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AN INVERSE PROBLEM WITH UNKNOWN RADIATION TERM

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Abstract

In this paper, we consider an inverse problem of linear heat equation with nonlinear boundary condition. We identify the temperature and the unknown radiation term from an overspecified condition on the boundary.

Introduction

This paper considers the determination of an unknown function $p(u)$ from a linear heat equation which is defined on [0, 1], and a function $u(x, t)$ satisfying

$$\partial_t u(x,t) = \partial_{xx} u(x,t), \quad 0<x<1, \quad 0<t<T, \quad (1)$$

$$u(x,0) = f(x), \quad 0<x<1, \quad (2)$$

$$u(0,t) = g(t), \quad 0<t<T, \quad (3)$$

$$\partial_x u(1,t) - p(u(1,t)) = h(t), \quad 0<t<T, \quad (4)$$

and the overspecified condition

$$u(1,t) = \phi(t), \quad 0<t<T, \quad (5)$$

where T is a given constant and $f(x)$, $g(t)$, h(t) and $\phi(t)$ are continuous functions.

If the function $p(u)$ is given, then there may be no solution for the problem (1) - (5). On the other hand, when $p(u)$ is known *a priori*, then under certain conditions there may exist a unique solution $u(x, t)$ for the problem (1) - (4), but this solution may not satisfy the overspecified condition (5). In this case, we say that the pair of function (u, p) provides a solution to the inverse problem (1) - (5).

In this article, we will show that under certain conditions on data f, g, h and ϕ there exists a unique solution pair (u, p) to the inverse problem (1) - (5).

This linear heat equation with nonlinear boundary condition was investigated previously in some detail [1-8]. Hence, we may assume that under reasonable conditions on the data a solution of (1) - (4) exists and is unique.

Equation (1) describes the evaluation of the temperature in a homogeneous rod of constant conductivity. Hence, we may think of this problem as the problem of determining the unknown radiation term of a rod. The functions g and h are known heat flux at position $x = 0$ and $x = 1$ respectively, while f measures the initial temperature in the rod.

The plan of this paper is as follows. In the following section, we will consider the existence and uniqueness solution of the inverse problem. In the final section, we compare the analytical solution of the problem (1) - (5) with some experimental results.

Keywords: Heat equation; Inverse problem; Overspecified condition; Radiation terms

Determination of a Unique Solution for the Inverse Problem (1) - (5)

To solve the inverse problem (1) - (5) let us consider the following auxiliary problem

$$\partial_t u(x,t) = \partial_{xx} u(x,t), \quad 0<x<1, \quad 0<t<T, \quad (6)$$

$$u(x,0) = f(x), \quad 0<x<1, \quad (7)$$

$$u(0,t) = g(t), \quad 0<t<T, \quad (8)$$

$$u(1,t) = \phi(t), \quad 0<t<T. \quad (9)$$

For any piecewise-continuous functions f, g and ϕ this problem has a unique solution [1]

$$u(x,t)=\int_0^1 \{\theta(x-\xi,t)+\theta(x+\xi,t)\} f(\xi)\,d\xi$$
$$-2\int_0^t \frac{\partial\theta}{\partial x}(x,t-\tau)\,g(\tau)\,d\tau$$
$$+2\int_0^t \frac{\partial\theta}{\partial x}(x-1,t-\tau)\,\phi(\tau)\,d\tau. \tag{10}$$

where

$$\theta(x,t)=\sum_{m=-\infty}^{\infty} k(x+2m,t)$$

and

$$k(x,t)=\frac{1}{\sqrt{4\pi t}}\exp\left(-\frac{x^2}{4t}\right), \quad t>0.$$

Differentiating with respect to x, we obtain

$$u_x(x,t)=\int_0^1 \{\frac{\partial\theta}{\partial x}(x-\xi,t)+\frac{\partial\theta}{\partial x}(x+\xi,t)\} f(\xi)\,d\xi$$
$$-2\int_0^t \frac{\partial^2\theta}{\partial x^2}(x,t-\tau)\,g(\tau)\,d\tau$$
$$+2\int_0^t \frac{\partial^2\theta}{\partial x^2}(x-1,t-\tau)\,\phi(\tau)\,d\tau. \tag{11}$$

Using the properties of $\theta(x, t)$ function

$$\frac{\partial\theta}{\partial x}(x-\xi,t)=-\frac{\partial\theta}{\partial\xi}(x-\xi,t),$$
$$\frac{\partial\theta}{\partial x}(x+\xi,t)=\frac{\partial\theta}{\partial\xi}(x-\xi,t),$$
$$\frac{\partial^2\theta}{\partial x^2}=\frac{\partial\theta}{\partial t}=-\frac{\partial\theta}{\partial\tau},$$
$$\lim_{\tau\to t}\theta(x,t-\tau)=0, \quad 0<x<1,$$
$$\theta(x-1,t)=\theta(x+1,t)$$

and integrating by parts yields

$$u_x(x,t)=\int_0^1 \{\theta(x-\xi,t)+\theta(x+\xi,t)\} f'(\xi)\,d\xi$$
$$-2\int_0^t \theta(x,t-\tau)g'(\tau)\,d\tau+2\int_0^t \theta(x-1,t-\tau)\,\phi'(\tau)\,d\tau. \tag{12}$$

From $\theta(1 - \xi, t) = \theta(1 + \xi, t)$, (4) and (12) we obtain

$$p(\phi(t))=2\int_0^1 \theta(\xi,t)f'(\xi)\,d\xi$$
$$-2\int_0^t \theta(1,t-\tau)\,g'(\tau)\,d\tau$$
$$+2\int_0^t \theta(0,t-\tau)\,\phi'(\tau)\,d\tau-h(t). \tag{13}$$

If $s = \phi(t)$ is an invertible function, then we find

$$p(s)=\int_0^1 \theta(\xi+1,\phi^{-1}(s))f'(\xi)\,d\xi$$
$$-2\int_0^{\phi^{-1}(s)} \theta(1,\phi^{-1}(s)-\tau)\,g'(\tau)\,d\tau$$
$$+2\int_0^{\phi^{-1}(s)} \theta(0,\phi^{-1}(s)-\tau)\,\phi'(\tau)\,d\tau-h(\phi^{-1}(s)). \tag{14}$$

Now we can summarize the above results in the following statement.

Theorem 2.1 For any piecewise - continuous function h and any continuously differentiable functions f, g and ϕ there is a unique solution pair (u, p), where u and p can be represented in the form (10) and (14), respectively, for the inverse problem (1) - (5).

Numerical Procedure

In this section, we compare the analytical solution

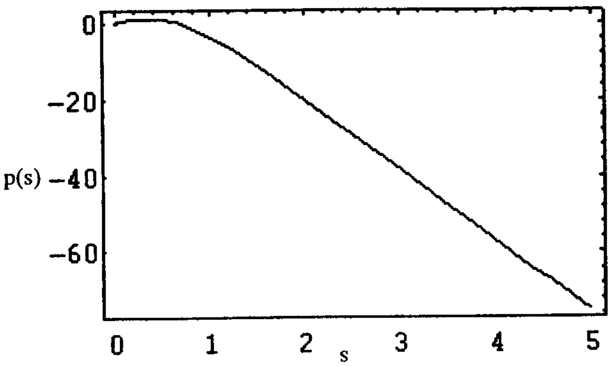

Figure 1. The graph of the analytical solution $p(s)$

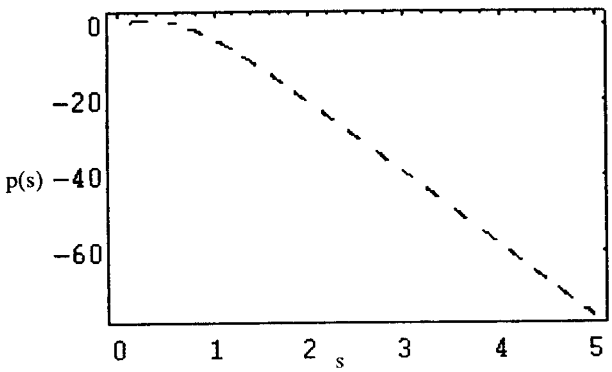

Figure 2. The graph of the numerical solution $p(s)$ by using C.N. method

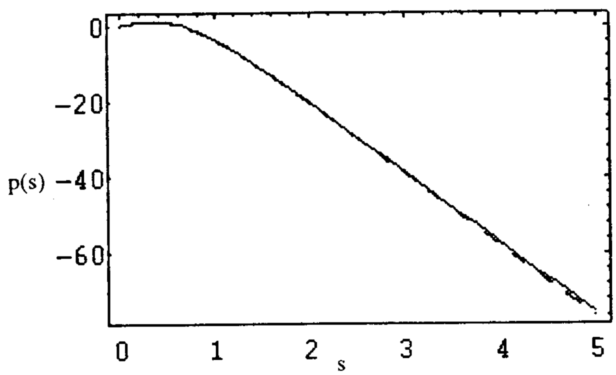

Figure 3. The graph of the analytical and numerical solution of $p(s)$

(14) of problem (1) - (5) with some experimental results.

We apply the Crank-Nicolson method by choosing $f = h = 0$, $g(t) = 100t$, $\phi(t) = 5t$, $\delta_x = 0.05$ and $\delta_t = 0.0025$. For calculation of $\theta(x, t)$, we use the first 51 terms of its series. Then (14) can be written in the form

$$p(s) = \frac{5\left(\frac{2\sqrt{s}}{\sqrt{5}} + 10\left(2\sqrt{\pi} + \mathrm{Gamma}\left(-\frac{1}{2}, 0, \frac{5}{4s}\right)\right)\right)}{\sqrt{\pi}} + \frac{5}{\sqrt{\pi}}\left(\sum_{m=1}^{50}\left\{-\left(\mathrm{Abs}(m)\left(2\sqrt{\pi} + \mathrm{Gamma}\left(-\frac{1}{2}, 0, \frac{5m^2}{s}\right)\right)\right) - 20\left(-\left(\sqrt{\pi}\,\mathrm{Abs}(-1+2m)\right)\right) - \frac{\mathrm{Abs}(-1+2m)\,\mathrm{Gamma}\left(-\frac{1}{2}, 0, \frac{-5(-1+4m-4m^2)}{4s}\right)}{2}\right\}\right). \quad (15)$$

The results for p is plotted in Figures 1 and 2. After 400 time steps, in Figure 3, the numerical and analytical results completely covered each other.

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