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INCLUSION RELATIONS CONCERNING WEAKLY ALMOST PERIODIC FUNCTIONS AND FUNCTIONS VANISHING AT INFINITY

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Abstract

We consider the space of weakly almost periodic functions on a transformation semigroup (S, X, π) and show that if X is a locally compact noncompact uniform space, and π is a separately continuous, separately proper, and equicontinuous action of S on X, then every continuous function on X, vanishing at infinity is weakly almost periodic. We also use a number of diverse examples to show that the conditions we have imposed on the transformation semigroup are almost essential for the inclusion to hold.

Introduction

Let (S, X, π) be a separately continuous transformation semigroup. This means that S is a semitopological semigroup, X is a Hausdorff topological space, and $\pi: S \times X \to X$, called the action of S on X, is a separately continuous mapping such that

$$\pi(s, \pi(t,x)) = \pi(st,x) \qquad (s,t \in S,\ x \in X)$$

This gives rise to the following continuous mappings:

If $s \in S$ then ${}_s\pi : X \to X$ is defined by ${}_s\pi(y) = \pi(s,y)$.

If $x \in X$ then $\pi_x : S \to X$ is defined by $\pi_x(t) = \pi(t,x)$.

In the notation of a transformation semigroup, normally the letter π is suppressed, and (S, X, π) is simply denoted by (S, X). Accordingly, $\pi(s,x)$ is denoted by sx. It should also be noted that a semitopological semigroup is a transformation semigroup under its own multiplication. The action is called *separately proper* if for each $s \in S$ and each $x \in X$ both ${}_s\pi$ and π_x are proper mappings, in the sense that the inverse image of a compact set is compact. If X is a uniform space, the action of S on X is said to be *equicontinuous at a point* $x_0 \in X$, if for each vicinity V of X there exists a neighbourhood U of x_0 such that $(sx, sx_0) \in V$ for all $x \in U$ and $s \in S$. The action is called equicontinuous if it is equicontinuous at every point of X.

We shall be following the terminology and notations of [1]. In particular, $C(X)$ denotes the C*-algebra of bounded continuous complex-valued functions on X, with the supremum norm $\|.\|_\infty$. When X is locally compact, $C_0(X)$ denotes the C*-subalgebra of $C(X)$ consisting of all functions tending to zero at infinity.

Each $s \in S$ induces a continuous mapping λ_s on $C(X)$ defined by

$$(\lambda_s f)(x) = f(sx) \qquad (x \in X\ ,\ f \in C(X)).$$

A function $f \in C(X)$ is called *weakly almost periodic*, with respect to S, if $\lambda_S(f) = \{\lambda_s f : s \in S\}$ is relatively compact in the weak topology of $C(X)$. The set of weakly almost periodic functions on X, with respect to S, is denoted by $WAP(X,S)$. When there is no risk of confusion we denote it by $WAP(X)$.

It can be shown that $WAP(X)$ is a C*-subalgebra of

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$C(X)$ containing the constant functions, see [9]. The extent of $WAP(X)$, and its inclusion relations with various C*-subalgebras of $C(X)$, has its roots in the semigroup case, which has been the center of attention of a good number of mathematicians over the past half century. Those who would like to follow up the historical background of these problems in the semigroup case can refer to [1], §4.13.

$WAP(X,S)$ was first introduced and its main properties were studied in [4] and [7]. The inclusion problems concerning this space have also received some attention; in this regard we cite [8,9,10], and also [6] which contains some indirect results.

The inclusion relation between $WAP(X,S)$ and $C_0(X)$ is an important one and deserves special treatment. In this article we will set down the conditions which imply the relation $C_0(X) \subset WAP(X,S)$. We will also show that the conditions we impose are more or less essential for the above inclusion to hold.

Preliminary Examples

First we give some examples of inclusion relations which illustrate the diversity of situations involved.

(a) By a standard proof similar to the semigroup case ([2] theorem 1.7) we can show that if S and X are both compact then $WAP(X) = C(X)$. Notice that since X is compact $C_0(X) = C(X)$.

(b) If S is a finite semigroup and X is any locally compact topological space, then for each $f \in C(X)$, $\lambda_s f$ is a finite, hence compact, set. Therefore

$$C_0(X) \subset C(X) = WAP(X).$$

The inclusion relation between $C_0(X)$ and $WAP(X)$ can sometimes be in reverse order, as the following example reveals.

(c) Consider the action of Z, the group of integers, on R, the space of real numbers, by ordinary addition. This action can be extended by continuity to an action of Z on βR, the Stone-Cech compactification of R. In [9] it is shown that $WAP(\beta R, Z)$ is a proper subset of $C(\beta R) = C_0(\beta R)$.

(d) Let X be the set of vectors $\begin{pmatrix} x \\ 1 \end{pmatrix}$, where $x \in R$, and let S be the group of 2×2 matrices of the form $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$, where $a,b \in R$, and $a \neq 0$. Then S acts on X by matrix multiplication. In fact X is homeomorphic to R and S can be considered as the group of affine transformations of the real line. In [8] it is shown that for this X and S, $WAP(X,S)$ is equal to the set of constant functions on R. This means that in this case $WAP(X,S) \cap C_0(X)$ consists of the constant function 0 only.

(e) If S is a locally compact topological group, or a locally compact semitopological semigroup, such that $C_0(S)$ is translation invariant, we know that each element of $C_0(S)$ is weakly almost periodic, see [1] chapter 4, corollary 2.13.

We are now in a position to state our main result.

Theorem

If (S,X) is a transformation semigroup, where X is a locally compact noncompact uniform space and the action is separately proper and equicontinuous, then $C_0(X) \subset WAP(X)$.

Proof

Let $g \in C_0(X)$. To show that g is weakly almost periodic, by the Eberlein-Smulian theorem, it is sufficient to prove that $\lambda_s g$ is weakly relatively sequentially compact. But since this set is uniformly bounded (in fact $\|\lambda_s g\|_\infty \leq \|g\|_\infty$ for each $s \in S$) by Grothendieck's Theorem [3], it is sufficient to show that for each sequence $\{\lambda_{s_n} g\}$ of translates of g there exists a subsequence $\{\lambda_{t_n} g\}$, and a function $f \in C_0(X)$ such that for each $x \in X$, $\{\lambda_{t_n} g(x)\}$ tends to $f(x)$. Of course Grothendieck's Theorem applies to certain subsets of the space of continuous functions on a compact topological space, but here as we are confined to $C_0(X)$, the same proof applies to this situation as well. Equivalently we may adapt the proof of Corollary A.7, in [1].

Since each $s \in S$ acts as a proper mapping, each $\lambda_{s_n} g$ vanishes at infinity. Hence, for arbitrary positive integers m, n there exists a compact subset $A_{m,n}$ of X such that $|\lambda_{s_n} g| < 1/m$ for all $x \in X \setminus A_{m,n}$. Now let $A = \bigcup_{m,n=1}^{\infty} A_{m,n}$. Then $\lambda_{s_n} g(x) = 0$ for all n and all $x \in X \setminus A$.

Each $A_{m,n}$ is compact, hence a uniform space. Since the $\lambda_{s_n} g$ are equicontinuous and uniformly bounded, by an application of Ascoli's Theorem ([5], p. 223), for each k, l we can find a subsequence of $\{\lambda_{s_n} g\}$ converging uniformly on A_{kl}. These subsequences can be chosen so that by a diagonal process, we can find a subsequence $\{\lambda_{t_n} g\}$ and a function $f \in C(X)$ such that

$$\lim_n \lambda_{t_n} g(x) = f(x) \quad (\forall\, x \in X).$$

Of course, for this, we have to define $f(x)$ equal to zero for each $x \in X \setminus A$.

We now prove that $f \in C_0(X)$. Two cases are to be considered.

(a) If there exists a compact set $K \subset S$ containing an infinite number of elements of $\{t_n\}$, then K contains an accumulation point s of $\{t_n\}$. Therefore, for each $x \in X$

$$f(x) = \lim_n \lambda_{t_n} g(x) = \lim_n g(t_n x) = g(sx) = \lambda_s g(x) \quad (x \in X),$$

i.e. $f = \lambda_s g$ for some $s \in S$. Hence $f \in C_0(X)$.

(b) If no compact set contains more than a finite number of points of $\{t_n\}$, then $\lim_n t_n = \infty$. As for each $x \in X$, the mapping π_x is a proper mapping of S into X, we have

$$\lim_n t_n x = \infty \qquad (\forall x \in X).$$

So for each $x \in X$,

$$f(x) = \lim_n \lambda_{t_n} g(x) = \lim_n g(t_n x) = 0\,.$$

i.e. f is identically zero, hence it belongs to $C_0(X)$.

It seems that the conditions imposed on the action in this theorem cannot be removed very effectively. Apart from the case of a semigroup, example (e), which reveals the necessity of some of our conditions, in example (d) above we noticed that no member of $C_0(X)$, except the constant function zero, can be weakly almost periodic. In this case, for each $s \in S$, ${}_s\pi$ is a proper mapping, but it is not equicontinuous, and also no mapping π_x for $x \in R$ is proper.

The next example shows that the condition of π_x being proper, is an essential condition for the theorem to hold.

Example

Let S be the open interval $(0, 1)$ with ordinary multiplication, and let $X = R$, both with their ordinary topologies, and let S act on X by ordinary multiplication. This action is jointly continuous and equicontinuous, and for each $s \in S$, ${}_s\pi$ is proper. However, π_0 fails to be proper as $\pi_0^{-1}(\{0\}) = (0,1)$. In fact if a noncompact topological space has a fixed point or a compact invariant subspace under the action of a noncompact semigroup, then π_x is not proper for some $x \in X$.

Now let $f \in C_0(R)$ with $f(0) = 1$, and let $\{s_\alpha\} \subset (0,1)$ tend to 0. We can easily see that $\lambda_{s_\alpha} f \to 1$ on compact sets in R. If f were weakly almost periodic then there would be a subnet $\{\lambda_{s_\beta} f\}$ of $\{\lambda_{s_\alpha} f\}$, and a function $g \in C(R)$ with $\lambda_{s_\beta} f$ tending weakly to g. This implies that $\lambda_{s_\beta} f$ tends in the pointwise topology to g. So $g \equiv 1$. However, $C_0(R)$ is a norm closed, hence weakly closed subspace of $C(R)$, so $g \in C_0(R)$ which is a contradiction.

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