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QUASI-PERMUTATION REPRESENTATIONS OF SUZUKI GROUP

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Abstract

By a quasi-permutation matrix we mean a square matrix over the complex field **C** with non-negative integral trace. Thus every permutation matrix over **C** is a quasi-permutation matrix. For a given finite group G, let $p(G)$ denote the minimal degree of a faithful permutation representation of G (or of a faithful representation of G by permutation matrices), let $q(G)$ denote the minimal degree of a faithful representation of G by quasi-permuation matrices over the rational field **Q**, and let $c(G)$ be the minimal degree of a faithful representation of G by complex quasi-permuatation matrices. Let $r(G)$ denote the minimal degree of a faithful rational valued character of G. In this paper we will calculate $c(G), q(G), p(G)$ and $r(G)$ where $G= Sz(q)$ is the Suzuki group. Also we will show that $lim_{q \to \infty} \frac{c(G)}{r(G)} = 1$.

Introduction

By a quasi-permutation matrix we mean a square matrix over the complex field **C** with non-negative integral trace. Thus every permutation matrix over **C** is a quasi-permutation matrix. For a given finite group G, let $p(G)$ denote the minimal degee of a failthful permutation representation of G (or of a faithful representation of G by permutation matrices), let $q(G)$ denote the minimal degree of a faithful representation of G by quasi-permutation matrices over the rational field **Q**, and let $c(G)$ be the minimal degree of a faithful representation of G by complex quasi-permutation matrices. See [1].

By a rational valued character we mean a character χ corresponding to a complex representation of G such that $\chi(g) \in \mathbf{Q}$ for all $g \in G$. As the values of the character of a complex representation are algebraic numbers, a rational valued character is in fact integer valued. A quasi-permutation representation of G is then simply a complex representation of G whose character values are rational and non-negative. The module of such a representation will be called a quasi-permutation module. We will call a homomorphism from G to $GL(n, \mathbf{Q})$ a rational representation of G and its corresponding character will be called a rational character of G. Let $r(G)$ denote the minimal degree of a faithful rational valued character of G. It is easy to see that

$$r(G) \leq c(G) \leq q(G) \leq p(G)$$

where G is a finite group.

Let $Sz(q)$ denote the Suzuki group [4] where q is a power of 2. We will apply the algorithms we developed in [1] to the group $Sz(q)$. We will show that

$$lim_{q \to \infty} \frac{c(G)}{r(G)} = 1, \text{ where } G= Sz(q).$$

Keywords: Suzuki group; Quasi-permutation; Quasi-permutation representation; Representation theory

2. Algorithms for $p(G)$, $c(G)$ and $q(G)$

Lemma 2.1. *Let G be a finite group with a unique minimal normal subgroup. Then p(G) is the smallest index of a subgroup with a trivial core (that is, containing no non-trivial normal subgroup).*

Proof. See [1, Corollary 2.4].

Definition 2.2. *Let χ be a character of G such that, for all $g\in G$, $\chi(g)\in \mathbf{Q}$ and $\chi(g)\geq 0$. Then we say that χ is a non-negative rational valued character.*

Notation. Let $\Gamma(\chi)$ be the Galois group of $\mathbf{Q}(\chi)$ over $\mathbf{Q}$.

Definition 2.3. *Let G be a finite group. Let χ be an irreducible complex character of G. Then define*

(1) $d(\chi)=|\Gamma(\chi)|\,\chi(1)$

(2) $m(\chi)=\begin{cases}0 & \text{if } \chi=1_G\\ \left|\min\{\sum_{\alpha\in\Gamma(\chi)}\chi^{\alpha}(g): g\in G\}\right| & \text{otherwise}\end{cases}$

(3) $c(\chi)=\sum_{\alpha\in\Gamma(\chi)}\chi^{\alpha}+m(\chi)1_G$.

Corollary 2.4. Let $\chi\in \mathrm{Irr}(G)$. Then $\sum_{\alpha\in\Gamma(\chi)}\chi^{\alpha}$ is a rational valued character of G. Moreover $c(\chi)$ is a non-negative rational valued character of G and $c(\chi)(1)=d(\chi)+m(\chi)$.

Proof. See [1, Corollary 3.7].

Now we will give algorithms for calculating $c(G)$ and $q(G)$ where G is a finite group with a unique minimal normal subgroup.

Lemma 2.5. *Let G be a finite group with a unique minimal normal subgroup. Then*
(1) $c(G)=\min\{c(\chi)(1): \chi$ *is a faithful irreducible complex character of G*$\}$;
(2) $q(G)=\min\{m_Q(\chi)c(\chi)(1): \chi$ *is a faithful irreducible complex character of G*$\}$.

Proof. See [1, Corollary 3.11].

Lemma 2.6. Let G be a finite group. If the Schur index of each non-principal irreducible character is equal to m, then $q(G)=mc(G)$.

Proof. See [1, Corollary 3.15].

3. Quasi-Permutation Representations

Definition 3.1. *A group G is called a (ZT)-group if there is a positive integer N such that*

(1) *G is a doubly transitive group on 1+N symbols;*
(2) *The identity is the only element which leaves three distinct symbols invariant;*
(3) *G contains no normal subgroup of order 1+N;*
(4) *N is even.*

We shall use the following notations:
G: a (ZT)-group;
H: the subgroup of G consisting of the elements leaving one fixed symbol, say a, invariant;
Q: a Sylow 2-subgroup of H;
K: the subgroup of H consisting of the elements leaving an additional fixed symbol invariant;
x: an involution in the normalizer $N_G(K)$.

Lemma 3.2. *Q contains two elements y and z such that y is an involution and $xyx=z^{-1}xz$.*

Proof. See [4, Proposition 9].

It is a theorem that there is a unique (ZT)-group of order $q^2(q-1)(q^2+1)$ for any odd power q of 2(see [4, Theorem 8]). This group will be denoted here as $Sz(q)$ and called a Suzuki group. The Suzuki groups are simple for all $q>2$.
Let $G=Sz(q)$ where $q=2^{2s+1}$ and $s\geq 1$; let $r=2^{s+1}$. Then

$$|Sz(q)|=q^2(q-1)(q^2+1)=q^2(q-1)(q+r+1)(q-r+1).$$

The group G by [4, Theorem 9] has the following among its subgroups:
H: a Frobenius group of order $q^2(q-1)$;
B_0: a dihedral group of order $2(q-1)$;
A_0: a cyclic subgroup of order $q-1$;
A_i: cyclic groups of order $q\pm r+1$ for $i=1,2$ respectively;
B_i: the normalizer $N_G(A_i)$, which has order $4(q\pm r+1)$, $i=1,2$;
$Sz(q_1)$: for some q_1 such that $q=q_1^m$.

Lemma 3.3. Let $G=Sz(q)$, $q\geq 8$. Then the maximal order of a proper subgroup of G is equal to $q^2(q-1)$.

Proof. The maximal order of a proper subgroup of G is either $q^2(q-1)$ or $4(q+r+1)$ or $|Sz(q_1)|$ for some q_1 by [4, Theorem 9].

Now $q^2(q-1)=\frac{r^4}{4}(\frac{r^2}{2}-1)$ and $4(q+r+1)=4(\frac{r^2}{2}+r+1)$.
Since $s\geq 1$ so $r>4$. Therefore $\frac{r^2}{2}\geq 2r>r+1$ so we have

$\frac{r^4}{4}(\frac{r^2}{2}-1) > \frac{r^4}{4} \geq r^3 \geq 4r^2 = 4\frac{r^2}{2} + 4\frac{r^2}{2} > 4\frac{r^2}{2} + 4(r+1)$.
Thus $q^2(q-1) > 4(q+r+1)$.

Let $Sz(q_1) < Sz(q)$ for some $q_1 = 2^m$. As q is an odd power of 2, m is odd. Therefore $q_1^2 \leq q$.

Now we have to show that $|Sz(q_1)| \leq q^2(q-1)$. We know that $|Sz(q_1)| = q_1^2(q_1-1)(q_1^2+1)$, so $|Sz(q_1)| \leq q(\sqrt{q}-1)(q+1)$. We have to show that $q(\sqrt{q}-1)(q+1) \leq q^2(q-1)$ or equivalently, $(\sqrt{q}-1)(q+1) \leq q(q-1)$ the latter is equivalent to $\sqrt{q}(q-1) \leq q^2+1$. But $\sqrt{q}(q+1) \leq (q-1)(q+1) = q^2-1 < q^2+1$.

Theorem 3.4. Let $G = Sz(q)$, $q \geq 8$. Then $p(G) = q^2+1$.

Proof. This follows from Lemmas 3.3 and 2.1.

Theorem 3.5. *The character table of* $Sz(q)$, $q \geq 8$ *is given in Table 1.*

Table 1. Character Table of $Sz(q)$

	1	y	z	b_0	b_1	b_2
χ	q^2	0	0	1	-1	-1
χ_i	q^2+1	1	1	$\varepsilon_0^i(b_0)$	0	0
ψ_j	$(q-r+1)(q-1)$	$r-1$	-1	0	$-\varepsilon_1^j(b_1)$	0
η_k	$(q+r+1)(q-1)$	$-r-1$	-1	0	0	$-\varepsilon_2^k(b_2)$
ξ_l	$\frac{r(q-1)}{2}$	$-\frac{r}{2}$	$\pm\frac{r\sqrt{-1}}{2}$	0	1	-1

In this table $2q = r^2$ *and* y, z *are as Lemma* 3.2, *and* b_i *is a non-identity element of* A_i *for* $i = 0,1,2$. *Let* ε_0 *be a primitive* $(q-1)$*-th root of 1. If* a_0 *is a generator of* A_0 *the function* ε_0^i *is defined by*

$$m(\chi) = \varepsilon_0^i(a_0^j) = \varepsilon_0^{ij} + \varepsilon_0^{-ij} \text{ for } i = 1,\ldots,\frac{q}{2}-1;$$

(it is a character of A_0*).*

Let ε_1 *be a primitive* $(q+r+1)$*-th root of* 1. *If* a_1 *is a generator of* A_1 *the function* ε_1^i *is defined by*

$$\varepsilon_1^i(a_1^k) = \varepsilon_1^{ik} + \varepsilon_1^{ikq} + \varepsilon_1^{-ik} + \varepsilon_1^{-ikq} \text{ for } i = 1,\ldots,q+r;$$

(it is a character of A_1*).*

Let ε_2 *be a primitive* $(q-r+1)$*-th root of* 1. *If* a_2 *is a generator of* A_2 *the function* ε_2^i *is defined by*

$$\varepsilon_2^i(a_2^k) = \varepsilon_2^{ik} + \varepsilon_2^{ikq} + \varepsilon_2^{-ik} + \varepsilon_2^{-ikq} \text{ for } i = 1,\ldots,q-r;$$

(it is character of A_2*).*

Finally, $l = 1,2$.

Proof. See [4, Theorem 13].

Theorem 3.6. *Let G be a simple Suzuki group. Then all characters of G have Schur index* 1.

Proof. See [2, Theorem 9].

Theorem 3.7. *Let* $G = Sz(q)$ *where* $q = 2^{2s+1}$, $s \geq 1$. *Then* $c(G) = q(G) = rq = \frac{r^3}{2}$ *where* $r = 2^{s+1}$.

Proof. Since by Theorem 3.6 the Schur index for each irreducible character of G is 1 so by Lemma 2.6 we have $c(G) = q(G)$. In order to calculate $c(G)$ we apply Lemma 2.5. Now we have Table 2.

Now

$(q-1+1)(q-1) = q^2 - rq + r - 1 = \frac{r^4}{4} - \frac{r^3}{2} + r - 1 > \frac{r^4}{4} - \frac{r^3}{2} \geq 4\frac{r^3}{4} - \frac{r^3}{2} = \frac{r^3}{2} = rq$ and

Table 2.

θ	$d(\theta)$	$c(\theta)(1)$
χ	q^2	q^2+1
χ_i	$\geq q^2+1$	$\geq q^2+1$
ψ_j	$\geq(q-r+1)(q-1)$	$\geq(q-r+1)(q-1)$
η_k	$\geq(q+r+1)(q-1)$	$\geq(q+r+1)(q-1)$
ξ_l	$r(q-1)$	rq

$(q+r+1)(q-1) > (q-r+1)(q-1) > rq$
and
$q^2+1=\frac{r^4}{4}+1>\frac{r^4}{4}\geq 4\frac{r^3}{4}>\frac{r^3}{2}=rq.$
So $c(G)=rq$.

4. Rational Valued Characters

Lemma 4.1. *Let G be a finite group. Let G have a unique minimal normal subgroup. Then*

$r(G)=\min\{d(\chi): \chi$ *is a faithful irreducible character of* $G\}$.

Proof. Let $\chi\in Irr(G)$. Then $\sum_{\alpha\in\Gamma}\chi^\alpha$, where $\Gamma=\Gamma(Q(\chi):Q)$, is an irreducible rational valued character by [3, Corollary 10.2].
Let ϕ be a faithful rational valued character such that $r(G)=\phi(1)$. Since G has a unique minimal normal subgroup so there exists a faithful irreducible character, say χ, such that $[\phi,\chi]\neq 0$. So $\phi=\sum_{\alpha\in\Gamma}\chi^\alpha+\psi$, for some rational valued character ψ. Hence $\phi(1)\geq\sum_{\alpha\in\Gamma}\chi^\alpha(1)=d(\chi)$. So $r(G)=d(\chi)$.

Lemma 4.2. *Let* $G=Sz(q)$, $q\geq 8$. *Then* $r(G)=r(q-1)$.

Proof. This follows from Table 2.

Theorem 4.3. *Let* $G=Sz(q)$. *Then*

$$\lim_{q\to\infty}\frac{c(G)}{r(G)}=1.$$

Proof. Let $G=Sz(q)$. Then by Thorem 3.7 and Lemma 4.2 we have $\frac{c(G)}{r(G)}=\frac{q}{q-1}$. Hence $\lim_{q\to\infty}\frac{c(G)}{r(G)}=1$.

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