CHARACTERIZATIONS OF EXTREMELY AMENABLE FUNCTION ALGEBRAS ON A SEMIGROUP

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Abstract

Let S be a semigroup. In certain cases we give some characterizations of extreme amenability of S and we show that in these cases extreme left amenability and extreme right amenability of S are equivalent. Also when S is a compact topological semigroup, we characterize extremely left amenable subalgebras of $C(S)$, where $C(S)$ is the space of all continuous bounded real valued functions on S.

Introduction

Let S be a semigroup, B(S) the Banach algebra of all bounded real valued functions on S, βS the Stone-Čech compactification of S, $Ml(S) \subseteq B(S)^*$ the linear span of the set of left invariant means on S and $Mr(S) \subseteq B(S)^*$ the linear span of the set of right invariant means on S. Granirer [2, Theorem 3] has shown that S is extremely left amenable if and only if βS has a right zero. We show that when S has a cancellative left ideal, S is extremely left amenable if and only if βS has a right zero. Also when S has a cancellative ideal (or when $0<\dim Ml(S)<\infty$ and $0<\dim Mr(S)<\infty$), extreme left amenability, extreme right amenability and existence of a unique multiplicative invariant mean on S are equivalent. Similar results are proved for compact topological semigroups.

In addition, if S is a compact topological semigroup and $C(S)$ the set of all continuous functions in $B(S)$, two characterizations of extremely left amenable subalgebras of $C(S)$, which had been given for $B(S)$ by Granirer [3, Theorem 5], are given with a different proof.

Some Notations

Let K be the intersection of all ideals of the compact semigroup S. By [4, Theorem 9.21] $K \neq \varnothing$ and it is the minimal ideal of S.

Let $f \in B(S)$, for $a \in S$ define ${}_af(t)=f(at)$ $[f_a(t)=f(ta)]$ the left [right] translation of f by a. If A is a left invariant subalgebra of $B(S)$ (i.e. A is an algebra and ${}_af \in$ A for all $f \in$ A and all $a \in S$), then the ideal H_l(A) of A is the set of all $h \in A$ which have a representation $h=\sum_{i=1}^{n} f_i(g_i - {}_{a_i}g_i)$ where $f_i, g_i \in A, a_i \in S$, $i=1, \ldots, n$. The ideal $H_r(A)$ is defined in a similar way and $H(A)$ is the ideal of A containing all h of the form $h=\sum_{i=1}^{n} f_i(g_i - {}_{a_i}g_i)+\sum_{j=1}^{m} s_j(t_j - t_{j\,b_j})$ where $f_i, g_i, s_j, t_j \in A, a_i, b_j \in S$.

A (left, right) invariant subalgebra A of $B(S)$ which contains constants is called extremely (left, right) amenable denoted by EA (ELA, ERA), if it admits a multiplicative (left, right) invariant mean. When A = B(S) is extremely (left, right) amenable, we say that S is extremely (left, right) amenable.

ELA Semigroups

Throughout this section S denotes a semigroup.

Theorem 3.1. If the semigroup S has a right [left] cancellative ideal I then the following are equivalent:

(i) S is extremely amenable,

(ii) S is ELA [ERA],

(iii) $|I| = 1$,

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(iv) S has a zero.

Proof. (i) $\Rightarrow$ (ii) Trivial.

(ii) $\Rightarrow$ (iii) Let $x_1, x_2 \in I$, by [2, Theorem 3] there is a $z \in S$ such that $x_1 z = x_2 z = z$ thus $z \in I$ and by assumption we can cancel z from both sides of $x_1 z = x_2 z$ i.e. $x_1 = x_2$.

(iii) $\Rightarrow$ (iv) Clearly the single element of I is a zero of S.

(iv) $\Rightarrow$ (i) Let $a \in$ S be the zero element, then M defined by $M(f) = f(a)$ for $f \in B(S)$ is obviously a multiplicative invariant mean.

Corollary 3.2. If the semigroup S has a cancellative ideal I or if $0 < dim\, ML(S) < \infty$ and $0 < dim\, Mr(S) < \infty$, then the following are equivalent:

(i) S is extremely amenable,

(ii) S is ELA,

(iii) S is ERA

(iv) S has a zero.

Proof. If $0 < dim\, Ml(S) < \infty$ and $0 < dim\, Mr(S) < \infty$, then by [1, Theorem 1] S has an ideal which is a group. Therefore, the corollary follows from Theorem 3.1.

ELA Function Algebras on Compact Semigroups

Theorem 4.1. The following conditions on a compact topological semigroup S are equivalent:

(i) $B(S)$ has a unique multiplicative invariant mean,

(ii) S is extremely amenable,

(iii) $|K| = 1$,

(iv) S has a zero element,

(v) K has a zero element.

Proof. (i) $\Rightarrow$ (ii) Trivial.

(ii) $\Rightarrow$ (iii) Since $H(CB(S)) \subseteq H(B(S))$, then by [3, Theorem 2] $C(S)$ is extremely amenable and by [5, Corollary 1] K is a group. Let $x_1, x_2 \in K$, by [2, Theorem 3] there is a $z \in S$ such that $x_1 z = x_2 z = z$ thus $z \in K$ and since K is a group, then $x_1 = x_2$.

(iii) $\Rightarrow$ (i) Clearly the single element z of K is a zero of S, hence M on $B(S)$ defined by $M(f) = f(z)$ is a multiplicative invariant mean on $B(S)$. To prove the uniqueness of M let M_1 be another multiplicative invariant mean on $B(S)$. Since $l_z(\chi_{S-\{z\}}) = 0$, then $M_1(\chi_{S-\{z\}}) = M_1(l_z \chi_{S-\{z\}}) = 0$, hence $M_1(X_{\{z\}} + X_{S-\{z\}}) = M_1(1) = 1$. Therefore, $M_1(\chi_{\{z\}}) = 1$. Thus for all $f \in B(S)$. $M_1(f) = M_1(f\chi_{\{z\}}) = f(z) M_1(\chi_{\{z\}}) = f(z) = M(f)$ i.e. $M_1 = M$.

(iii) $\Rightarrow$ (iv) Trivial.

(iv) $\Rightarrow$ (v) Let z be the zero of S and $a \in K$, since $z = az \in K$, then z is the zero element of K.

(v) $\Rightarrow$ (iii) By [4, Corollary 9.24] K is a union of pair wise disjoint groups. Let z be the zero element of K and $G \subseteq K$ be a group that contains z, since for all $g \in G$ we have $gz = z = z^2$ then $g = z$ i.e. $G = \{z\}$. Suppose $G' \subseteq K$ is another group of the above type with identity e. Since $ze = z = ez$ and K is completely simple [4, Theorem 9.21], then $z = e$. Therefore $G' = G = \{z\}$ i.e. $K = \{z\}$.

Remark. The following theorem has been proved by E. Granirer for $B(S)$ [3, Theorem 5]. But we state it for uniformly closed subalgebras of $C(S)$. Part (i) $\Rightarrow$ (ii) of proof depends on the compactness of S and is completely different from the work of Granirer, but other parts are similar.

Theorem 4.2. Let S be a compact right [left] topological semigroup and A be a uniformly closed left [right] invariant subalgebra of $CB(S)$. The following are equivalent:

(i) A is ELA [ERA],

(ii) For any finite subset $\{g_1, ..., g_n\}$ of A and any $\{a_1, ..., a_n\} \subseteq S$ there is an $a \in S$ such that for $1 \leq i \leq$ n we have: $\left(g_i - {}_{a_i}g_i\right)(a) = 0 \left[\left(g_i - g_{i a_i}\right)(a) = 0\right]$

(iii) Every $h \in H_l(A)$ [$h \in H_r(A)$] has a zero in S.

Proof. (i) $\Rightarrow$ (ii) Let $\{g_i, ..., g_n\} \subseteq$ A and $\{a_l, ..., a_n\} \subseteq S$. Since $h = \sum_{i=1}^{n} -(g_i - {}_{a_i}g_i)^2 \in H_l(A)$, then by [3, Theorem 2] we have $\sup_{x \in S} h(\chi) \geq 0$ and hence $\sup_{x \in S} h(\chi) = 0$. Since h is continuous and S is compact, then h takes its supremum at a point $a \in S$. Thus $\sum_{i=1}^{n} -(g_i - {}_{a_i}g_i)^2(a) = h(a) = 0$. Therefore $(g_i - {}_{a_i}g_i)(a) = 0$ for $i = 1, ..., n$.

(ii) $\Rightarrow$ (iii) Let $h = \sum_{i=1}^{n} f_i(g_i - {}_{a_i}g_i) \in H_l(A)$ and a be a common zero of $(g_l - {}_{a_l}g_l), ..., (g_n - {}_{a_n}g_n)$. Clearly $h(a) = 0$.

(iii) $\Rightarrow$ (i) This follows from [3, Theorem 2].

Remark. Compactness in Theorem 4.1 is necessary since if $S = R$ with the usual topology and multiplication given by $x, y = max\{x, y\}$, then S is extremely amenable but does not have any zero element.

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