Topological Lumpiness and Topological Extreme Amenability

A.H. Riazi*

Faculty of Mathematics and computer science, Amirkabir University of Technology, Tehran, Islamic Republic of Iran

Abstract

In this paper we give some characterizations of topological extreme amenability. Also we answer a question raised by Ling [5]. In particular we prove that if $T$ is a Borel subset of a locally compact semigroup $S$ such that $M(S)^*$ has a multiplicative topological left invariant mean then $T$ is topological left lumpy if and only if there is a multiplicative topological left invariant mean $M$ on $M(S)^*$ such that $M(\chi_T)=1$, where $\chi_T$ is the characteristic functional of $T$. Consequently if $T$ is a topological left lumpy locally compact Borel subsemigroup of a locally compact semigroup $S$, then $T$ is extremely topological left amenable if and only if $S$ is.

Keywords: Mean; Topological extreme amenability; Left lumpy

1. Introduction

Let $S$ be a locally compact (Hausdorff) semigroup. Let $C_0(S)$ be the subalgebra of $CB(S)$ consisting of functions which vanish at infinity. Let $M(S)^*$ be the Banach space of all bounded regular Borel (signed) measures on $S$ with total variation norm.

Let $M_0(S) = \{ \mu \in M(S) : \mu \geq 0 \text{ and } \|\mu\| = 1 \}$ be the set of all probability measures in $M(S)$. It is known that $M(S)^* = C_0(S)^*$ via the correspondence $\mu \mapsto \overline{\mu}$ where $\overline{\mu}(f) = \int f \, d\mu$ for any $f$ in $C_0(S)$ [4, § 14]. Consider the continuous dual $M(S)^*$ of $M(S)$. Denote by $1$ the element $1$ in $M(S)^*$ such that such that $1(\mu) = \mu(S)$ for any $\mu$ in $M(S)$.

Also if $T$ is a Borel subset of $S$ we define the Borel characteristic functional $\chi_T$ of $T$ in $M(S)^*$ by $\chi_T(\mu) = \mu(T)$, $\mu \in M(S)$. An element $M$ in $M(S)^{**}$ is called a mean on $M(S)$ if $M(1)=1$ and $M(F) \geq 0$, whenever $F \geq 0$. An equivalent definition for a mean is that

$$\inf\{F(\mu) : \mu \in M_+(S)\} \leq M(F) \leq \sup\{F(\mu) : \mu \in M_+(S)\}$$

for any $F$ in $M(S)^*$. We also note that $M \in M(S)^{**}$ is a mean if and only if $\|M\|=M(1)=1$. Each probability measure $\mu$ in $M_0(S)$ is a mean on $M(S)^*$ if we put $\mu(F) = F(\mu)$, for any $F$ in $M(S)^*$. An application of Hahn-Banach separation theorem shows that $M_0(S)$ is weak* dense in the set of all means on $M(S)^*$.

Under pointwise operations and supremum norm $C_0(S)$ becomes a Banach algebra. Arens product can thus be defined in $C_0(S)^{**}$. In particular, we have the

*E-mail: riazi@aut.ac.ir
following defining formulas for any \( f, g \) in \( C_a(S) \), \( m \) in \( C_a(S) \)' and \( \theta, \varphi \) in \( C_a(S)' \).

\[
(m \circ f)(g) = m(fg)
\]

\[
(\varphi \circ m)(f) = \varphi(m \circ f)
\]

\[
(\theta \circ \varphi)(m) = \theta(\varphi(m))
\]

This product induces a multiplication in \( M(S)' \) via the identification \( M(S) = C_a(S)' \). For \( F, G \) in \( M(S)' \) we denote the multiplication of \( F \) and \( G \) by \( F \times G \). In [5] it is shown that \( F \times G \) is defined via the following three steps:

(i) For any \( \mu \in M(S) \) and \( f \in C_o(S) \), \( \mu_f \in M(S) \) is defined by

\[
\int g d \mu_f = \int g f d \mu \quad \text{for all } g \in C_o(S)
\]

(ii) For any \( \mu \in M(S) \) and \( G \in M(S)' \), \( G \times \mu \in M(S) \) is defined by

\[
\int f d (G \times \mu) = G(\mu_f) \quad \text{for all } f \in C_o(S)
\]

(iii) For any \( F, G \in M(S)' \), \( F \times G \in M(S)' \) is defined by

\[
(F \times G)(\mu) = F(\mu \circ G) \quad \text{for all } \mu \in M(S).
\]

Then \( M(S)' \) becomes a commutative Banach algebra with identity [5, theorem 1.2.3].

For each \( \mu \) in \( M(S) \) define an operator \( l_{\mu} : M(S)' \to M(S)' \) by \( l_{\mu}(F)(\nu) = F(\mu \circ \nu) \), \( \nu \in M(S) \). We denote \( l_{\nu}F \) by \( \mu \circ F \). A mean \( \mu \) on \( M(S)' \) is called topological left invariant (TLIM) if \( M(\mu \circ F) = M(F) \) for all \( F \in M(S)' \) and for all \( \mu \in M_o(S) \). A topological left invariant mean \( \mu \) on \( M(S)' \) is called a multiplicative topological left invariant mean (MTLIM) if

\[
M(F \times G) = M(F)M(G) \quad \text{for all } \mu \in M(S)'
\]

If there is a MTLIM on \( M(S)' \) we say that \( S \) is extremely topological left amenable (ETLA). For results concerning ETLA semigroups see [5] and [6].

2. Main Results

Note that for elements \( M, N \) in \( M(S)' \) their Arens product is denoted by \( M \circ N \) and is defined by

\[
(M \circ N)(F) = M(N(F)) \quad \text{for all } F \in M(S)'
\]

where \( N_L : M(S)' \to M(S) \) is defined by \( N_L(\mu) = N(\mu \circ F) \), \( \mu \in M(S) \). See [1] and [2].

First we prove two Lemmas.

**Lemma 2.1.** Suppose \( M \) and \( N \) are functionals in \( M(S)' \).

(i) If \( M \) and \( N \) are means on \( M(S)' \) then \( M \circ N \) is also a mean on \( M(S)' \).

(ii) For each \( \mu \in M(S) \) and each \( F \in M(S)' \) we have

\[
M_L(\mu \circ F) = \mu \times M_L(F)
\]

(iii) If \( M \) is a topological left invariant mean, then \( M \circ N \) is also topological left invariant.

**Proof.** (i) It is easy to see that for each \( \mu \in M(S) \) and \( 1 \in M(S)' \) we have \( \mu \circ 1 = 1(\mu) \), hence

\[
(M \circ N)(1)(1) = M(N(1)) = M(1) = 1.
\]

Also \( \|M \circ N\| \leq \|M\| \|N\| \), hence \( M \circ N \) is a mean on \( M(S)' \).

(ii) For each \( \nu \in M(S) \)

\[
M_L(\mu \circ F)(\nu) = M(\nu \circ (\mu \circ F))
\]

\[
= M((\mu \circ \nu) \circ F)
\]

\[
= M_L(F)(\mu \circ \nu)
\]

\[
= (\mu \times M_L(F)) (\nu)
\]

Thus \( M_L(\mu \circ F) = \mu \times M_L(F) \).

(iii) Suppose \( M \) is topological left invariant, then for each \( \mu \in M_o(S) \) and \( F \in M(S)' \) we have

\[
(M \circ N)(\mu \circ F) = M(N_L(\mu \circ F))
\]

\[
= M(\mu \circ N_L(F))
\]

\[
= M(N_L(F))
\]

\[
= (M \circ N)(F)
\]

where we have used (ii) in the second equality. So \( M \circ N \) is topological left invariant, whenever \( M \) is.

**Lemma 2.2.** For each \( s \in S \), \( F \in M(S)' \) and \( \mu \in M(S)' \) we have

(i) \( (\varepsilon_s)_L(F) = F \circ \varepsilon_s \)

(ii) \( (M \circ \varepsilon_s)(F) = M(F \circ \varepsilon_s) \)

(iii) \( (\varepsilon_s)(F \times G) = (F \times G) \circ \varepsilon_s \)
\begin{align*}
\text{(iv) If } M \text{ is multiplicative, then } M \odot e_s \text{ is so.}
\end{align*}

**Proof.** (i) \[
(e_s)\cdot (F)(\mu) = e_s (\mu \odot F) = (\mu \odot F)(e_s)
\]

hence \((e_s)F = F \odot e_s\).

(ii) \((M \odot e_s)(F) = M ((e_s)\cdot (F)) = M (F \odot e_s)\)

where we have used (i) in the second equality.

(iii) the first equality follows from (i) and the second follows from [5, p.27]

(iv) Suppose \(M \in M (S)^\ast\) is multiplicative. Then:

\[
(M \odot e_s)(F \times G) = M ((e_s)\cdot (F \times G))
\]

\[
= M((F \odot e_s) \times (G \circ e_s))
\]

\[
= (M \odot e_s)(F)(G).
\]

where we have used (iii) in the second equality and (ii) in the last equality.

The following theorem is an extension of [5, theorem 3.2.1]. But first we need a definition.

**Definition 2.3.** Let \(S\) be a locally compact semigroup and \(T\) a Borel subset of \(S\). \(T\) is said to be topological left lumpy in \(S\) if it satisfies the following condition.

\((\text{TLL})\) For each \(\delta > 0\) and \(\mu \in M_\delta (S)\) with compact support, there exists \(a \in S\) such that \(\mu \cdot e_s (T) > 1 - \delta\).

It is known that (TLL) is equivalent to each of the following conditions:

\((\text{TLL}1)\) For any \(\delta > 0\) and \(\nu \in M_\delta (S)\) with compact support, there exists \(\mu \in M_\delta (S)\) with compact support such that

\[\mu(T) > 1 - \delta\] \(\quad \text{and} \quad (\nu \ast \mu)(T) > 1 - \delta\]

\((\text{TLL}2)\) For any \(\delta > 0\) and \(\nu \in M_\delta (S)\) with compact support, there exists \(\mu \in M_\delta (S)\) with compact support such that

\[\mu(T) > 1 - \delta\] \(\quad \text{and} \quad (\nu \ast \mu)(T) > 1 - \delta\]

See [7, pp. 571-574 and addendum on p.585] for more details. See also [3].

**Theorem 2.4.** Suppose \(T\) is a Borel subset of a locally compact semigroup \(S\). Suppose \(M(S)^\ast\) has a MTLIM then the following statements are equivalent:

(i) \(T\) is topological left lumpy.

(ii) There is a MTLIM on \(M(S)^\ast\) such that \(M(\chi_T) = 1\).

**Proof.** (i) \(\Rightarrow\) (ii). Let \(F = \{ \mu_1, \ldots, \mu_k \}\) be a finite subset of \(M_\delta (S)\) (The elements in \(M_\delta (S)\) with compact support). For each \(\epsilon > 0\) there is \(s = s(F, \epsilon) \in S\) such that

\[
\frac{\mu_1 + \ldots + \mu_k \cdot e_s (T)}{k} > 1 - \frac{\epsilon}{2}
\]

(by TLL), in particular \(\mu_i \cdot e(T) > 1 - \epsilon, 1 \leq i \leq k\).

Let \(F\) be the collection of all finite (nonempty) subsets of \(M_\delta (S)\). Put \(\Delta = F \times (0, \infty)\) and order \(\Delta\) as follows:

\[\langle F_i, \alpha_i \rangle \geq \langle F_j, \alpha_j \rangle \Leftrightarrow F_i \subseteq F_j \quad \text{and} \quad \alpha_i < \alpha_j\]

By above discussion there is a net \(\{ s_\alpha \}\) of elements of \(S\) with \(\gamma = (F, \alpha) \in \Delta\). Since the set of means on \(M(S)^\ast\) is weak* compact the net \(\{ e_s \}\) has a subnet \(\{ e_{s_\alpha} \}\) which converges weak* to a mean \(N\) on \(M(S)^\ast\) and also for each \(\mu \in M_\delta (S)\) we have

\[N(\mu \cdot \chi_T) = \lim_{\beta} (\mu \cdot \chi_T)(e_{s_\beta}) = \lim_{\beta} (\mu \ast e_{s_\beta})(T) = 1\] (1)

Now suppose \(M\) is MTLIM on \(M(S)^\ast\). Since the Arens product is weak* continuous in the second variable and using Lemma 2.2 (iv) we conclude that \(M \odot N\) is multiplicative. Also since \(M\) and \(N\) are means and \(M\) is topological left invariant, by using Lemma 2.1 we conclude that \(M \odot N\) is a MTLIM on \(M(S)^\ast\). Now since \(M_\delta (S)\) is weak* dense in the set of means on \(M(S)^\ast\), by using (1) we obtain

\[(M \odot N)(\chi_T) = 1\]

(iii) \(\Rightarrow\) (i). Suppose \(M\) is a MTLIM on \(M(S)^\ast\) such that \(M(\chi_T) = 1\). If \(\{ \mu_\alpha \}\) is a net in \(M_\delta (S)\) which converges to \(M\) in weak* topology, then for each \(\nu \in M_\delta (S)\) we have

\[
\omega^* \lim_{\alpha} (\nu \ast \mu_\alpha - \mu_\alpha) = \nu \odot M - M = 0
\]
Since \( \lim_{n \to \infty} \mu_n(T) = M(\chi_T) = 1 \) and for each \( v \in M_0^\alpha(S) \)
\[
(v \ast \mu_\alpha)(\chi_T) = \chi_T(v \ast \mu_\alpha) = (v \ast \mu_\alpha)(T)
\]
We conclude that for each \( v \in M_0^\alpha(S) \),
\[
\lim_{n \to \infty} (v \ast \mu_\alpha)(T) = 1.
\]
So for each \( v \in M_0^\alpha(S) \) and each \( \delta > 0 \) there is \( \mu = \mu_\alpha \in M_0^\alpha(S) \) such that \( (v \ast \mu)(T) > 1 - \delta \).
Therefore by (TLL), \( T \) is topological left lumpy.

Let \( S \) be a locally compact semigroup and \( T \) a locally compact Borel subsemigroup of \( S \). We recall some of the constructions in [8] and [9].

Let \( B(S) \) be the \( \sigma \)-algebra of Borel subsets of \( S \).

1. Let \( \mu \in M(S) \), then \( \mu_T \) is the restriction of \( \mu \) to \( B(T) \) and \( \mu_T \in M(T) \).
2. Let \( F \in M(T) \), then \( F' \in M(S) \) is well-defined by \( F' = \mu(T) \) for any \( \mu \in M(S) \).
3. Let \( M \in M(S) \) is well-defined by \( M_0(F) = M(F') \) for any \( F \in M(T) \).

Lemma 2.5. (a) \( F \times \mu_T = (F' \times \mu)_T \) for \( F \in M(T) \) and \( \mu \in M(S) \).
(b) \( (F \times G)' = F' \times G' \) for any \( F,G \in M(T) \).

Proof. (a) For any \( A \in B(T) \) we denote \( \xi_A \) for characteristic function of \( A \) in \( T \) and \( \chi_T \) for characteristic function of \( A \) in \( S \).
\[
(F \times \mu_T)(A) = \int \xi_A d(F \times \mu_T) = F(\mu_T) \chi_T = (F'(\mu_T)) = F'(\mu_T)
\]
\[
= \int \chi_T d(F' \times \mu) = (F' \times \mu)(A)
\]
\[
(\text{b) For any } \mu \in M(S) \text{ by (a) we have}
(F' \times G')_\mu = F'(G' \times \mu) = F((G' \times \mu)_T)
\]
\[
= F(G \times \mu_T)
\]
\[
= (F \times G)(\mu_T) = (F \times G)'(\mu)
\]

We now state the main result of this paper which answers a question raised by J.M. Ling, See [5, then P. 51].

**Theorem 2.6.** Let \( T \) be a topological left lumpy locally compact Borel subsemigroup of a locally compact semigroup \( S \). Then \( T \) is ETLA if and only if \( S \) is ETLA.

**Proof.** Suppose \( T \) is ETLA, then by [5, Theorem 3.2.3] \( S \) is ETLA.

Conversely suppose \( S \) is ETLA, by theorem 2.4 there is a MTLIM on \( M(S) \) such that \( M(\chi_T) = 1 \). Then \( M_0(F) = M(F') \) defines a TLIM on \( M(T) \), we show that \( M_0 \) is multiplicative
\[
M_0(F \times G) = M((F \times G)) = M(F' \times G') = M(F')M(G') = M_0(F)M_0(G)
\]

**Corollary 2.7.** Let \( T \) be a left ideal of a locally compact semigroup \( S \). Then \( M(T) \) has a MTLIM if and only if \( M(S) \) has a MTLIM.

**Proof.** It suffices to show that every left ideal is topological left lumpy. Let \( t \in T \). If \( K \subseteq S \) is compact then \( Kt \subseteq ST \subseteq T \). Consider the Dirac measure \( \e \) at \( t \).
For any \( \mu \in M_0(S) \) with \( \mu(K) = 1 \), we have
\[
\mu \ast \e(T) = \int \chi_T(x) d\mu(x) = \left[ \int \xi_T(x) d\mu(x) \right] = \mu(K) = 1, \text{ hence } T \text{ is topological left lumpy}.
\]

**References**