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Non-equilibrium Thermodynamics and Superfluid He-II

P. Havvaie and A. Rajabtabar
Department of physics, Tabriz University, Tabriz Iran
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Abstract

We adopt the general theory of Non-equilibrium thermodynamics to the two-fluid Model of He-II. In this Model superfluid is considered as a fluid of two-components each of with its own density and velocity fields. In contrast to the earlier works, our model assumes that superfluid carries small entropy ( ), owing to the Fliessbach. Since the fluid has two independent velocity fields, there is no preferred frame of references and hence, no unique definition of internal energy. The explicit forms of this quantity are worked out in different models and compared with our definition of internal energy and Gibbs function. The existing relation and unification together with the equivalency of London and Landau's theories is also introduced.

Keywords: *Two-fluid Model, Superfluid Helium, Lagrangian Density, Chemical reaction, Internal convection.*

Introduction

The two-fluid Model is concerned with the macroscopic theory of superfluid ***He-II,*** i.e. the liquid phase of He^4 at $T \leq 2.17^0 K$. Several authors have used this Model (Donnelly and Roberts, 1974; Londau, 1978; London, 1954; Tilley, D.R. and Tilley, J., 1990) and considered He-II as a mixture of two fluids***, normal fluid*** and ***superfluid***, each of with its own density and velocity fields. It is assumed that superfluid has no viscosity and no entropy and hence, heat is only transferred by normal fluid. Recently, Fliessbach (Fliessbach, 1991; Fliessbach and Schaefer, 1994) has pointed out that the superfluid can carry small amount of entropy, which is an evidence for the existence of sixth sound and it has been supported experimentally. Owing to Fliessbach and in the framework of two-fluid Model, we identify ***He-II*** with a two-component mixture behaving independently and undergoing to the immediate chemical reaction $n \leftrightarrow s$. Applying the basic conservation relations of non-equilibrium thermodynamics (de Groot, and Mazur, 1984) in which conventionally employs **center-of-mass** velocity, we attempt to find relations between internal energies and Gibbs's func-tions appearing in different theories of superfluid heluim.

a) Basic conservation Laws

I) Mass balance:

For each component separately:

$$\left\{ \begin{array}{l} \partial_t \rho_n + \partial_j (\rho_n \overrightarrow{v_{n,j}}) = \nu_n j_c \\ \partial_t \rho_s + \partial_j (\rho_s \overrightarrow{v_{s,j}}) = \nu_s j_c \end{array} \right\} \tag{1}$$

With

$$\partial_t \rho + \partial_j (\rho \overrightarrow{v_j}) = 0 \tag{2}$$

Where

$$\left\{ \begin{array}{l} \rho = \rho_n + \rho_s \\ \rho \vec{v} = \rho_n \overrightarrow{v_n} + \rho_s \overrightarrow{v_s} \\ \overrightarrow{J_n} = \rho_n (\overrightarrow{v_n} - \vec{v}) \quad , \quad \overrightarrow{J_s} = \rho_s (\overrightarrow{v_s} - \vec{v}) \end{array} \right\} \tag{3}$$

In above relations, $\rho_n, \overrightarrow{v_n}, \overrightarrow{J_n}$ and $\nu_n j_c$ are the density, velocity, mass current relative to the center-of-mass velocity and the rate of production of normal component due to the chemical reaction $n \leftrightarrow s$, respectively. For the superfluid component similar quantities but with subscript s, can be defined. Of course, ρ is total density and $\vec{v}$ is the center-of-mass velocity and for the present single reaction, $\nu_n = -\nu_s \equiv \nu$.

II) Momentum balance:

$$\rho D_t \vec{v} = -\partial_j P_i + \rho_n \overrightarrow{F_n} + \rho_s \overrightarrow{F_s} \tag{4}$$

Where D_t stands for substantial time derivative, P is hydrostatic pressure and $\overrightarrow{F_n}$ and $\overrightarrow{F_s}$ are external forces per unit mass acting on the normal and superfluid components, respectively. Note that we have omitted the viscous pressure tensor from r.h.s. of (4) since viscosity effects are not being considered in the present work.

III) Energy balance

$$\rho D_t (\frac{1}{2} v^2 + u) = -\partial_j (P\overrightarrow{v_j} + \overrightarrow{J_{q,j}}) + \rho_n \overrightarrow{F_n}.\overrightarrow{v_n} + \rho_s \overrightarrow{F_s}.\overrightarrow{v_s} \tag{5}$$

Where $\overrightarrow{J_q}$ is the flux of heat and the total energy per unit mass of whole fluid is separated in two parts:

$$E = \frac{1}{2} v^2 + u \tag{6}$$

In eq. (6)**,** the first term is the kinetic energy of center-of-mass motion and u is the internal energy per unit mass.

IV) Gibbs equation and thermodynamic parameters

The local equilibrium hypothesis implies that the intensive form of the Gibbs relation can be used:

$$T\,ds = du + P\,dV - \mu_n\,dc_n - \mu_s\,dc_s \tag{7}$$

Where V is specific volume $(V = 1/\rho)$, ***s*** is specific entropy and μ_n, μ_s, $c_n (= \rho_n/\rho\ )$ and $c_s (= \rho_s/\rho\ )$ are the chemical potentials and concentrations of normal and superfluid, respectively.

The specific Gibbs function g is related to the chemical potentials of the two fluids (partial specific Gibbs functions) in the form:

$$g = c_n \mu_n + c_s \mu_s = u - Ts + PV \tag{8}$$

b) Landau's Theory

In the framework of the two-fluid model, there are two velocity fields within the system and hence there is no preferred frame of references for ***He-II***. Different frame of references can be used to define the thermodynamic variables of the whole fluid taking account of the separate motion of the normal and superfluid components; and the relative motion of the two fluids. There is no unique definition of internal energy when there is relative motion. In addition the Gibbs relation appropriate for ***He-II*** can be exp-ressed in different forms depending on the frame of references. Landau uses k_0-frame, which is the rest frame relative to the superfluid fraction. In this frame, total momentum density of the whole fluid is that of normal fluid moving at $(\overrightarrow{v_n} - \overrightarrow{v_s})$ relative to the superfluid:

$$\overrightarrow{J_0} = \rho c_n (\overrightarrow{v_n} - \overrightarrow{v_s}) \tag{9}$$

If we call the total energy density E_0 in k_o-frame, the internal energy per unit mass is defined by:

$$\frac{E_0}{\rho} = \frac{J_0^2}{2\rho^2} + u_L \tag{10}$$

Since the first term on the r.h.s is the kinetic energy of the fluid, u_L is the internal energy when both components are at rest.

Starting from (9) and (10), total momentum and energy can be transferred to the Laboratory frame using a Galilean transformation. Denoting these by quantities $\vec{J}$ and E in the k-frame respectively, we have:

$$\vec{J} = \overrightarrow{J_0} + \rho \overrightarrow{v_s} \tag{11}$$

And

$$\frac{E}{\rho} = \frac{J^2}{2\rho^2} + u_L \tag{12}$$

We now substitute (9) and (11) in (12) and make use of the identity,

$$\frac{1}{2}c_n v_n^2 + \frac{1}{2}c_s v_s^2 = \frac{1}{2}v^2 + \frac{1}{2}c_n c_s(\overrightarrow{v_n} - \overrightarrow{v_s})^2 = \frac{1}{2}v^2 + k_i \quad (13)$$

Where the quantity k_i is called " kinetic energy of internal convection" then we find:

$$\varepsilon = \frac{E}{\rho} = \frac{1}{2}v^2 + u_L \quad (14)$$

Thus, our definition of internal energy (6) is the same as Landau's. Landau defines thermodynamic variables of the fluid in k_0-frame too. He suggests the following Gibbs relation:

$$dE_0 = Td(\rho s) + \varphi d\rho + (\overrightarrow{v_n} - \overrightarrow{v_s}).d\overrightarrow{J_0} \quad (15)$$

Where φ is the Gibbs function per unit mass in superfluid rest frame? From (15) and taking in account the definition of pressure as $P = \left[-\partial(E_0 V)/\partial V\right]_{\rho V,\ \rho s V,\ J_0 V}$ it can be shown that:

$$\rho\varphi = E_0 - T\rho s + P - (\overrightarrow{v_n} - \overrightarrow{v_s}).\overrightarrow{J_0} \quad (16)$$

The Gibbs relation (15) indicates that in Landau's theory momentum density (drift momentum of excitation viewed in k_0-frame) is considered as an independent thermodynamic variable, where its conjugate is relative velocity $(\overrightarrow{v_n} - \overrightarrow{v_s})$.

c) London's Theory

London employs k-frame and expresses total energy per unit mass in the form:

$$\varepsilon = \frac{1}{2}c_n v_n^2 + \frac{1}{2}c_s v_s^2 + u_{\text{Lond.}} \quad (17)$$

In fact he adapts a variational principle to the two-fluid Model and starts with Lagrangian density:

$$L_{\text{Lond.}} = \frac{1}{2}c_n v_n^2 + \frac{1}{2}c_s v_s^2 + u_{\text{Lond.}} \quad (18)$$

Where $u_{\text{Lond.}}$ is the internal energy of the whole fluid? Performing the variations and treating c_n as an independent variable, London derives a Gibbs equation in the form:

$$du_{\text{Lond.}} = Tds - PdV + \frac{1}{2}(\overrightarrow{v_n} - \overrightarrow{v_s})^2 dc_n \qquad (19)$$

Which shows that the quantity $\frac{1}{2}(\overrightarrow{v_n} - \overrightarrow{v_s})^2$ is thermodynamic conjugate of normal fluid concent-ration c_n. If we now use the identity (13) in our definition of total energy (6), we will find that:

$$u_{\text{Lond.}} = u - k_i \qquad (20)$$

Equation (20) indicates that London internal energy is different from u by an amount of k_i, i.e. the kinetic energy of internal convection. This difference appears in the Gibbs function through the relation:

$$g_{\text{Lond.}} = u_{\text{Lond.}} - Ts + PV \qquad (21)$$

Which makes $g_{\text{Lond.}}$ different from g.

d) Lin's Theory

Within the context of a general variational principle, Lin considers liquid ***He-II*** as a single component fluid and proposes the following Lagrangian density (Lin, 1963):

$$L_{\text{Lin.}} = \frac{1}{2}v^2 - u_{\text{Lin.}} \qquad (22)$$

When the identity (13) is used, (22) transforms as:

$$L_{\text{Lin.}} = \frac{1}{2}c_n v_n^2 + \frac{1}{2}c_s v_s^2 - k_i - (u_{\text{Lond.}} - k_i) \qquad (23)$$

Where

$$u_{\text{Lin.}} = u_{\text{Lond.}} - k_i \qquad (24)$$

Equation (23) means that Lin's Lagrangian density is algebrically equal to London's, but is regarded as the difference of kinetic energy of center-of-mass motion and internal energy defined in (24). If we substitute (20) in (24), we find:

$$u_{\mathrm{Lin.}} = u - 2k_i \tag{25}$$

Also with (6) and (25) it can be shown that:

$$\frac{1}{2}v^2 + u_{\mathrm{Lin.}} \neq \varepsilon \tag{26}$$

Which states that the sum of kinetic and internal energy is not equal to total energy in this case.

Starting with Lagrangian density (23) and using ordinary procedure of the variational method, Lin derives a Gibbs relation in the form:

$$du_{\mathrm{Lin.}} = Tds - PdV + (c_n/c_s)d\left[\frac{1}{2}c_s(\overrightarrow{v_n} - \overrightarrow{v_s})^2\right] \tag{27}$$

Indicating that the pairs of c_n/c_s and k_i/c_n are conjugate variables. It should be mentioned that there is an additional constraint in Lin's variational approach, which is associated with vorticity (e.g., see: Jackson, 1978; Geurst, 1979; Geurst 1980).

e) Equivalence of Landau and London Theories

The theories of Landau and London were thought for a long time to be alternative formulations of two-fluid thermohydrodynamics. The following discussion shows that they are equivalent.

We start with the expression for total energy density in both methods. The first is the one presented by Landau:

$$E = \frac{1}{2}\rho v_s^2 + \overrightarrow{v_s}.\overrightarrow{J_0} + E_0 \tag{28}$$

The second is London's:

$$E = \frac{1}{2}\rho_n v_n^2 + \frac{1}{2}\rho_s v_s^2 + \rho u_{\mathrm{Lond.}} \tag{29}$$

If we equate (29) with (28), gives:

$$u_{\mathrm{Lond.}} = \frac{E_0}{\rho} - \frac{1}{2}c_n(\overrightarrow{v_n} - \overrightarrow{v_s})^2 \tag{30}$$

We now differentiate (30) and use (15), after arranging the terms yields:

$$du_{\text{Lond.}} = T\,ds + \left[T\rho s + \rho\varphi + (\overrightarrow{v_n} - \overrightarrow{v_s}).\overrightarrow{J_0} - E_0\right]\frac{d\rho}{\rho^2} + \frac{1}{2}(\overrightarrow{v_n} - \overrightarrow{v_s})^2\,dc_n \qquad (31)$$

Because of (16) the bracketed expression in (31) is equal to pressure P. Using this together with the identity $(dV = -d\rho/\rho^2)$, equation (31) reads:

$$T\,ds = du_{\text{Lond.}} + PdV - \frac{1}{2}(\overrightarrow{v_n} - \overrightarrow{v_s})^2\,dc_n \qquad (32)$$

Therefore, we arrived at the same Gibbs relation (19) derived by London. It should be mentioned that Jackson (Jackson, 1978**)** and Geurest (Geurst, 1979) had also reached the same conclusion but using different approaches.

Finally equation (30) can be used to establish a relationship between the Gibbs functions in these methods. In order to show this we make use of (16) and the thermodynamic relation (21) in (30), we obtain:

$$g_{\text{Lond.}} = \varphi + \frac{1}{2}c_n(\overrightarrow{v_n} - \overrightarrow{v_s})^2 \qquad (33)$$

Having shown the eqivalence of London and Landau's theories, we shall in future make detailed comparisons between the present work and London's theory.

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