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M-IDEAL STRUCTURE IN UNIFORM ALGEBRAS

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Abstract

It is proved that if A is a regular uniform algebra on a compact Hausdorff space X in which every closed ideal is an M-ideal, then A = C(X).

Introduction

Alfsen and Effros [1] introduced M-ideals for a real Banach space and showed that the M-ideals behave in many ways like the closed ideals of a C^*-algebra. Smith and Ward [9] investigated the M-ideal structure of a Banach algebra and proved that the M-ideals in a C^*-algebra are exactly the closed two-sided ideals.

In this paper we consider a uniform algebra A on a compact Hausdorff space X in which every closed ideal is an M-ideal and prove that if A is regular, then A = C(X).

2. Preliminaries

Suppose X is a compact Hausdorff space and let C(X) denote the space of all continuous complex functions on X. With the sup-norm and the pointwise operations, C(X) is a commutative Banach algebra. A subalgebra A of C(X) is said to be a **uniform algebra** if it contains the identity, separates the points of X and is uniformly closed.

Let A be a uniform algebra. A closed subset E of X is called a **peak set** for A if there is a function f in A such that;

$f(x) = 1$ for all x in E, and $|f(y)| < 1$ for all $y \in X \setminus E$. A closed subset E of X is said to be a p-set or **generalized peak set**, if it is the intersection of peak sets. The notions of p-set and peak set coincide when X is metrizable [4].

A net $\{e_\lambda\}_{\lambda\in\Lambda}$ is said to be an **approximate identity** for A if

$$\lim_\lambda \| e_\lambda f - f\| = 0, \text{ for all } f \in A.$$

If there is also a constant K such that

$$\|e_\lambda\| \leq K \text{ for all } \lambda \in \Lambda,$$

then $\{e_\lambda\}_{\lambda\in\Lambda}$ is called a **bounded approximate** identity [2].

A closed subspace N_1 of A is called an **L-summand** if there is a closed subspace N_2 of A such that $A = N_1 \oplus N_2$, and if $f_1 \in N_1$, $f_2 \in N_2$ then

$$\|f_1 + f_2\| = \|f_1\| + \|f_2\|$$

A closed subspace J of A is an **M-ideal** if J^0, the annihilator of J in A^*, is an L-summand in A^*.

The next result gives a characterization of the M-ideals in a uniform algebra in terms of the p-sets and bounded approximate identities.

2.1. **Theorem** [5,8]. Suppose A is a uniform algebra on a compact Hausdorff space X and J is a closed subspace of A. Then the following statements are equivalent:

(i) J is an M-ideal in A,

(ii) $J = \{f \in A; f(E) = \{0\}\}$ where E is a p-set for A,

(iii) J is a closed ideal with a bounded approximate identity.

3. Main Result

We recall that a uniform algebra A is said to be

regular on X if for each closed subset E of X and each $x\in X\backslash E$, there is a function f in A such that $f(E)=\{0\}$ and $f(x)=1$.

Next we apply theorem 2.1 to prove our main result.

3.1. **Theorem.** Suppose X is a compact Hausdorff space and A is a regular uniform algebra on X. If each closed ideal in A is an M-ideal then A = C(X).

Proof. Suppose E is a closed set in X. Let

$$J=\{f\epsilon A;\ f(E)=\{O\}\},$$

then J is a closed ideal in A and by assumption J is an M-ideal. Therefore, by regularity of A and theorem 2.1, E is a p-set for A, The result now follows from the Glicksberg peak set Theorem [4, Theorem 12.7, p. 58].

Remarks:

(i). R. McKissick [6] constructs a uniform algebra A which is regular on X and such that A = C(X). By theorem 3.1, there are closed ideals in A which are not M-ideals.

(ii). It is natural to ask whether theorem 3.1 is valid without the regularity assumption. An inspection of the proof of theorem 3.1 reveals that a key element was the fact that if the kernel of a closed set E in X is an M-ideal in a uniform algebra A, then E is a p-set. In fact, P.C. Curtis and A. Figa-Talamanca [3, Theorem 4.1.] suggest that it is true in general that E is a p-set, however, the next example shows that on the contrary E is not a p-set even if E is A-convex. An additional assumption (E must also be closed in hull-kernel topology or A must be regular) is necessary. First we need a definition.

3.2. **Definition.** Let A be a uniform algebra on a compact Hausdorff space X and E be a non-empty subset of X. The A-hull of E is the set,

$$\tilde{E}=\{a\epsilon\Phi_A;\ |f(a)|<\sup_{x\epsilon E}|f(x)| \text{ for all f in A}\},$$

where Φ_A is the maximal ideal space of A and f is the Gelfand transform of f [2]. A subset E is said to be A-convex if and only if $E=\tilde{E}$.

3.3. **Counterexample.** Let $D=\{z\epsilon C: |z|\leqslant 1\}$ be the closed unit disc and A (D) be the **disc algebra** the space of all continuous functions on D which are analytic in the interior of D. Let

$$X=\{z\epsilon C; |z|<2\} \text{ and } A=\{f\epsilon C(X);\ f|_D\epsilon A(D)\}.$$

Let $E=\{z\ \epsilon\ C;\ |z|<\frac{1}{2}\}$ Then the following are true;

(i) A is a uniform algebra on X.

(ii) $\Phi_A \doteq X$, where Φ_A is the maximal ideal space of A.

(iii) If $J_E=\{f\epsilon A;\ f(E)=\{O\}\}$, then J_E is a closed ideal such that $hull(J_E)=D$, where

$$hull(J_E)=\{\phi\ \epsilon\ \Phi_A:\ \phi(f)=O \text{ for all f in } J_E\}.$$

(iv) J_E contains a bounded approximate identity.

(v) E is A-convex.

(vi) E is not a p-set.

Proof. (i) Obvious.

(ii) The map $A \longrightarrow A(D)$, $(f \longrightarrow f|D)$, is a continous surjection with kerenel J_D. Therefore, $A/J_D=A(D)$. We can apply [7, Theorem 3.1.17] to obtain

$$D\doteq\Phi_{A(D)}=\text{hull}\ (J_D) \text{ in } \Phi_A.$$

Now by [7, Theorem 3.1.18],

$$\Phi_A\backslash \text{hull}\ (J_D) \cong \Phi_{J_D},$$

so

$$\Phi_A\backslash D\cong\Phi_{J_D}.$$

Hence

$$\Phi_{J_D}=\{z\epsilon C;\ 1<|z|\leqslant 2\}.$$

Therefore, $\Phi_A \subset X$. Clearly $X\subset\Phi_A$, so $\Phi_A \doteq X$.

(iii) Note that, $J_E=J_D$, so $hull(J_E)=hull(J_D)=D$.

(iv) Since $J_E = J_D$ and D is a peak set for A, therefore J_E has a bounded approximate identity by Theorem 2.1..

(v) Since $\Phi_A \doteq X$, it follows that the A-hull of E is

$$\tilde{E}=\{z\epsilon X;|f(z)|\leqslant \sup_{x\epsilon E} |f(x)| \text{ for all f in A}\}.$$

Clearly $E\subset\tilde{E}$. for the reverse inclusion it suffices to consider $f_O(z)=z$. Hence $E=\tilde{E}$ and therefore E is A-convex.

(vi) Since X is a metrizable space, it follows that the notion of peak set and p-set coincide. Suppose E is a peak set. Then, by defintion, there is a function f in A such that

$$f(z)=1 \text{ for all z in E, and } |f(\lambda)|<1 \text{ if } \lambda\epsilon X\backslash E.$$

Since f(z)=1 for z in D, we obtain a contradiction. Thus E is not a peak set.

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