**Contra-α-Continuous Functions between Topological Spaces**

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(received: 1/9/2000; accepted: 4/3/2001)

**Abstract**
In this paper, we apply the notion of α-open sets in topological spaces to present and study contra-α-continuity as a new generalization of contra-continuity (Dontchev, 1996).

**Keywords**: α-open, α-closed, contra-α-closed, α-compact, strongly S-closed, contra-α-continuity.

**1. Introduction**
In 1996, Dontchev (Dontchev, 1996) introduced a new class of functions called contra-continuous functions. Recently, Dontchev and Noiri (Dontchev and Noiri, 1999) introduced and studied, among others, a new weaker form of this class of functions called contra-semicontinuous functions. They also introduced the notion of RC-continuity (Dontchev and Noiri, 1999) which is weaker than contra-continuity and stronger than α-continuity (Tong, 1998). The present authors (Jafari and Noiri, 1999) introduced and studied a new class of functions called contra-super-continuous functions which lies between classes of RC-continuous functions and contra-continuous functions.

This paper is devoted to introduce and investigate a new class of functions called contra-α-continuous functions which is weaker than contra-continuous functions and stronger than both contra-semicontinuous functions and contra-precontinuous functions (Jafari and Noiri, 2001).
2. Preliminaries

Throughout this paper, all spaces $X$ and $Y$ (or $(X, t)$ and $(Y, \sigma)$) are topological spaces. A subset $A$ is said to be regular open (resp. regular closed) if $A = \text{Int}(\text{CI}(A))$ (resp. $A = \text{CI}(\text{Int}(A))$) where $\text{CI}(A)$ and $\text{Int}(A)$ denote the closure and interior of $A$.

**Definition 2.1.** A subset $A$ of a space is called:

1. $\mathcal{R}$-open (Abd El-Monsef et al., 1983) if $A \subseteq \text{CI}(\text{Int}(\text{CI}(A)))$,
2. preopen (Mashhour et al., 1982) if $A \subseteq \text{Int}(\text{CI}(A))$,
3. semi-open (Levine, 1963) if $A \subseteq \text{CI}(\text{Int}(A))$,
4. $\mathcal{P}$-open (Njåstad, 1965) if $A \subseteq \text{Int}(\text{CI}(\text{Int}(A)))$.

The complement of a preopen (resp. semi-open, $\mathcal{P}$-open) set is said to be preclosed (resp. semi-closed, $\mathcal{P}$-closed). The collection of all closed (resp. preopen, semi-open, $\mathcal{P}$-open and $\mathcal{P}$-open) subsets of $X$ will be denoted by $C(X)$ (resp. $\text{PO}(X)$, $\text{SO}(X)$, $\mathcal{P}(X)$, $\mathcal{P}\text{O}(X)$).

It is shown in (Njåstad, 1965) that $\mathcal{P}\text{O}(X)$ (or $\alpha\text{CI}(A)$) is a topology for $X$ and it is stronger than the given topology on $X$. By $\alpha\text{CI}(A)$, we denote the closure of a subset $A$ with respect to $\mathcal{P}\text{O}(X)$. We set $C(X, x) = \{ V \in C(X) \mid x \in V \}$ for $x \in X$. We define similarly $\text{PO}(X, x)$, $\text{SO}(X, x)$, $\alpha(X, x)$ and $\mathcal{P}\text{O}(X, x)$. Recall that a subset $A$ of $X$ is said to be generalized closed (briefly g-closed (Levine, 1970)) (resp. $\mathcal{P}$-generalized closed (briefly $\mathcal{P}$g-closed) (Maki et al., 1994) if $\text{CI}(A) \subseteq U$ (resp. $\alpha\text{CI}(A) \subseteq U$) whenever $A \subseteq U$ and $U$ is open. Recall that a subset $A$ of $X$ is called NDB-set (Dontchev, preprint), if it has nowhere dense boundary. A subset $A$ of $X$ is called $\mathcal{P}$-open if it is the union of regular open sets. The complement of a $\mathcal{P}$-open set is called $\mathcal{P}$-closed. Equivalently, $A \subseteq X$ is called $\mathcal{P}$-closed (Velicko, 1968) if $A = \text{Cl}_t(A)$, where $\text{Cl}_t(A) = \{ x \in X \mid \text{Int}(\text{CI}(U)) \cap A \neq \emptyset, U$ is an open set and $x \in U \}$. A subset $A$ of $X$ is called $\mathcal{P}$-generalized closed (Dontchev and Ganster, 1996) if $\text{Cl}_t(A) \subseteq U$, whenever $A \subseteq U$ and $U$ is open in $X$.

**Definition 2.2.** A function $f : X \to Y$ is called perfectly continuous (Noiri, 1984) (resp. $\mathcal{R}$c-continuous (Dontchev & Noiri, 1999) if for each open set $V$ of $Y$, $f^{-1}(V)$ is clopen (resp. regular closed) in $X$. 


**Definition 2.3.** A function $f: X \to Y$ is called *precontinuous* (Mashhour *et al.*, 1982) (resp. *semi-continuous* (Levine, 1963), *?-continuous* (Abd El-Monsef *et al.*, 1983) if for each $x \in X$ and each open set $V$ of $Y$ containing $f(x)$, there exists $U \in \mathcal{P}(X, x)$ (resp. $U \in \mathcal{S}(X, x)$, $U \in ?\mathcal{P}(X, x)$) such that $f(U) \subset V$.

**Definition 2.4.** A function $f: X \to Y$ is called *contra-super-continuous* (Jafari & Noiri, 1999) if for each $x \in X$ and each closed set $V$ of $Y$ containing $f(x)$, there exists a regular open set $U$ in $X$ containing $x$ such that $f(U) \subset V$.

**Definition 2.5.** A function $f: X \to Y$ is called *contra-?-continuous* (resp. *contra-continuous* (Dontchev, 1996), *contra-semi-continuous* (Dontchev & Noiri, 1999), *contra-precontinuous* (Jafari & Noiri, 2001) if $f^{-1}(V)$ is *?-closed* (resp. closed, semi-closed, preclosed) in $X$ for each open set $V$ of $Y$.

**Remark 2.1.** Every contra-continuous function is contra-?-continuous but not conversely as the following example shows.

**Example 2.1.** Let $X = \{a, b, c\}$, $\mathcal{A} = \{X, \varnothing, \{a\}\}$ and $\sigma = \{X, \varnothing, \{b\}, \{c\}, \{b, c\}\}$. Then the identity function $f: (X, \sigma) \to (X, \mathcal{A})$ is contra-?-continuous but not contra-continuous.

### 3. Some properties

**Definition 3.1.** Let $A$ be a subset of a space $(X, \sigma)$. The set $\cap \{U \in \mathcal{A} \mid A \subset U\}$ is called the kernel of $A$ (Mrsevic, 1986) and is denoted by $\text{Ker}(A)$.

**Lemma 3.1.** The following properties hold for subsets $A, B$ of a space $X$:
1. $x \in \text{Ker}(A)$ if and only if $A \cap F \neq \varnothing$ for any $F \in C(X, x)$.
2. $A \subset \text{Ker}(A)$ and $A = \text{Ker}(A)$ if $A$ is open in $X$.
3. $A \subset B$, then $\text{Ker}(A) \subset \text{Ker}(B)$.

**Theorem 3.1.** The following are equivalent for function $f: X \to Y$:
1. $f$ is contra-?-continuous;
(2) for every closed subset $F$ of $Y$, $f^{-1}(F)$ is a closed subset of $X$;
(3) for each $x \in X$ and each $F \in C(Y, f(X))$, there exists $U \in \mathcal{F}(X, x)$ such that $f(U) \subset F$;
(4) $f(\overline{\text{Cl}(A)}) \subset \text{Ker}(f(A))$ for every subset $A$ of $X$;
(5) $\overline{\text{Cl}(f^{-1}(B))} \subset f^{-1}(\text{Ker}(B))$ for every subset $B$ of $Y$.

**Proof.** The implications (1) $\iff$ (2) and (2) $\implies$ (3) are obvious.

(3) $\implies$ (2): Let $F$ be any closed set of $Y$ and $x \in f^{-1}(F)$. Then $f(x) \in F$ and there exists $U_x \in \mathcal{F}(X, x)$ such that $f(U_x) \subset F$. Therefore, we obtain $f^{-1}(F) = \{ U_x | x \in f^{-1}(F) \} \in \mathcal{F}(X)$.

(2) $\implies$ (4): Let $A$ be any subset of $X$. Suppose that $y \notin \text{Ker}(f(A))$. Then by Lemma 3.1 there exists $F \in C(X, y)$ such that $f(A) \cap F = \emptyset$. Thus, we have $A \cap f^{-1}(F) = \emptyset$ and $\overline{\text{Cl}(A)} \cap f^{-1}(F) = \emptyset$. Therefore, we obtain $f(\overline{\text{Cl}(A)}) \cap F = \emptyset$ and $y \notin f(\overline{\text{Cl}(A)})$. This implies that $f(\overline{\text{Cl}(A)}) \subset \text{Ker}(f(A))$.

(4) $\implies$ (5): Let $B$ be any subset of $Y$. By (4) and Lemma 3.1 we have $f(\overline{\text{Cl}(f^{-1}(B))}) \subset \text{Ker}(B)$ and $\overline{\text{Cl}(f^{-1}(B))} \subset f^{-1}(\text{Ker}(B))$.

(5) $\implies$ (1): Let $V$ be any open set of $Y$. Then, by Lemma 3.1 we have $\overline{\text{Cl}(f^{-1}(V))} \subset f^{-1}(\text{Ker}(V)) = f^{-1}(V)$ and $\overline{\text{Cl}(f^{-1}(V))} = f^{-1}(V)$. This shows that $f^{-1}(V)$ is $\mathcal{F}$-closed in $X$.

**Theorem 3.2.** A function $f : (X, \mathcal{A}) \to (X, \sigma)$ is contra-$\mathcal{F}$-continuous if and only if $f : (X, \mathcal{A}) \to (X, \sigma)$ is contra-continuous.

Recall that a subset of a topological space $(X, \mathcal{A})$ is called a $\mathcal{F}$-set if it is the intersection of open sets.

**Theorem 3.3.** A function $f : (X, \mathcal{A}) \to (X, \sigma)$ is contra-$\mathcal{F}$-continuous if and only if inverse images of $\mathcal{A}$-sets are closed.

**Lemma 3.2.** (Mashhour et al., 1983). Let $A \in \mathcal{PO}(X)$ and $B \in \mathcal{F}(X)$. Then $A \cap B \in \mathcal{F}(A)$.

**Theorem 3.4.** If $f : X \to Y$ is contra-$\mathcal{F}$-continuous and $U \in \mathcal{PO}(X)$, then $f|_U : U \to Y$ is contra-$\mathcal{F}$-continuous.

**Lemma 3.3.** (Mashhour et al., 1983). If $A \in \mathcal{F}(Y)$, and $Y \in \mathcal{F}(X)$, then $A \in \mathcal{F}(X)$. 


**Theorem 3.5.** Let \( f: X \to Y \) be a function and \( \{ U_i \mid i \in I \} \) be a cover of \( X \) such that \( U_i \in \mathcal{K}(X) \) for each \( i \in I \). If \( f \mid U_i: U_i \to Y \) is contra-\( \mathcal{K} \)-continuous for each \( i \in I \), then \( f \) is contra-\( \mathcal{K} \)-continuous.

**Proof.** Suppose that \( F \) is any closed set of \( Y \). We have

\[
 f^{-1}(F) = \bigcup_{i \in I} f^{-1}(F) \cap U_i = \bigcup_{i \in I} (f \mid U_i)^{-1}(F)
\]

Since \( f \mid U_i \) is contra-\( \mathcal{K} \)-continuous for each \( i \in I \), it follows that \( f \mid (U_i)^{-1}(F) \in \mathcal{K}(U_i) \). Then, as a direct consequence of Lemma 3.3 we have \( f^{-1}(F) \in \mathcal{K}(X) \) which means that \( f \) is contra-\( \mathcal{K} \)-continuous.

Now we mention the following well-known result:

**Lemma 3.4.** The following properties are equivalent for a subset \( A \) of a space \( X \):
1. \( A \) is clopen;
2. \( A \) is ?-closed and ?-open;
3. \( A \) is ?-closed and preopen.

**Theorem 3.6.** For a function \( f: X \to Y \) the following continuous are equivalent:
1. \( f \) is perfectly continuous;
2. \( f \) is contra-\( \mathcal{K} \)-continuous and ?-continuous;
3. \( f \) is contra-\( \mathcal{K} \)-continuous and precontinuous.

**Proof.** The proof follows immediately from Lemma 3.4.

**Remark 3.1.** In Theorem 3.6, (2) and (3) are decompositions of perfect continuity. The following example shows that contra-\( \mathcal{K} \)-continuity and precontinuity (or ?-continuity) are independent of each other.

**Example 3.1.** The identity function on the real line with the usual topology is continuous and hence ?-continuous and precontinuous. The inverse image of \((0, 1)\) is not ?-closed and the function is not contra-\( \mathcal{K} \)-continuous.

**Example 3.2.** Let \((\mathbb{Z}, \kappa)\) be the digital line (Khalimsky et al., 1990) and define a function \( f: (\mathbb{Z}, \kappa) \to (\mathbb{Z}, \kappa) \) by \( f(n) = n + 1 \) for each \( n \in \mathbb{Z} \).
Then f is contra-$\neg$-continuous. But $\text{Int}(\text{Cl}(f^{-1}(\{1\})) = \emptyset$ and $f^{-1}(\{1\}) \notin \text{PO}(Z,\kappa)$, hence f is neither precontinuous nor $\neg$-continuous.

**Theorem 3.7.** Let Y be a regular space. For a function $f: X \to Y$, the following properties are equivalent:
1. f is perfectly continuous;
2. f is RC-continuous;
3. f is contra-continuous;
4. f is contra-$\neg$-continuous.

**Proof.** The following implications are obvious: perfect continuity $\Rightarrow$ RC-continuity $\Rightarrow$ contra-continuity $\Rightarrow$ contra-$\neg$-continuity. We show the implication (4) $\Rightarrow$ (1). Let $x$ be an arbitrary point of $X$ and $V$ an open set of $Y$ containing $f(x)$. Since $Y$ is regular, there exists an open set $W$ in $Y$ containing $f(x)$ such that $\text{Cl}(W) \subset V$. Since $f$ is contra-$\neg$-continuous, so by Theorem 3.1 there exists $U \in \mathcal{O}(X, x)$ such that $f(U) \subset \text{Cl}(W)$. Then $f(U) \subset \text{Cl}(W) \subset V$. Hence, $f$ is $\neg$-continuous. Since $f$ is contra-$\neg$-continuous and $\neg$-continuous, by Theorem 3.6 $f$ is perfectly continuous.

**Corollary 3.1.** If a function $f: X \to Y$ is contra-$\neg$-continuous and $Y$ is regular, then $f$ is continuous.

**Remark 3.2.** The converse of corollary 3.1 is not true. Example 3.1 shows that continuity does not necessarily imply contra-$\neg$-continuity even if the range is regular. Recall that a space $X$ is said to be rim-compact if each point of $X$ has a base of neighborhoods with compact frontiers.

**Lemma 3.5** (Noiri (1976), Theorem 4]). Every rim-compact Hausdorff space is regular.

**Corollary 3.2.** If a function $f: X \to Y$ is contra-$\neg$-continuous and $Y$ is rim-compact Hausdorff, then $f$ is continuous.

**Definition 3.2.** A function $f: X \to Y$ is called contra-$\neg$ $g$-continuous if the preimage of every open subset of $Y$ is $\neg$-$g$-closed.
Recall that a space $X$ is $T_{1/2}$-space (Levine, 1961) if every generalized closed set is closed.

**Lemma 3.6** (Dontchev, 1997). For a space $X$ the following conditions are equivalent:
1. $X$ is $T_{1/2}$-space.
2. Every $g$-closed subset of $X$ is $\gamma$-closed.

**Theorem 3.8.** If a function $f: X \to Y$ is contra-$g$-continuous and $X$ is $T_{1/2}$-space, then $f$ is contra-$\gamma$-continuous.

Recall that a function $f: X \to Y$ is NDB-continuous (Dontchev, preprint) if the preimage of every open set is an NDB-set.

**Lemma 3.7** (Dontchev, preprint) For a subset $A$ of a space $X$ the following conditions are equivalent:
1. $A$ is $\gamma$-closed.
2. $A$ is a preclosed NDB-set.

**Theorem 3.9.** For a function $f: X \to Y$, the following conditions are equivalent:
1. $f$ is contra-$\gamma$-continuous.
2. $f$ is contra-precontinuous and NDB-continuous.

**Definition 3.3.** A function $f: X \to Y$ is said to be
1. I.c.-$\gamma$-continuous if for each $x \in X$ and each closed set $F$ of $Y$ containing $f(x)$, there exists an $\gamma$-open set $U$ in $X$ containing $x$ such that $\text{Int}[f(U)] \subseteq F$.  
2. $(\gamma, s)$-open if $f(U) \in SO(Y)$ for every $U \in \gamma(X)$.

**Theorem 3.10.** If a function $f: X \to Y$ is I.c.$\gamma$-continuous and $(\gamma, s)$-open, then $f$ is is contra-$\gamma$-continuous.

**Proof.** Let $x$ be an arbitrary point of $X$ and $V \in C(Y, f(x))$. By hypothesis $f$ is I.c.$\gamma$-continuous which implies the existence of a set $U \in \gamma(X, x)$ such that $\text{int}[f(U)] \subseteq V$. Since $f$ is $(\gamma, s)$-open, then $f(U) \in SO(Y)$. It follows that $f(U) \subseteq \text{Cl}(\text{Int}(f(U))) \subseteq \text{Cl}(V)$ and therefore $f$ is contra-$\gamma$-continuous.
Definition 3.4. A filter base $\Lambda$ is said to be $\beta$-convergent (Jafari, 2001) (resp. $c$-convergent) to a point $x$ in $X$ if for any $U \in \beta(X, x)$ (resp. $U \in C(X, x)$), there exists $B \in \Lambda$ such that $B \subset U$.

Theorem 3.11. A function $f: X \rightarrow Y$ is contra-$\beta$-continuous if and only if for each point $x \in X$ and each filter base $\Lambda$ in $X$ $\beta$-converging to $x$, the filter base $f(\Lambda)$ is $c$-convergent to $f(x)$.

Proof. Necessity. Let $x \in X$ and $\Lambda$ be any filter base in $X$ $\beta$-converging to $x$. Since $f$ is contra-$\beta$-continuous, then for any $V \in C(Y, f(x))$, there exists $U \in \beta(X, x)$ such that $f(U) \subset V$. Since $\Lambda$ is $\beta$-converging to $x$, there exists a $B \in \Lambda$ such that $B \subset U$. This means that $f(B) \subset V$ and therefore the filter base $f(\Lambda)$ is $c$-convergent to $f(x)$.

Sufficiency. Let $x \in X$ and $V \in C(Y, f(x))$. If we take $\Lambda$ to be the set of all sets $U$ such that $U \in \beta(X, x)$, then $\Lambda$ will be a filter base which $\beta$-converges to $x$. Thus, there exists $U \in \Lambda$ such that $f(U) \subset V$.

4. Contra-$\beta$-closed graphs

We begin with the following notion:

Definition 4.1. The graph $G(f)$ of a function $f: X \rightarrow Y$ is said to be contra-$\beta$-closed if for each $(x, y) \in (X \times Y) - G(f)$, there exist $U \in \beta(X, x)$ and $V \in C(Y, y)$ such that $(U \times V) \cap G(f) = \emptyset$.

Lemma 4.1. The graph $G(f)$ of a function $f: X \rightarrow Y$ is said to be contra-$\beta$-closed in $X \times Y$ if and only if for each $(x, y) \in (X \times Y) - G(f)$, there exist $U \in \beta(X, x)$ and $V \in C(Y, y)$ such that $f(U) \cap V = \emptyset$.

Theorem 4.1. If $f: X \rightarrow Y$ is contra-$\beta$-continuous and $Y$ is Urysohn, then $G(f)$ is contra-$\beta$-closed in $X \times Y$.

Proof. Let $(x, y) \in (X \times Y) - G(f)$, then $y \neq f(x)$ and there exist open sets $V, W$ such that $f(x) \in V$, $y \in W$ and $Cl(V) \cap Cl(W) = \emptyset$. Since $f$ is contra-$\beta$-continuous, there exists $U \in \beta(X, x)$ such that $f(U) \subset Cl(V)$. Therefore, we obtain $f(U) \cap Cl(W) = \emptyset$. This shows that $G(f)$ is contra-$\beta$-closed.
**Theorem 4.2.** If \( f: X \to Y \) is \( \tau \)-continuous and \( Y \) is \( T_1 \), then \( G(f) \) is contra-\( \tau \)-closed in \( X \times Y \).

**Proof.** Let \( (x, y) \in (X \times Y) - G(f) \), then \( f(x) \neq y \) and there exists an open set \( V \) of \( Y \) such that \( f(x) \in V \) and \( y \not\in V \). Since \( f \) is \( \tau \)-continuous, there exists \( U \in \tau(X, x) \) such that \( f(U) \subseteq V \). Therefore, we obtain \( f(U) \cap (Y - V) = \emptyset \) and \( Y - V \in C(Y, y) \). This shows that \( G(f) \) is contra-\( \tau \)-closed in \( X \times Y \).

**Definition 4.2.** A space \( X \) is said to be \( \tau \)-compact (Maheshwari & Thakur, 1985) (resp. strongly \( S \)-closed (Dontchev, 1996)) if every \( \tau \)-open (resp. closed) cover of \( X \) has a finite subcover.

A subset \( A \) of a space \( X \) is said to be \( \tau \)-compact relative to \( X \) (Noiri & Di Maio, 1988) if every cover of \( A \) by \( \tau \)-open sets of \( X \) has a finite subcover. A subset \( A \) of a space \( X \) is said to be strongly \( S \)-closed if the subspace \( A \) is strongly \( S \)-closed.

**Theorem 4.3.** If \( f: X \to Y \) has a contra-\( \tau \)-closed graph, then the inverse image of a strongly \( S \)-closed set \( K \) of \( Y \) is \( \tau \)-closed in \( X \).

**Proof.** Assume that \( K \) is a strongly \( S \)-closed set of \( Y \) and \( x \not\in f^{-1}(K) \). For each \( k \in K \), \( (x, k) \not\in G(f) \). By Lemma 4.1, there exist \( U \subseteq \tau(X, x) \) and \( V \subseteq C(Y, k) \) such that \( f(U) \cap V = \emptyset \). Since \( \{K \cap V \mid k \in K\} \) is a closed cover of the subspace \( K \), there exists a finite subset \( K_1 \subseteq K \) such that \( k \subseteq U \cap \{V \mid k \in K_1\} \). Set \( U = \cap \{U_k \mid k \in K_1\} \), then \( U \subseteq \tau(X, x) \) and \( f(U) \cap K = \emptyset \). Therefore \( U \cap f^{-1}(K) = \emptyset \) and hence \( x \not\in C(f^{-1}(K)) \). This shows that \( f^{-1}(K) \) is \( \tau \)-closed in \( X \).

**Theorem 4.4.** Let \( Y \) be a strongly \( S \)-closed space. If a function \( f: X \to Y \) has a contra-\( \tau \)-closed graph, then \( f \) is contra-\( \tau \)-continuous.

**Proof.** Suppose that \( Y \) is strongly \( S \)-closed and \( G(f) \) is contra-\( \tau \)-closed. First, we show that an open set of \( Y \) is strongly \( S \)-closed. Let \( V \) be an open set of \( Y \) and \( \{H \mid \alpha \in \nabla\} \) be a cover of \( V \) by closed sets \( H \) of \( V \). For each \( \alpha \in \nabla \), there exists a closed set \( K_\alpha \) of \( X \) such that \( H_\alpha = K_\alpha \cap V \). Then, the family \( \{K_\alpha \mid \alpha \in \nabla\} \cup (Y - V) \) is a closed cover of \( Y \). Since \( Y \) is strongly \( S \)-closed, there exists a finite subset \( \nabla^* \subseteq \nabla \) such
that $Y = U \{ K_{\alpha} \mid \alpha \in \mathcal{V} \} \cup (Y - V)$. Therefore we obtain $V = (U \{ H_{\alpha} \mid \alpha \in \mathcal{V} \})$. This shows that $V$ is strongly $S$-closed. For any open set $V$, by Theorem 4.3 $f^{-1}(V)$ is $?-\alpha$-closed in $X$ and $f$ is contra-$?-\alpha$-continuous.

5. Covering properties

**Theorem 5.1.** If $f: X \rightarrow Y$ is contra-$?-\alpha$-continuous and $K$ is $?-\alpha$-compact relative to $X$, then $f(K)$ is strongly $S$-closed in $Y$.

**Proof.** Let $\{ H_{\alpha} \mid \alpha \in \mathcal{V} \}$ be any cover of $f(K)$ by closed sets of the subspace $f(K)$. For each $\alpha \in \mathcal{V}$, there exists a closed set $K_{\alpha}$ of $n Y$ such that $H_{\alpha} \subseteq K_{\alpha} \cap f(K)$. For each $x \in K$, there exists $\alpha(x) \in \mathcal{V}$ such that $f(x) \in K_{\alpha(x)}$. Since the family $\{ U_{\alpha} \mid x \in K \}$ is a cover of $K$ by $?-\alpha$-open sets of $X$, there exists a finite subset $K_0$ of $K$ such that $K \subseteq U\{ U_{\alpha(x)} \mid x \in K \}$. Therefore, we obtain $f(K) \subseteq \bigcup \{ f(U_{\alpha(x)}) \mid x \in K \}$ which is a subset of $\bigcup \{ K_{\alpha(x)} \mid \alpha \in K \}$. Thus, $f(K) = \bigcup \{ H_{\alpha(x)} \mid x \in K \}$ and hence $f(K)$ is strongly $S$-closed.

**Corollary 5.1.** If $f: X \rightarrow Y$ is a contra-$?-\alpha$-continuous surjection and $X$ is $?-\alpha$-compact, then $Y$ is strongly $S$-closed.

**Definition 5.1.** A topological space $X$ is said to be

1. **$S$-closed** (Thompson, 1976) if for every semi-open cover $\{ V_{\alpha} \mid \alpha \in \mathcal{V} \}$ of $X$, there exists a finite subset $\mathcal{V}^* \subseteq \mathcal{V}$ such that $X = \bigcup \{ \text{Cl}(V_{\alpha}) \mid \alpha \in \mathcal{V}^* \}$, equivalently if every regular closed cover of $X$ has a finite subcover.

2. **nearly compact** (Singal & Mathur, 1969) if every regular open cover of $X$ has finite subcover.

3. **almost compact** (Singal & Mathur, 1969) if for every open cover $\{ V_{\alpha} \mid \alpha \in \mathcal{V} \}$ of $X$, there exists a finite subset $\mathcal{V}^* \subseteq \mathcal{V}$ such that $X = \bigcup \{ \text{Cl}(V_{\alpha}) \mid \alpha \in \mathcal{V}^* \}$.

4. **mildly compact** (Staum, 1974) if every clopen cover of $X$ has a finite subcover.
Remark 5.1. For the spaces defined above, we have the following implications:

\[ \text{compact} \Rightarrow \text{nearly compact} \]

Strongly S-closed \( \Rightarrow \) S-closed \( \Rightarrow \) almost compact \( \Rightarrow \) mildly compact

Theorem 5.2. If \( f: X \rightarrow Y \) is contra-\( ? \)-continuous and \( X \) is an S-closed space, then \( Y \) is compact.

Proof. Let \( \{ V_\alpha \mid \alpha \in \mathcal{V} \} \) be any open cover of \( Y \). Then \( \{ f^{-1}(V_\alpha) \mid \alpha \in \mathcal{V} \} \) is a cover of \( X \). Since \( f \) is contra-\( ? \)-continuous, \( f^{-1}(V_\alpha) \) is \( ? \)-closed and \( ? \)-open in \( X \) for each \( \alpha \in \mathcal{V} \). This implies that \( \{ f^{-1}(V_\alpha) \mid \alpha \in \mathcal{V} \} \) is a regular closed cover of the S-closed space \( X \). We have \( X = \bigcup \{ f^{-1}(V_\alpha) \mid \alpha \in \mathcal{V} \} \) for some finite \( \mathcal{V}^* \) of \( \mathcal{V} \). Since \( f \) is surjective, \( Y = \bigcup \{ V_\alpha \mid \alpha \in \mathcal{V}^* \} \). This shows that \( Y \) is compact.

Corollary 5.2. (Dontchev, 1996). Contra-continuous \( ? \)-continuous images of S-closed spaces are compact.

Theorem 5.3. If \( f: X \rightarrow Y \) is contra-\( ? \)-continuous precontinuous surjection and \( X \) is mildly compact, then \( Y \) is compact.

Proof. Let \( \{ V_\alpha \mid \alpha \in \mathcal{V} \} \) be any open cover of \( Y \). Since \( f \) is contra-\( ? \)-continuous precontinuous, by Theorem 3.4 \( \{ f^{-1}(V_\alpha) \mid \alpha \in \mathcal{V} \} \) is a clopen cover of \( X \) and there exists a finite subset \( \mathcal{V}^* \) of \( \mathcal{V} \) such that shows that \( Y \) is compact.

Corollary 5.3. (Dontchev 1996). The image of an almost compact space under contra-continuous, nearly continuous (= precontinuous) function is compact.

6. Connected spaces

Theorem 6.1. Let \( X \) be connected and \( Y \) be \( T_1 \). If \( f: X \rightarrow Y \) is contra-\( ? \)-continuous, then \( f \) is constant.
Proof. Since $Y$ is $T_1$-space, $\Omega = \{f^{-1}(\{y\}) \mid y \in Y\}$ is a disjoint open partition of $X$. If $|\Omega| \geq 2$, then there exists a proper open closed set $W$. By Lemma 3.4, $W$ is clopen in the connected space $X$. This is a contradiction. Therefore $|\Omega| = 1$ and hence $f$ is constant.

Corollary 6.1. (Dontchev and Noiri, 1999). Let $X$ be connected and $Y$ be $T_1$. If $f: X \to Y$ is contra-continuous, then $f$ is constant.

Theorem 6.2. If $f: X \to Y$ is a contra-$\sigma$-continuous precontinuous surjection and $X$ is connected, then $Y$ has an indiscrete topology.

Proof. Suppose that there exists a proper open set $V$ of $Y$. Then, since $f$ is contra-$\sigma$-continuous precontinuous, $f^{-1}(V)$ is $\sigma$-closed and preopen in $X$. Therefore, by Lemma 3.4 $f^{-1}(V)$ is clopen in $X$ and proper. This shows that $X$ is a connected which is a contradiction.

Theorem 6.3. If $f: X \to Y$ is contra-$\sigma$-continuous surjection and $X$ is connected, then $Y$ is connected.

Proof. Suppose that $Y$ is not connected. There exist nonempty disjoint open sets $V_1$ and $V_2$ such that $Y = V_1 \cup V_2$. Therefore, $V_1$ and $V_2$ are clopen in $Y$. Since $f$ is contra-$\sigma$-continuous, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are $\sigma$-closed and $\sigma$-open in $X$ and hence clopen in $X$ by Lemma 3.4. Moreover, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are nonempty disjoint and $X = f^{-1}(V_1) \cup f^{-1}(V_2)$. This shows that $X$ is not connected.

A space $(X, \sigma)$ is said to be hyperconnected (Steen & Seebach, 1970) if the closure of every open set is the entire set $X$. It is well-known that every hyperconnected space is connected but not conversely.

Remark 6.1. In Example 2.1, $(X, \sigma)$ is hyperconnected and $f: (X, \sigma) \to (X, \sigma)$ is a contra-$\sigma$-continuous surjection, but $(X, \sigma)$ is not hyperconnected. This shows that contra-$\sigma$-continuous surjection do not necessarily preserve hyperconnectedness.

A function $f: X \to Y$ is said to be weakly continuous (Levine, 1961) if for each point $x \in X$ and each open set $V$ of $Y$ containing $f(x)$, there exists an open set $U$ containing $x$ such that $f(U) \subseteq \text{Cl}(V)$. It is shown in
(Noiri, 1974, Theorem 3) that if \( f: X \to Y \) is a weakly continuous surjection and \( X \) is connected, then \( Y \) is connected. However, it turns out that contra-\( ? \)-continuity and weak continuity are independent of each other. In Example 2.1, the function \( f \) is contra-\( ? \)-continuous but not weakly continuous. The following example shows that not every weakly continuous function is contra-\( ? \)-continuous.

**Example 6.1.** Let \( X = \{a, b, c, d\} \) and \( \mathcal{X} = \{\emptyset, X, \{b\}, \{c\}, \{b, c\}, \{a, b\}, \{a, c\}, \{b, c, d\}\} \). Define a function \( f: (X, \mathcal{X}) \to (X, \mathcal{X}) \) as follows: \( f(a) = c, f(b) = d, f(c) = b \) and \( f(d) = a \). Then \( f \) is weakly continuous (Neubrunnova, 1980). However, \( f \) is not contra-\( ? \)-continuous since \( \{a\} \) is a closed set of \( (X, \mathcal{X}) \) and \( f^{-1}(\{a\}) = \{d\} \) is not \( ? \)-open in \( (X, \mathcal{X}) \).

**References**


Jafari, S., Rare \( ? \)-continuity (submitted).


