4. If \( A \subseteq \mathcal{M}^{-1}(X, S_1) \cap \mathcal{M}^{-1}(X, S_2) \) and \((X, S)\) is \( A^{[\overline{M}]} \) distal then there exists \( i \in \{1, 2\} \) such that \( \overline{F}(A, E(X, S)) = \overline{F}(A, E(X, S_i)) \).

Proof.
1. By Theorem 3, \((X, S_1), (X, S_2)\) are distal therefore \( E(X, S_1), E(X, S_2) \) and \( E(X, S) \) are groups. By Lemma 2 we have \( E(X, S) = E(X, S_1) \cup E(X, S_2) \) Thus \( E(X, S_1) \subseteq E(X, S_2) \) or \( E(X, S_2) \subseteq E(X, S_1) \).
2. By Theorem 3, \((X, S_1), (X, S_2)\) are \( A \)-distal therefore for each \( a \in A \), \( F(a, E(X, S_1)) \), \( F(a, E(X, S_2)) \) and \( F(a, E(X, S)) \) are groups (Sabbaghan and Shirazi, 2001a, Theorem 18), moreover by Lemma 2, \( F(a, E(X, S)) = F(a, E(X, S_1)) \cup F(a, E(X, S_2)) \), Thus \( F(a, E(X, S_1)) \subseteq F(a, E(X, S_2)) \) or \( F(a, E(X, S_2)) \subseteq F(a, E(X, S_1)) \).
3. \( F(A, E(X, S_1)), F(A, E(X, S_2)) \) and \( F(A, E(X, S_3)) \) are groups (Sabbaghan and Shirazi, 2001a, Theorem 18), now use a similar method described in (2).

4. \( \overline{F}(A, E(X, S_1)), \overline{F}(A, E(X, S_2)) \) and \( \overline{F}(A, E(X, S_3)) \) are groups (Sabbaghan and Shirazi, 2001a, Theorem 18), now use a similar method described in (2).

**Theorem 5.** Let \( A \) be a nonempty subset of \( X \), we have:

1. \( P(X,S) = \bigcup_{i=1}^{n} P_i(X,S) \).

2. \( P_A(X,S) = \bigcup_{i=1}^{n} P_A(X,S_i) \).

3. \( \bigcup_{i=1}^{n} P_A(X,S_i) \subseteq P_A(X,S) \).

4. If \( A \subseteq \bigcap_{i=1}^{n} \overline{M}(X,S_i) \cap \overline{M}(X,S) \). Then \( \overline{P}(X,S) = \bigcup_{i=1}^{n} \overline{P}(X,S_i) \).

**Proof.** In all items we use Lemma 2 and (Sabbaghan and Shirazi, 2001b, Theorem 4).

1. \( P(X,S) = P(X, S) \), so this is a special case of (2).

2. Let \( x, y \in X \):

\((xy) \in P_A(X, S) \) \(\iff\) \( \exists a \in A \) \( \exists p \in F(a, E(X, S)) \) \( xp = yp \)

\( \iff\) \( \exists a \in A \) \( \exists p \in F(a, \bigcup_{i=1}^{n} E(X, S_i)) \) \( xp = yp \)

\( \iff\) \( \exists a \in A \) \( \exists p \in \bigcup_{i=1}^{n} F(a, E(X, S_i)) \) \( xp = yp \)

\( \iff\) \( \exists a \in A \) \( \exists i \in \{1, \ldots, n\} \) \( \exists p \in F(a, E(X, S_i)) \) \( xp = yp \)

\( \iff\) \( \exists i \in \{1, \ldots, n\} \) \((xy) \in P_A(X, S) \)

\( \iff\) \( (xy) \in \bigcup_{i=1}^{n} P_A(X, S_i) \).
Therefore $P_A(X, S) = \bigoplus_{i=1}^{n} P_A(X, S_i)$.

3. For $i \in \{1, \ldots, n\}$, if $K \in \overline{M_{(X,S_i)}}(A)$, then $\overline{K E(X,S)}$ is a closed right ideal of $E(X, S)$ and for each $a \in A$ we have $a \overline{K E(X,S)} = a E(X, S)$, thus there exists $L \in \overline{M_{(X,S)}}(A)$ such that $L \subseteq \overline{K E(X,S)}$ (Sabbaghan and Shirazi, 2001a, Corollary 3). Let $(x, y) \in X$, we have:

$$(x, y) \in \bigoplus_{i=1}^{n} \overline{P_A}(X, S_i).$$

Then there exists $L \in \overline{M_{(X,S)}}(A)$ such that $L \subseteq \overline{K E(X,S)}$.

4. Let $x, y \in X$, we have:

$$(x, y) \in \bigoplus_{i=1}^{n} P_A(X, S) \iff \exists p \in F(A, E(X, S)) \iff \exists p \in F(A, \bigoplus_{i=1}^{n} E(X, S)) \iff \exists p \in F(A, \bigoplus_{i=1}^{n} E(X, S)) \iff \exists i \in \{1, \ldots, n\} \iff (x, y) \in \bigoplus_{i=1}^{n} P_A(X, S).$$
Therefore $\overline{P}_A(X, S) = \bigwedge_{i=1}^{n} \overline{P}_A(X, S_i)$.

**Corollary 6.** Let $\phi : (X, S) \rightarrow (Y, S)$ be an onto homomorphism, $A$ be a nonempty subset of $X$ and $B$ be a nonempty subset of $Y$, then (with all the factors and extensions being under $\phi$):

a. “$(Y, S)$ is a distal factor of $(X, S)$” if and only if “for each $i \in \{1, \ldots, n\}$, $(Y, S_i)$ is a distal factor of $(X, S)$”.

b. “$(Y, S)$ is an $A$-distal factor of $(X, S)$” if and only if “for each $i \in \{1, \ldots, n\}$, $(Y, S_i)$ is an $A$-distal factor of $(X, S)$”.

c. Let $A \subseteq \bigcap_{i=1}^{n} \overline{M}(X, S) \cap \overline{M}(X, S)$, then “$(Y, S)$ is an $A$-distal factor of $(X, S)$” if and only if “for each $i \in \{1, \ldots, n\}$, $(Y, S_i)$ is an $A$-distal factor of $(X, S)$”.

d. “$(X, S)$ is a distal extension of $(Y, S)$” if and only if “for each $i \in \{1, \ldots, n\}$, $(X, S_i)$ is a distal extension of $(Y, S)$”.

e. “$(X, S)$ is a $B$-distal extension of $(Y, S)$” if and only if “for each $i \in \{1, \ldots, n\}$, $(X, S_i)$ is a $B$-distal extension of $(Y, S)$”.

f. Let $\phi^{-1}(B) \subseteq \bigcap_{i=1}^{n} \overline{M}(X, S) \cap \overline{M}(X, S)$, then “$(X, S)$ is a $B$-distal extension of $(Y, S)$” if and only if “for each $i \in \{1, \ldots, n\}$, $(X, S_i)$ is a $B$-distal extension of $(Y, S)$”.

**Proof.** Use Theorem 5.

**Note 7.** Let $A_1, \ldots, A_n$ be nonempty subsets of $X$. We have:

1. If $\bigwedge_{i=1}^{n} A_i \subseteq \overline{M}(X, S)$ and for each $j \in \{1, \ldots, n\}$, $(X, S_j)$ is $A_j^{(\overline{M})}$ distal, then $(X, S)$ is $\bigwedge_{i=1}^{n} A_i^{(\overline{M})}$ distal.
2. If \( \bigoplus_{i=1}^{n} A_i \in \overline{\text{M}}(X,S) \) and for each \( j \in \{1, \ldots, n\}, (X,S) \) is \( A_j^{[\overline{M}]} \) distal, then \( (X,S) \) is \( \bigoplus_{i=1}^{n} A_i^{[\overline{M}]} \) distal.

(Compare with Theorem 3).

**Proof.** In (1) and (2) we have (by Sabbaghan and Shirazi, 2001a, Theorem 18) and Lemma 2):

\[
\{e\} \subseteq J(F(\bigoplus_{i=1}^{n} A_i, E(X,S))) = \bigoplus_{j=1}^{n} J(F(\bigoplus_{i=1}^{n} A_i, E(X,S))) \subseteq \bigoplus_{j=1}^{n} J(F(\bigoplus_{i=1}^{n} A_i, E(X,S))) = \{e\}. 
\]

So \( J(F(\bigoplus_{i=1}^{n} A_i, E(X,S))) = J(F(\bigoplus_{i=1}^{n} A_i, E(X,S))) = \{e\} \). Therefore \( (X,S) \) is \( \bigoplus_{i=1}^{n} A_i^{[\overline{M}]} \) distal in (1) and \( (X,S) \) is \( \bigoplus_{i=1}^{n} A_i^{[\overline{M}]} \) distal in (2).

**Theorem 8.** Let \( A \) be a nonempty subset of \( X \), we have:
1. \( E(X, S_0) \subseteq E(X, S) \).
2. If \( (X,S) \) is distal, then \( (X,S_0) \) is distal.
3. If \( (X,S) \) is \( A \)-distal, then \( (X,S_0) \) is \( A \)-distal.
4. If \( (X,S) \) is \( A^{[\overline{M}]} \) distal and \( A \in \overline{\text{M}}(X,S_0) \), then \( (X,S_0) \) is \( A^{[\overline{M}]} \) distal.
5. If \( (X,S) \) is \( A^{[\overline{M}]} \) distal and \( A \in \overline{\text{M}}(X,S_0) \), then \( (X,S_0) \) is \( A^{[\overline{M}]} \) distal.
6. Let \( Z \) be a closed invariant subset of \( (X,S) \) and \( h_{x,S}(Z) \leq h_{x,S_0}(Z) \).
7. \( P(X, S_0) \subseteq P(X, S) \), \( P_A(X, S_0) \subseteq P_A(X, S) \) and \( \overline{P}_{A}(X, S_0) \subseteq \overline{P}_{A}(X, S) \).

**Proof.** Take \( S = S \cup S_0 \) and use Lemma 2, Theorem 3 and Theorem 5.

**Corollary 9.** Let \( (X,S) \) be distal, then for each \( s \in S - \{e\} \) and each \( m \in \mathbb{N} \), there exists a net \( \{m\}_{\gamma \in \Gamma} \) in \( \mathbb{N} \) such that \( \lim_{\gamma \in \Gamma} s^m = s^{-m} \).
Proof. Let \( m \in \mathbb{N} \) and \( s \in S - \{ e \} \), by Theorem 8 \((X, \{ s^k \mid k \in \mathbb{N} \} \cup \{ e \})\) is distal, therefore \( E(X, \{ s^k \mid k \in \mathbb{N} \} \cup \{ e \}) \) is a group, so there exists a net \( \{ m \}_\gamma \) in \( \mathbb{N} \cup \{ 0 \} \) such that \( \lim_{\gamma \to \Gamma} s^m = s^{-m} (s^0 = e) \), since \( s \neq e \) we can take \( m \in \mathbb{N} \) (\( \gamma \in \Gamma \)).

Corollary 10. Let \( ?(X, S) \to (Y, S) \) be an onto homomorphism and let \( A \) be a nonempty subset of \( X \) and \( B \) be a nonempty subset of \( Y \), then (with all the factors and extensions being under ?):

a. If \( (Y, S_0) \) is a proximal factor of \( (X, S_0) \), then \( (Y, S) \) is a proximal factor of \( (X, S) \).

b. If \( (Y, S_0) \) is an \( A \)-proximal factor of \( (X, S_0) \), then \( (Y, S) \) is an \( A \)-proximal factor of \( (X, S) \).

c. If \( (Y, S_0) \) is an \( A^{[\Omega]} \) proximal factor of \( (X, S_0) \), then \( (Y, S) \) is an \( A^{[\Omega]} \) proximal factor of \( (X, S) \).

d. If \( (X, S_0) \) is a proximal extension of \( (Y, S_0) \), then \( (X, S) \) is a proximal extension of \( (Y, S) \).

f. If \( (X, S_0) \) is a \( B \)-proximal extension of \( (Y, S_0) \), then \( (X, S) \) is a \( B \)-proximal extension of \( (Y, S) \).

e. If \( (X, S_0) \) is a \( B^{[\Omega]} \) proximal extension of \( (Y, S_0) \), then \( (X, S) \) is a \( B^{[\Omega]} \) proximal extension of \( (Y, S) \).

Proof. Use Theorem 8.

Theorem 11. Let \( A \) be a nonempty subset of \( X \).

1. Let \( a_1, \ldots, a_p, b_1, \ldots, b_q \in S \) be such that \( S = (\bigcup_{i=1}^{p} S_0 a_i) \cup (\bigcup_{i=1}^{q} b_i S_0) \), then:

a. \( E(X, S) = (\bigcup_{i=1}^{p} E(X, S_0 a_i) \cup (\bigcup_{i=1}^{q} b E(X, S_0)) \).

b. \( (X, S) \) is distal if and only if \( a_1, \ldots, a_p, b_1, \ldots, b_q \) are one to one and \( (X, S_0) \) is distal.
c. Suppose $a_1, \ldots, a_p, b_1, \ldots, b_q \in F(A,S)$. Then $(X,S)$ is $A$-distal if and only if $a_1, \ldots, a_p, b_1, \ldots, b_q$ are one to one and $(X,S_0)$ is $A$-distal.

d. Suppose $a_1, \ldots, a_p, b_1, \ldots, b_q \in F(A,S)$ and $A? \bar{M}^{-}(X,S) \cap \bar{M}^{-}(X,S_0)$. Then $(X,S)$ is $A^{(3)}$ distal if and only if $a_1, \ldots, a_p, b_1, \ldots, b_q$ are one to one and $(X,S)$ is $A^{(3)}$ distal.

e. Suppose $a_1, \ldots, a_p, b_1, \ldots, b_q \in \bar{F}(A,S)$ and $A? \bar{M}^{-}(X,S) \cap \bar{M}^{-}(X,S_0)$. Then $(X,S)$ is $A^{(3)}$ distal if and only if $a_1, \ldots, a_p, b_1, \ldots, b_q$ are one to one and $(X,S_0)$ is $A^{(3)}$ distal.

2. Let $\mathcal{S} \subseteq S$ be such that $s^1 \in \mathcal{S}$, then:
   a. $E(X, s^1 \mathcal{S} \mathcal{S}) = s^1 E(X, \mathcal{S} \mathcal{S})$.
   b. $(X, \mathcal{S})$ is distal if and only if $(X, s^1 \mathcal{S} \mathcal{S})$ is distal.
   c. Suppose $\mathcal{S} \subseteq F(A,S)$. Then $(X, \mathcal{S})$ is $A$-distal if and only if $(X, s^1 \mathcal{S} \mathcal{S})$ is $A$-distal.
   d. Suppose $\mathcal{S} \subseteq F(A,S)$ and $A? \bar{M}^{-}(X,S_0) \cap \bar{M}^{-}(X, s^1 \mathcal{S} \mathcal{S})$. Then $(X,\mathcal{S})$ is $A^{(3)}$ distal if and only if $(X, s^1 \mathcal{S} \mathcal{S})$ is $A^{(3)}$ distal.
   e. Suppose $\mathcal{S} \subseteq \bar{F}(A,S)$ and $A? \bar{M}^{-}(X, S_0) \cap \bar{M}^{-}(X, s^1 \mathcal{S} \mathcal{S})$. Then $(X, \mathcal{S})$ is $A^{(3)}$ distal if and only if $(X, s^1 \mathcal{S} \mathcal{S})$ is $A^{(3)}$ distal.

**Proof.**

1. 
   a. Let $r \in E(X,S)$, then there exists a net $\{s_\gamma\}_{\gamma \in \Gamma} \subseteq S$ such that $\lim s_\gamma = r$. There exists a subnet $\{s_{\gamma_\lambda}\}_{\lambda \in \Lambda}$ of $\{s_\gamma\}_{\gamma \in \Gamma}$ and $\{s_{\gamma_\lambda}\}_{\lambda \in \Lambda} \subseteq \mathcal{S}$ such that:

   \[
   (\exists i \in \{1, \ldots, p\} \ \forall \lambda \in \Lambda \ \ s_{\gamma_\lambda} \ ? \ a_i ) \lor (\exists i \in \{1, \ldots, q\} \ \forall \lambda \in \Lambda
   s_{\gamma_\lambda} \ ? \ b_i )\).

   There exists a subnet $\{t_{i\lambda}\}_{\lambda \in \Lambda} \subseteq \{t_i\}_{i \in \Omega}$ such that $\lim t_{i\lambda} \in E(X,\mathcal{S})$, therefore $r E(X,\mathcal{S}) = (\bigwedge_{i=1}^{p} E(X,\mathcal{S}) a_i) \cup (\bigwedge_{i=1}^{q} b_i E(X,\mathcal{S}))$.

   b. If $(X,S)$ is distal, then $E(X,S)$ is a group, so $a_1, \ldots, a_p, b_1, \ldots, b_q$ are one to one, also $(X,\mathcal{S})$ is distal by Theorem 8.
Conversely suppose \(a_1, \ldots, a_p, b_1, \ldots, b_q\) be one to one and \((X,S_0)\) be distal, then \(E(X,S_0)\) is a group, so the elements of \((\bigcup_{i=1}^{p} E(X,S_0) a_i) \cup (\bigcup_{i=1}^{q} b_i E(X,S_0))\) are one to one, thus by (a) the elements of \(E(X,S)\) are one to one and \(J(E(X,S)) = \{e\}\). Therefore \((X,S)\) is distal.

c. If \((X,S)\) is \(A\)-distal and \(a \in A\), then \(F(a, E(X,S))\) is group and \(a_1, \ldots, a_p, b_1, \ldots, b_q \in F(a, E(X,S))\), so \(a_1, \ldots, a_p, b_1, \ldots, b_q\) are one to one, also \((X,S_0)\) is \(A\)-distal by Theorem 8.

Conversely suppose \(a_1, \ldots, a_p, b_1, \ldots, b_q\) be one to one and \((X,S_0)\) is \(A\)-distal, then for each \(a \in A\), then \(F(a, E(X,S_0))\) is a group, so the elements of \((\bigcup_{i=1}^{p} F(a, E(X,S_0)) a_i) \cup (\bigcup_{i=1}^{q} b_i F(a, E(X,S_0)))\) are one to one, thus the elements of \(F(a, E(X,S))\), are one to one (by using (a) we have \((\bigcup_{i=1}^{p} F(a, E(X,S_0)) a_i) \cup (\bigcup_{i=1}^{q} b_i F(a, E(X,S_0))) = F(a, E(X,S))\)

and, \(J(F(a, E(X,S))) = \{e\}\). Therefore \((X,S)\) is \(A\)-distal.

d. Use a similar method described in (c).

e. Use a similar method described in (c).

2. Use a similar method described in (1).

**Corollary 12.**

If \(S\) is a group and \(S_0\) is a normal subgroup of \(S\) such that \(\frac{S}{S_0}\) is finite, then \((X,S)\) is distal if and only if \((X,S_0)\) is distal.

**References**
