ALGEBRAIC NONLINEARITY IN VOLTERRA-HAMMERSTEIN EQUATIONS

K. Maleknejad[1] and M. Hadizadeh[2]

School of Mathematics, Iran University of Science and Technology, Tehran-16, Islamic Republic of Iran

Abstract

Here a posteriori error estimate for the numerical solution of nonlinear Volterra-Hammerstein equations is given. We present an error upper bound for nonlinear Volterra-Hammerstein integral equations, in which the form of nonlinearity is algebraic and develop a posteriori error estimate for the recently proposed method of Brunner for these problems (the implicitly linear collocation method). We also generalize this upper bound for nonlinear Volterra integro-differential and Volterra-Hammerstein integral equations of mixed type. Finally, several numerical examples are given to show effectiveness of these bounds.

1. Introduction

Nonlinear Volterra integral equations of the second kind often occur in Hammerstein form,

$$y(t)=f(t)+\int_0^t k(t,s)\,G(s,y(s))\,ds, \quad 0\le t\le T, \quad (1.1)$$

where f,k and G are smooth and known functions. These equations arise in chemical engineering [9], cell membrane theory [16], and other branches of science. Another rich source of Hammerstein integral equations is the reformulation of boundary value problems for both ordinary and partial differential equations.

Several numerical methods for approximating the solution of (1.1) are known. The classical method of successive approximations [10], a variation of the Nystrom method was presented in [12]. Other works about classical numerical solutions of these equations can be found in [1,2,6,13] and references therein.

Kumar and Sloan [11] recommend a new collocation type method for the numerical solution of the Hammerstein equation. Brunner [4] applied this method to nonlinear Volterra equations. Recently, Frankel [8] established a posteriori error estimate by symbolic manipulation for the method of Kumar and Sloan [11]. This type of estimate is highly desirable from the practical point of view. In the present paper we have used this idea for establishing the error estimates of nonlinear Volterra-Hammerstein equations, in which the form of nonlinearity is algebraic. Development of an integral equation for the error in the approximate solution has been illustrated in Delves and Walsh [7], and Delves and Mohamed [6].

This paper is divided into five sections. In section 2, we briefly review the implicitly linear collocation method, that proposed by Brunner [4], for nonlinear Volterra integral equations.

In section 3, we give an error upper bound for application of implicitly linear collocation method for these equations. In section 4, we developed this idea to

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[1] *E-mail: maleknejad @ sun. iust. ac. ir*
[2] *hadizadeh @ sun. iust. ac. ir*

determine an error estimate for nonlinear Volterra integro-differential equations and Volterra-Hammerstein integral equations of mixed type. In section 5, we present some numerical examples.

2. Preliminaries

Consider the nonlinear Volterra-Hammerstein equation (1.1), where f, k and G are smooth and known functions, $G(s,v)$ nonlinear in v, and y is the solution to be determined.

Let $C \equiv C[0,T]$ denote the Banach space of continuous real-valued functions on [0, T], with the uniform norm

$$\|x\| \equiv \sup_{t\in I} |x(t)|, \quad x \in C, \quad I=[0,T].$$

In order to guarantee the existence of a unique solution to equation (1.1), we assume throughout this paper that the following conditions (1-4) be satisfied

1. $f \in C$,

2. The kernel k satisfies $sup_{t\in I} \int_0^t |k(t,s)|\, ds < \infty$

and

$$\lim_{\delta\to 0} \int_{t_1}^{t_1+\delta} |k(t,s)|\, ds = 0 \quad (0 \le t_1 < t_1+\delta \le t \le T),$$

uniformly in t_1 and t.

3. The function $G(t,v)$ is defined and continuous on $[0,T]\times \mathbb{R}$.

4. The partial derivative $G_v(t,v) = (\frac{\partial}{\partial v})\, G(t,v)$ exists and is continuous on $[0,T]\times\mathbb{R}$.

Under these conditions there exists a unique solution in C for equation (1.1). (see [11])

Now, we consider the equation (1.1), when using a collocation method. We have a nonlinear system of equations. In the iteration solution of this system, many integrals will need to be computed, which usually becomes quite expensive. Kumar and Solan [11] recommend the following variant approach. Define $z(t') \equiv G(t, y(t))$, thus from (1.1), we have

$$z(t)= G(t,f(t) + \int_0^t k(t,s)z(s)ds), \quad 0 \le t \le T, \quad (2.1)$$

and obtain $y(t)$ from

$$y(t)= f(t) + \int_0^t k(t,s)z(s)ds, \qquad 0 \le t \le T.$$

If we solve (2.1) by collocation method, the integrals that appear in nonlinear system of equations, need to be evaluated only once, since they are dependent only on the basic function, not on the unknown collocation parameters (see[11]). Brunner [4] applied this method (which referred to as the implicitly linear collocation method) to nonlinear Volterra integral and integro-differential equations. We will use a uniform approximation in developing the approximate solution of (1.1).

3. Error Estimate with Algebraic Nonlinearity

In this section we assume that the form of nonlinearity in (1.1) is algebraic (i.e. $G(t, y(t))= y^p(t),\ p \in N$). To discretize (1.1) let $S_m^{(0)}(Z_N)$ be the piecewise polynomial space, such that $S_m^{(0)}(Z_N) = \{y/y \in C, y_n = y\,|_{\sigma_n} \in \Pi_m,\ n=0,\dots,N-1\}$. Here $N\ge 1$ and $m\ge 1$ are positive integers, Π_m denotes the space of real polynomials of degree less than m, $h=\frac{T}{N}$, $t_0=0$, $t_n=t_0+nh\ (n=1,\dots,N-1)$, $Z_N:=\{t_n/n=1,\dots,N-1\}$, $\bar{Z}_N=Z_N\cup\{T\}$, and $\sigma_n:=[t_n,t_{n+1}]\ (0\le n\le N-1)$.

Let $X(N):=\cup_{n=0}^{N-1} X_n$, where $X_n:=\{t_{n,j}=t_n+c_jh\,/\, j=1,\dots,m;\ 0\le c_1\le\dots<c_m\le 1\}$, (n=0,…,N-1) denote the set of collocation points at which the approximate solution $y_n(t)\in S_m^{(0)}(Z_N)$ is to satisfy the equation (1.1). Following the method of [4], we define

$$z(t)= y^p(t), \quad p \in \mathrm{N}. \qquad (3.1)$$

By substitution (3.1) in equation (1.1), we have

$$y(t)= f(t) + \int_0^t k(t,s)z(s)ds, \quad 0 \le t \le T, \qquad (3.2)$$

the explicit form of (2.1) now becomes

$$z(t)=\left[f(t) + \int_0^t k(t,s)z(s)ds\right]^p, \quad p \in N, \ 0 \le t \le T. \ (3.3)$$

The solution of (3.3) is approximated by an element $z_n \in S_m^{(0)}(Z_N)$ and we have

$$z_n(t_n + \tau h) = \sum_{l=1}^{m} L_l(\tau)\, a_{n,l}, \quad \tau \in [0,1], \qquad (3.4)$$

where $L_l(\tau)$ denotes the lth Lagrange polynomial, and $a_{n,l}$ are expansion parameters. Define the local residual function by

$$R_n(t)+z_n(t)=\left[f(t)+\int_{t_n}^{t} k(t,s)\,z_n(s)\,ds+\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} k(t,s)\,z_i(s)ds\right]^p,$$
$$p\in \mathrm{N},\quad (n=0,\ldots,N-1). \tag{3.5}$$

Equation (3.5) can be written as

$$R_n(t_{n,j})=-z_n(t_{n,j})+\left[f(t_{n,j})+h\int_0^{c_j} k(t_{n,j},\,t_n+\tau h)z_n(t_n+\tau h)d\tau+\right.$$
$$\left.h\sum_{i=0}^{n-1}\int_0^1 k(t_{n,j},\,t_n+\tau h)\,z_i(t_i+\tau h)d\tau\right]^p$$
$$p\in N,\ (n=0,\ldots,N-1). \tag{3.6}$$

By substituting (3.4) in (3.6) we obtain

$$R_n(t_{n,j})=-\sum_{l=1}^{m} L_l(\tau)a_{n,l}+\left[f(t_{n,j})+h\sum_{l=1}^{m} A_{l,n}(c_j)a_{n,l}+h\sum_{i=0}^{n-1}\right.$$
$$\left.\sum_{l=1}^{m} B_{l,i}(t_{n,j})\,a_{n,l}\right]^p \tag{3.7}$$

where

$$A_{l,n}(c_j)=\int_0^{c_j} k(t_n+c_jh,\,t_n+\tau h)L_l(\tau)d\tau,\quad \tau\in[0,1],$$
$$t_n+\tau h\in(t_n,\,t_{n+1}], \tag{3.8}$$

$$B_{l,i}(t)=\int_0^1 k(t,t_i+\tau h)\,L_l(\tau)\,d\tau. \tag{3.9}$$

In most applications the integrals (3.8) and (3.9) cannot be evaluated analytically and must be determined numerically. We suppose that, the parameters $\{c_j\}_{j=1}^{m}$ are choosen such that (see [3])

$$0=-\sum_{l=1}^{m} L_l(\tau)a_{n,l}+\left[f(t_{n,j})+h\sum_{l=1}^{m} A_{l,n}(c_j)a_{n,l}+h\sum_{i=0}^{n-1}\sum_{l=1}^{m}\right.$$
$$\left.B_{l,i}(t_{n,j})\,a_{n,l}\right]^p,\quad p\in N. \tag{3.10}$$

Note that our aim here is a determination of a set of expansion coefficients $a_{n,l}$ $(l=1,\ldots,m;\ n=0,\ldots,N-1)$. As mentioned above, we assume that the values of $A_{l,n}(c_j)$, $B_{l,i}(t)$ are evaluated analytically or numerically, so for each $n=0,\ldots,N-1$, we obtained a nonlinear system of algebraic equations in $\mathbb{R}^m$ for expansion coefficients $\{a_n\}$ $(n=0,\ldots,N-1)$, where $a_n=(a_{n,1},\ldots,a_{n,m})^T$. Once a_n has been determined, the approximation z on the subinterval $(t_n,\,t_{n+1})$ is given by (3.4) and the local residual function $R_n(t)$ is determined from (3.7). Finally, the desired approximation to $y(t)$ is then obained by use of the equation

$$y_n(t)=f(t)+\int_0^1 k(t,s)z_n(s)ds, \tag{3.11}$$

or

$$y_n(t_n+\tau h)=f(t_n+c_jh)+h\sum_{l=1}^{m} A_{l,n}(c_j)a_{n,l}+h\sum_{i=0}^{n-1}\sum_{l=1}^{m} B_{l,i}(t_{n,j})a_{n,l},$$
$$\tau\in[0,1],\quad t_n+\tau h\in(t_n,\,t_{n+1}], \tag{3.12}$$

where $A_{l,n}(c_j)$ and $B_{l,i}(t)$ as defined in (3.8) and (3.9). Let $\varepsilon_n(t)$ be the local error in the $z_n(t)$

$$\varepsilon_n(t)=z(t)-z_n(t),\qquad 0\le t\le T. \tag{3.13}$$

Our aim here is to obtain an error upper bound for the exact solution of (1.1) (i.e. $y(t)$), so we have

$$y_n(t_{n,j})=f(t_{n,j})+h\int_0^{c_j} k(t_{n,j},\,t_n+\tau h)z_n(t_n+\tau h)\,d\tau+$$
$$h\sum_{i=0}^{n-1}\int_0^1 k(t_{n,j},\,t_n+\tau h)z_i(t_n+\tau h)d\tau. \tag{3.14}$$

We may write the equation (3.2) in the (exact) form, (see [6] pp. 123)

$$y(t)=f(t)+\int_{t_n}^{t} k(t,s)z(s)ds+\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} k(t,s)z(s)ds\quad t\in X_n,$$
$$(n=0,\ldots,N-1), \tag{3.15}$$

or

$$y(t_{n,j})=f(t_{n,j})+h\int_0^{c_j} k(t_{n,j},t_n+\tau h)\,z\,(t_n+\tau h)d\tau+$$
$$h\sum_{i=0}^{n-1}\int_0^1 k\,(t_{n,j},\,t_n+\tau h)z_i(t_n+\tau h)d\tau, \tag{3.16}$$

by subtracting (3.16) from (3.14), we get

$$\delta_n(t_{n,j})\equiv y(t_{n,j})-y_n(t_{n,j})=h\int_0^{c_j} k(t_{n,j},t_n+\tau h)\,\varepsilon_n(t_n+\tau h)d\tau+$$
$$h\sum_{i=0}^{n-1}\int_0^1 k(t_{n,j},t_n+\tau h)\varepsilon_i(t_n+\tau h)d\tau,\quad t\in X_n,\ \tau\in[0,1]. \tag{3.17}$$

Now, we define the integral operatuor $T_{n,i}$: $\mathrm{C}\to\mathrm{C}$ by

$$T_{n,i}(\alpha)=\begin{cases}\int_0^1 k(t_{n,j}, t_i+\tau h)\,\alpha(\tau)\,d\tau, & i<n\\ \int_0^{c_j} k(t_{n,j}, t_n+\tau h)\,\alpha(\tau)\,d\tau, & i=n\end{cases}$$

$(n=0,\ldots,N\text{-}1;\ j=1,\ldots,m)$. (3.18)

Therefore (3.17) in operator form is

$$\delta_n = hT_{n,n}(\varepsilon_n) + h\sum_{i=0}^{n-1} T_{n,i}(\varepsilon_i), \tag{3.19}$$

by substitution (3.17) and (3.13) into (3.1) we get

$$z_n + \varepsilon_n = [y_n + \delta_n]^p,\ p\in N,\ (n=0,\ldots,N\text{-}1). \tag{3.20}$$

Multiply (3.20) by $k(t_{n,j}, t_i+\tau h)$, and integrate over the domain of interest, we have in operator form

$$hT_{n,n}(z_n) + h\sum_{i=0}^{n-1} T_{n,i}(z_i) + \delta_n = hT_{n,n}((y_n+\delta_n)^p) + h\sum_{i=0}^{n-1} T_{n,i}((y_i+\delta_i)^p),\quad p\in N. \tag{3.21}$$

Expanding (3.21), we get

$$\delta_n = -hT_{n,n}(z_n) - h\sum_{i=0}^{n-1} T_{n,i}(z_i) + hT_{n,n}(y_n^p + py_n^{p-1}\delta_n) + h\sum_{i=0}^{n-1} T_{n,i}(y_i^p + py_i^{p-1}\delta_i) + hT_{n,n}\left(\frac{p(p-1)}{2}y_n^{p-2}\delta_n^2 + \ldots + \delta_n^p\right) + h\sum_{i=0}^{n-1} T_{n,i}\left(\frac{p(p-1)}{2}y_i^{p-2}\delta_i^2 + \ldots + \delta_i^p\right) \tag{3.22}$$

If $\delta_n \to 0$, then for n sufficiently large, we can assume

$$|y_n^p + py_n^{p-1}\delta_n| >> \left|\frac{p(p-1)}{2}y_n^{p-2}\delta_n^2 + \ldots + \delta_n^p\right|,$$

thus we have

$$\delta_n - hpT_{n,n}(y_n^{p-1}\delta_n) - hp\sum_{i=0}^{n-1} T_{n,i}(y_i^{p-1}\delta_i) = -hT_{n,n}(z_n) - h\sum_{i=0}^{n-1} T_{n,i}(z_i) + hT_{n,n}(y_n^p) + h\sum_{i=0}^{n-1} T_{n,i}(y_i^p); \tag{3.23}$$

It follows by using (3.11) and (3.5)

$$R_n + z_n = y_n^p,\ p\in N,\quad (n=0,\ldots,N\text{-}1). \tag{3.24}$$

Therefore (3.23) can be written as

$$\delta_n - hpT_{n,n}(y_n^{p-1}\delta_n) - hp\sum_{i=0}^{n-1} T_{n,i}(y_i^{p-1}\delta_i) = hT_{n,n}(R_n) + h\sum_{i=0}^{n-1} T_{n,i}(R_i), \tag{3.25}$$

so, we can establish the following error bound from (3.25)

$$\|\delta_n\| \le \frac{h\|T_{n,n}(R_n)\| + h\sum_{i=0}^{n-1}\|T_{n,i}(R_i)\|}{1 - hp\|T_{n,n}(y_n^{p-1})\| - hp\sum_{i=0}^{n-1}\|T_{n,i}(y_i^{p-1})\|},$$

$p\in N,\ (n=0,\ldots,N\text{-}1)$ (3.26)

where $1 - hp\|T_{n,n}(y_n^{p-1})\| - hp\sum_{i=0}^{n-1}\|T_{n,i}(y_i^{p-1})\| > 0$ and the norm used is the uniform norm, i.e.,

$$\|\delta_n\| \equiv \sup_{t\in \overline{Z}_N} |\delta_n(t)| \tag{3.27}$$

Thus, we can state this theorem for error estimates of algebraic nonlinearity in (1.1).

Theorem 3.1. *Consider the nonlinear Volterra-Hammerstein integral equation (1.1), with algebraic nonlinearity, and let functions f,k and G satisfied to the hypothesis (1-4) stated at begining of section 2. Moreover, suppose that R_n and δ_n are the local residual function in z_n and the local error in y_n, respectively. If the operator $T_{n,i}(\alpha)$ defined as (3.18), then δ_n can be satisfied in the error estimate (3.26).*

4. Generalization of Nonlinear Volterra Integro-Differential and Volterra-Hammestein Integral Equations of Mixed Type

The arguments of section 3, can be extended to nonlinear Volterra integro-differential equations of the form

$$y'(t) = f(t,y(t)) + \int_0^t k(t,s)\,G(s,y(s))ds,\quad 0\le t\le T, \tag{4.1}$$

subject to the initial condition $y(0)=y_0$. In this equation f,k and G are assumed smooth and known functions, and $f(t,v)$, $G(t,v)$ to be nonlinear in v. The form of the nonlinearity is again assumed to be algebraic. This problem may be written as

$$y(t) = y_0 + \int_0^1 f(s,y(s))ds + \int_0^1 K(t,s)G(s,y(s))ds, \tag{4.2}$$

where

$$K(t,s)=\int_0^1 k(\tau,s)d\tau, \quad 0\le s\le t\le T. \qquad (4.3)$$

Equation (4.2), may be viewed as a special case of Volterra-Hammerstein integral equations of the mixed type

$$y(t)= g(t) + \sum_{\mu=1}^{M} \int_0^t k_\mu(t,s)G_\mu(s,y(s))ds, \quad 0\le t\le T, \qquad (4.4)$$

where the functions g, k_μ and G_μ ($\mu= 1,\ldots,M$) are subject to the hypotheses stated at the beginning of section 2.

Ganesh and Joshi in [9] analyzed the implicitly linear collocation method for Fredholm-Hammerstein integral equations analogous to (4.4). Using the notation of (4.4) we let $M= 2, k_1(t,s)= 1, k_2(t,s)= K(t,s), G_1(s,y)=f(s,y)$ and $G_2(s,y)= G(s,y)$. So, as mentioned above, the nonlinear Volterra integro-differential equation (4.2) is a special case of (4.4). Since the error analysis is obvious, we refrain from going into details of the method, and using a similar procedure as outline in section 3, we have the following theorem, without the proof.

Theorem 4.1. *Let the form of nonlinearity in Volterra-Hammerstein integral equation of mixed type (4.4) be algebraic (e.g. $G_\mu(s,y(s))= y^{p_r}(s); r,\mu= 1,\ldots,M; p_r\in N$), furthermore let assumptions (1-4) for the functions g, k_μ and G_μ($\mu= 1,\ldots,M$) hold. If $R_{n,r}$ and δ_n are the local residual functions in z_n and the local error in y_n, respectively, then the upper bound for δ_n can be established as:*

$$\|\delta_n\| \le \frac{\sum_{\mu=1}^{M}\left(h\left\|T^{\mu}_{n,n} R_{n,r})\right\| + h\sum_{i=0}^{n-1}\left\|T^{\mu}_{n,i}(R_{i,r})\right\|\right)}{1-hp_r\sum_{\mu=1}^{M}\left(\left\|T^{\mu}_{n,n}(y_n^{p_r-1})\right\| + \sum_{i=0}^{n-1}\left\|T^{\mu}_{n,i}(y_i^{p_r-1})\right\|\right)},$$

$$r= 1,\ldots,M; \quad p_r\in N; \quad (n= 0,\ldots,N-1), \qquad (4.5)$$

where $1-hp_r\sum_{\mu=1}^{M}\left(\left\|T^{\mu}_{n,n}(y_n^{p_r-1})\right\| + \sum_{i=0}^{n-1}\left\|T^{\mu}_{n,i}(y_i^{p_r-1}\right\|\right)>0$

and the operator $T^{\mu}_{n,i}$ as defined

$$T^{\mu}_{n,i}(\alpha)=\begin{cases}\int_0^1 k_\mu(t_{n,j}, t_i+\tau h)\,\alpha(\tau)\,d\tau, & i<n\\ \int_0^{c_j} k_\mu(t_{n,j}, t_n+\tau h)\,\alpha(\tau)\,d\tau, & i=n\end{cases}$$

$$(\mu= 1,\ldots, M). \qquad (4.6)$$

Numerical Experimentation

Here we present some illustrative numerical results indicating the merit of the error estimates. For computational purposes, we consider two test problems.

Example 1. (From [3])

$$y(t)= 1+ \sin^2(t) - \int_0^t 3\sin(t-s)y^2(s)ds, \quad 0\le t\le 5$$

with solution y(t)= cos t.

Example 2.

$$y'(t)= 1-\frac{5t^2}{6} + y^2(t) - \int_0^t ts.y^4(s)ds, \quad 0\le t\le 1,$$
$$y(0)= 0$$

with solution $y(t)= t$. This problem may be written as

$$y(t)= t+\frac{t^8}{48}-\frac{t^3}{3}+\int_0^t y^2(s)ds - \int_0^t k(t,s).y^4(s)ds,$$

where $k(t,s)= \frac{st^2}{2}-\frac{s^3}{3}$.

We choose $m= 2$, $c_1= (3-\sqrt{3})/6$, $c_2= (3+\sqrt{3})/6$, $p= 2$ (example 1), $p_1= 2$ and $p_2= 4$ (example 2). Using the subroutine BRENTM [14] for solving the nonlinear algebraic system of equations, we summarize the results from the actual error of the implicitly linear collocation method [4], and estimated upper bounds (3.26) and (4.5) in Tables 1 and 2. All computations were carried out in double precision on an IBM-PC using a program written in the language Mathematica™, ver. 2.1, in which the norm used throughout the program is the uniform norm, as displayed in (3.27).

Conclusion

In most contemporary studies involving Volttera-

Table 1. Comparison between the actual error and the upper bound estimate (3.26) for Example 1

N	Actual error $\|y-y_n\|$	Estimated upper bound (4.5)
4	3.04×10^{-5}	3.24×10^{-5}
5	3.45×10^{-6}	3.54×10^{-6}
6	7.87×10^{-8}	14.88×10^{-8}
7	8.01×10^{-9}	15.73×10^{-9}

Table 2. Comparison between the actual error and the upper bound estimate (4.5) for Example 2

N	Actual error $\|y-y_n\|$	Estimated upper bound (4.5)
5	2.21×10^{-4}	2.74×10^{-4}
7	9.57×10^{-7}	18.07×10^{-7}
9	1.01×10^{-8}	3.29×10^{-8}

Hammerstein integral equations [1,2,13] a priori error estimate is being developed with the aid of interpolatory projections and the approximation theory. Here we consider algebraic nonlinearity for Volterra-Hammerstein integral and integro-differential equations and illustrate that both a uniform approximation and a posteriori error estimate may be obtained for the implicitly linear collocation method. In other papers we extend this idea for other types of nonlinearity.

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