THE PC-TAU METHOD ON THE SOLUTION OF A MODEL OF AN ISOLATED COSMIC STRING

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Abstract

In this paper, we adapt the operational Tau Method for personal computers and apply it to a system of two nonlinear second order ordinary differential equations which are related to general relativity. The interesting behaviour this problem exhibits in its numerical treatment is discussed. In this problem, we try to use the Tau perturbation term to locate the correct solution of this nonlinear problem. The results of this experiment and those which are going to be given in another paper show that in some problems the effect of nonlinearity could prevent the correct solution from being located properly.

1. Introduction

We first briefly introduce the operational formulation of the Tau Method and then as in [3] modify it to apply to the following system of nonlinear Odes [2,8]:

$$\begin{cases} x\dfrac{d}{dx}\left[\dfrac{1}{x}\dfrac{dz}{dx}\right]=y^2 z \\ x\dfrac{d}{dx}\left[x\dfrac{dy}{dx}\right]=y\,[4x^2(y^2-1)+z^2] \end{cases}$$

with conditions:

$z(0)=1,\quad z\to 0\ \ as\ x\to\infty$
$y(0)=0,\quad y\to 1\ \ as\ x\to\infty$

Let us consider the general Ode

$$Dy(x)=f(x)\quad x\in[-1,\,1] \tag{1}$$

$$B_i(y)=\gamma_i \tag{2}$$

γ_i *constant*, $i=0,1,\ldots,\nu-1$

where ν is the order of the Ode and

$$D:=\sum_{r=0}^{\nu} p_r(x)\frac{d^r}{dx^r} \tag{3}$$

$$p_r(x)=\sum_{j=0}^{\alpha_r} p_{rj}x^j=\underline{\underline{p}}_r\,\underline{X} \tag{4}$$

and α_r is the degree of $p_r(x)$ and

$$\underline{\underline{p}}_r=(p_{r0},p_{r1},\ldots,p_{r\alpha_r},0,0,\ldots),$$
$$\underline{X}=(1,x,x^2,\ldots)^T.$$

The operational Tau [8] is mainly concerned with three elementary matrices η, μ or ι which are used to reduce differential problems to linear algebraic problems as follows:

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$$\eta=\begin{pmatrix} 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & \dots \\ 0 & 2 & 0 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix},$$

$$\mu=\begin{pmatrix} 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix},$$

$$\iota=\begin{pmatrix} 0 & 1 & 0 & \dots \\ 0 & 0 & \frac{1}{2} & \dots \\ 0 & 0 & 0 & \frac{1}{3} \\ \dots & \dots & \dots & \dots \end{pmatrix},$$

The effect of differentiation, shifting or integration on the coefficient vector $\underline{\underline{a}}_n=(a_0,a_1,\dots,a_n,0,0,\dots)$ of a polynomial $y_n(x)=\underline{\underline{a}}_n\underline{X}$ is the same as that of post-multiplication of $\underline{\underline{a}}_n$ by the matrices η, μ, or ι respectively. So:

$$\frac{dy_n(x)}{dx}=\underline{\underline{a}}_n\eta\underline{X}, \quad xy_n(x)=\underline{\underline{a}}_n\mu\underline{X},$$

$$\int y_n(x)\,dx=\underline{\underline{a}}_n\,\iota\underline{X}.$$

Following Theorem 1 [8], we have

$$Dy(x)=\underline{\underline{a}}\,\Pi\,\underline{X}\equiv\underline{\underline{a}}\,\widehat{\Pi}\,\underline{V} \tag{5}$$

$$\Pi=\sum_{i=0}^{\nu}\eta^i p_i(\mu)\ ,\ \widehat{\Pi}=V\,\Pi\,V^{-1} \tag{6}$$

where $y(x):=\underline{\underline{a}}\,\underline{V}$, $\underline{\underline{a}}:=(a_0,a_1,\dots)$.

We introduce the vector

$$\underline{\underline{\gamma}}^0=(\gamma_1,\gamma_2,\dots,\gamma_\nu,0,0,\dots)$$

and the matrix $B=(b_{ij})$ such that $b_{ij}=B_i(v_j)$ for $i=1,2,\dots,\nu$ and $j\in N$. Then the supplementary conditions in the problem take the form

$$\underline{\underline{a}}\,B=\underline{\underline{\gamma}}^0 \tag{7}$$

Let $\hat{\underline{\pi}}_i$ denote the i^{th} column of $\widehat{\underline{\Pi}}$ and let $\hat{\underline{f}}=\underline{f}\,V^{-1}$. Then the coefficient $\underline{\underline{a}}$ of the exact solutions $y=\underline{\underline{a}}\,\underline{V}$ of that problem satisfy the following infinite algebraic system:

$$\begin{aligned} &\underline{\underline{a}}\hat{\underline{\pi}}_i=\hat{f}_i \quad i=0,1,\dots,d_f \\ &\underline{\underline{a}}\hat{\underline{\pi}}_i=0; \quad i\geq d_f+1 \\ &\underline{\underline{a}}\underline{B}_j=\gamma_{ji} \quad j=1,2,\dots,\nu \end{aligned} \tag{8}$$

where d_f = degree of $f(x)$ and $\underline{B}_i$ is the ith column vector of the matrix B. Setting

$$G=(\underline{B}_1,\underline{B}_2,\dots,\underline{B}_\nu,,\hat{\underline{\pi}}_0,\hat{\underline{\pi}}_1\dots)$$

and

$$\underline{\underline{\gamma}}=(\underline{\underline{\gamma}}^0;\hat{\underline{\underline{f}}}),$$

we can write instead of (8):

$$\underline{\underline{a}}\;G=\underline{\underline{\gamma}}. \tag{9}$$

Definition 1. The polynomial $y_n(x)=\underline{\underline{a}}_n\,\underline{V}$ will be called an approximate solution of (9) if the vector $\underline{\underline{a}}_n$ is the solution of the linear algebraic system of equations

$$\underline{\underline{a}}_n\;G_n=\underline{\underline{\gamma}} \tag{10}$$

where G_n is the matrix defined by a restriction of G to its first n+1 rows and columns. To apply it to systems of Odes, we follow the same procedure as in [3,4].

2. The Main Problem

We actually want to solve the following system:

$$\begin{cases} xz''-z'=xy^2z \\ x^2y''+xy'=yz^2+4x^2y^3-4x^2y \end{cases}$$

with conditions:

$$z(0)=1,\quad z\to 0\ \ as\ \ x\to\infty$$
$$y(0)=0,\quad y\to 1\ \ as\ \ x\to\infty$$

After linearizing this problem by Newton's method [3] we use the following two strategies.

First Strategy. For sufficiently large x we solve that problem over [0, x] as a boundary value problem and use the Segmented Tau Method [7,6].

For example, in this problem if we take x = 10, using five segments we find the accurate solution; Figure 1.

Remark (1). In [2] there is no numerical result available unless a graph of the solutions over [0, 3] is given. Hence, we decided to give graphs of our solutions and also check their numerical accuracies with the Tau perturbation terms $h_n(x)$. In the final result, the maximum absolute value of the perturbation term over [0, 10] is about 1.0×10^{-5}. For $x\geq1.0$ that value is about 1.0×10^{-10}.

Second Strategy. We consider the following strategy which can be useful for the problems in which we may have to take x too large. Let us consider that problem over a small interval [0,3.] with conditions:

$$z_{n+1}(0)=1, \quad z_{n+1}(3.)=\alpha$$
$$y_{n+1}(0)=0, \quad y_{n+1}(3.)\approx 1.$$

where α is to be found in such a way that the correct solution over that interval is obtained.

The reason that we consider

$$y_{n+1}(3)\approx 1$$

is because we know it tends to value 1 approximately as $x\to 3$.

Generally, choosing this interval and value of α is not so easy, but depending on the problem, we can usually take advantage of some theoretical results concerning its solution.

In this example we take $\alpha = 0.(0.01)0.3$ and $N = 9$ for the degree of the Tau approximants, Legendre basis [5], and initial guesses

$$z_0 = x - x^2, \; y_0 = x.$$

Then a spectrum of solutions with a maximum value of

$$[\text{abs}(hn(x))] \approx 0.01$$

is obtained; see Figure 2.

To increase the accuracy of the solution and thus probably make that behaviour disappear, we apply the Segmented Tau Method over [0, 3] and the same happens again, but this time with a maximum value of

$$[\text{abs}(hn(x))] \approx 0.5\times10^{-4}.$$

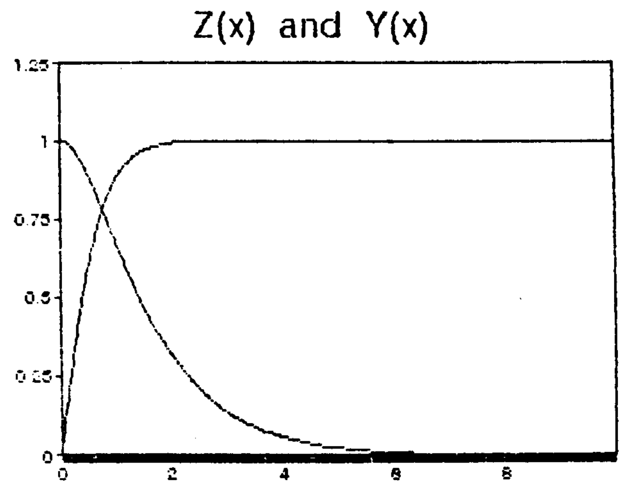

Figure 1. The $y(x)$ and $z(x)$ solution (first strategy)

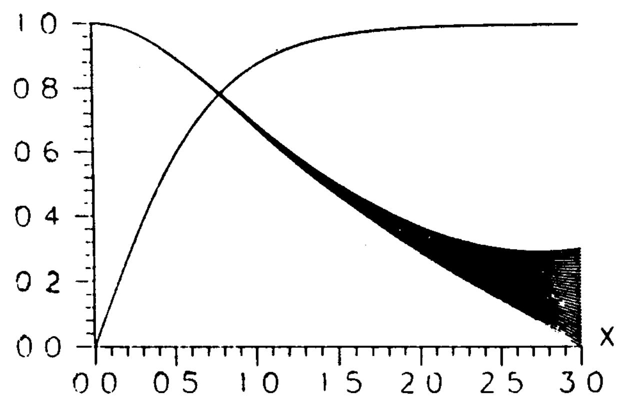

Figure 2. The spectrum (second strategy)

Hence, the effect of nonlinearity causes the perturbation terms, even with reasonable accuracies, not to be so effective in locating the correct solution in that spectrum. For more information concerning the effect of perturbation terms in locating the correct path of a solution see [9].

As we see in Figure 2, all these solutions are overlapping approximately over a relatively large segment of that interval and they behave roughly the same over the rest of it, and hence result in the same Hn(x). Therefore, we apply the following important remark:

Remark (2). When as in this problem, we have a com-

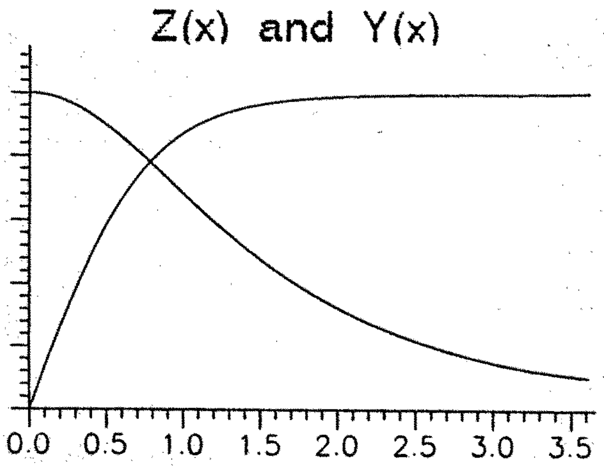

Figure 3. *y(x)* and *z(x)* solutions (second strategy)

mon piece, made by the solutions in the spectrum, which starts from one of the end points, the appropriate information of the solution (such as the values of the solution and its derivatives) at a point in that segment can be used to solve the problem over the rest of the interval as an initial value problem (Ivp).

Clearly, the Tau Method is one which provides such information with reasonable accuracy [1], by controlling the defect term in the Ode. Hence, using the computed approximate values of $y(x)$, $y'(x)$, $z(x)$, and $z'(x)$ at $x = .8$, and taking the steplength $h = 0.5$, the correct path of the solution in that spectrum is located; see Figure 3 which shows the solution over [0, 3.6].

3. Final Remarks

1- It is possible to apply the adaptive Tau Method to systems of Odes with normal sizes, using personal computers, provided that some necessary steps be taken to avoid limitations of personal computers. It should be noted that compared with some similar computation, using CDC main frame [4], here we had to apply some special treatment to be able to use a PC to find the correct solution. We will present some of those limitations in detail in another paper.

2- This sort of approach that we used to solve this problem can be followed by some other Ode numerical methods.

3- The second strategy is very useful to avoid unnecessary difficulties which may occur when using personal computers.

4- The Tau Method, along with the solution, usually provides good approximate values for derivatives. As in this problem it proved to be very essential in computing the solution over the rest of the interval.

References

1. El-Daou, M.K., Namasivayam, S. and Ortiz, E.L. Differential equations with piecewise approximate coefficients: Discrete and continuous estimation for initial and boundary value problems. *Computer Math. Applic.*, **24**, (4), 33-47, 1992.
2. Garfinkle, D. General relativistic strings. *Phys Rev. D*, **32**, (6), September 1985.
3. Hosseini Ali Abadi, M. and Ortiz, E.L. A Tau Method based on non-uniform space-time elements for the numerical simulation of solutions. *Computer Math. Applic.*, **22**, (9), 7-19, 1991.
4. Hosseini Ali Abadi, M. Ph.D Thesis, Imperial College, (1988).
5. Lanczos, C. Legendre vs. Chebyshev polynomials. *Topics in numerical analysis*, (ed. J.J.H. Miller), pp. 191-201. Academic Press, New York, (1973).
6. Onumanyi, P. and Ortiz, E.L. Numerical solution of stiff and singularly perturbed boundary value problems with a segmented-adaptive formulation of the Tau Method. *Math. Comput.*, **43**, 189-203, (1984).
7. Ortiz, E.L. Step by step Tau Method, piecewise polynomial approximations. *Comput. Math. Appl.*, **1**, 381-392, (1975).
8. Ortiz, E.L. and Samara, H. An operational approach to the Tau Method for the numerical solution of nonlinear differential equations. *Computing*, **27**, 15-25, (1981).
9. Hosseini Ali Abadi, M. *et al.* The Tau perturbation term in solving boundary value problems over semi-infinite intervals. (In press).