THE DERIVED GROUP OF A SEIFERT FIBRE GROUP

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Abstract

Every Seifert Fibre Group is the lift of a Fuchsian group to the universal covering group of PSL (2,**R**). From this, we work out a form of presentation for such a group. With the calculation of the Euler number, we can establish the presentation of the derived group of a Seifert Fibre Group.

Introduction (§ 1)

In [6] Milnor showed that the fundamental group of a 3-dimensional Brieskorn Manifold M(p,q,r) is the derived group of the lift of the triangle group T(p,q,r) to the universal covering group $\tilde{G}$ of the group G = PSL (2, R). A paper was issued on Fuchsian groups by A. M. MacBeath in 1965, [4], and under his supervision this author carried out further research on the subject as a background to the present work. Now, in this paper, the more general problem of finding the derived groups of lifts of Fuchsian groups Γ to $\tilde{G}$ is considered and is presented as an original work. The lift of a Fuchsian group Γ is a discrete subgroup $\tilde{\Gamma}$ of $\tilde{G}$ and is isomorphic to the fundamental group of $\tilde{G}/\tilde{\Gamma}$ which is a Seifert Fibre Space. For this reason we call the groups $\tilde{\Gamma}$ the **Seifert Fibre Groups (SF-GPs).**

Since Γ is acting on the upper half-plane $H^2 = \{z \in C: \mathrm{Im}(z) > 0\}$, it will be called **co-compact** if H^2/Γ is compact. With these hypotheses we start our work in section 2 by calculating the presentation of Γ' the derived group of a Fuchsian group Γ. Prior to the work we recall that:

1.1 Every finitely generated Fuchsian group Γ has a presentation of the following form:

generators: $a_1,b_1,a_2,b_2,\ldots,a_g,b_g$; $x_1,x_2,\ldots,x_r$; $p_1, p_2, \ldots, p_s$; $h_1,h_2,\ldots, h_t$; and

relations: $x_i^{m_i} = 1, i = 1,2,\ldots, r$;

$$\Pi_{i=1}^{r} x_i \Pi_{j=1}^{g}[a_j, b_j] \Pi_{k=1}^{s} p_k \Pi_{l=1}^{t} h_l = 1;$$

where $[a_j, b_j] = a_j b_j a_j^{-1} b_j^{-1}$; integers $m_i \geq 2$ called **periods of** Γ ; and the integer $g \geq 0$ the **genus of** Γ.

In the case of Γ a co-compact Fuchsian group we get s=t=0.

1.2 A **group H** is called a **Seifert Fibre Group** if it admits at least one presentation of the following form:

generators: $\xi_1,\xi_2,\ldots,\xi_r;\alpha_1,\beta_1,\alpha_2,\beta_2,\ldots\ldots,\alpha_g,\beta_g;\zeta$;

relations: $\xi_i^{m_i} = \zeta^{q_i}, i = 1,2,\ldots, r$;

$$\Pi_{i=1}^{r}\xi_i \Pi_{j=1}^{g}[\alpha_j, \beta_j] = \zeta^{q_0};$$

ζ commutes with every element;

where q_i also q_0 are integers. We will call the second relation the **long** relation, and all the others the **short** relations.

The Derived Group of a Co-Compact Fuchsian Group (§ 2)

Let Γ be a co-compact Fuchsian group with

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signature $(0; m_1, m_2, \ldots, m_r)$ and let

$$G = \langle s_i : m_i s_i = 0,\ i=1,2,\ldots,r \text{ and } \Sigma_{i=1}^{r} s_i = 0 \rangle$$

be the additive group isomorphic to the abelian group Γ / Γ'. By (p,q) and $[p,q]$ we mean the greatest common divisor (g.c.d.) and least common multiple (l.c.m.) of p and q, respectively, then we have the following:

2.1 Theorem

(a) If s_i for each $i \in \{1,2,\ldots,r\}$ has order l_i then

$$l_i = (m_i, [m_1, m_2, \ldots, m_{i-1}, m_{i+1}, \ldots, m_r]).$$

(b) If the index of Γ' is k then

$$k = \text{g.c.d.}\ \{\frac{m}{m_1}, \frac{m}{m_2}, \ldots, \frac{m}{m_r}\},$$

where $m = m_1 m_2 \ldots m_r$.

Proof of (a)

Suppose ϕ is the natural homomorphism $\phi : \Gamma \to \Gamma/\Gamma'$ defined by $\phi(x_i) = s_i$, $i=1,2,\ldots,r$ then

$l_i \mid m_i$ for every $i \in \{1,2,\ldots,r\}$,

and from the operation of ϕ on the relation $\Pi_{i=1}^{r} x_i = 1$ we get

$$s_i = s_r^{-1} \ldots s_{i+1}^{-1} s_{i-1}^{-1} \ldots s_1^{-1} \text{ for each i.}$$

Let us denote $[m_1, m_2, \ldots, m_{i-1}, m_{i+1}, \ldots, m_r]$ by $[\hat{m}_i]$ then

$$s_i^{[\hat{m}_i]} = (s_1^{[\hat{m}_i]} \ldots s_{i-1}^{[\hat{m}_i]}\ s_{i+1}^{[\hat{m}_i]} \ldots s_r^{[\hat{m}_i]})^{-1} = 1,$$

thus $l_i \mid [\hat{m}_i]$ for all i,

hence $l_i \mid (m_i, [\hat{m}_i])$ for all $i \in \{1,2,\ldots,r\}$......... $\oplus$

Next consider a cyclic group $\langle z \rangle$, say of order p^h for p any prime number and h chosen a follows:

For each i, let h_i be the greatest number such that $p^{h_i} \mid m_i$ but $p^{h_i+1} \nmid m_i$. Denote by $h_{j\hat{i}}$ the maximum of $\{h_1, \ldots, h_{i-1}, h_{i+1}, \ldots, h_r\}$, then h is taken to be the maximum of $\{h_i, h_{j\hat{i}}\}$.

Thus $p^h \mid (m_i, [\hat{m}_i])$ but $p^h \nmid (m_{i+1}, [\hat{m}_{i+1}])$. A homomorphism $\psi : \Gamma \to \langle z \rangle$ defined by

$$\psi(x_i) = z,$$
$$\psi(x_j) = z^{-1},\ j \neq i \text{ and}$$
$$\psi(x_\nu) = 1 \text{ for } \nu \neq i, j,$$

ensures that $\Gamma \supset \ker \psi \supset \Gamma'$.

Thus ψ is a homomorphic map from Γ onto the abelian group $\langle z \rangle$ with $\psi(x_i)$ of order p^h. Then l_i (the order of s_i) is divisible by p^h, i.e. $p^h \mid l_i$, hence $(m_i, [\hat{m}_i]) \mid l_i$ ....$\otimes$

The relations $\oplus$ and $\otimes$ together imply that

$$l_i = (m_i, [\hat{m}_i]) \text{ for each i.}$$

Proof of (b)

We proceed by calculating the order of the additive group G mentioned above. Since the group G is isomorphic to the direct sum of the groups $Z_{d_1}, Z_{d_2}, \ldots, Z_{d_r}$ where d_i are the elementary divisors of the matrix

$$\begin{bmatrix} m_1 & & & & & \\ & m_2 & & & & \\ & & \cdot & & 0 & \\ & & & \cdot & & \\ & 0 & & & \cdot & \\ & & & & & m_r \\ 1 & 1 & \cdot & \cdot & \cdot & 1 \end{bmatrix},$$ (see Hartley and Hawkes [2]),

then we can calculate the greatest common divisor of $r \times r$ minors to get the order of G. A straightforward calculation of d_i for each i leads us to the relation $d_1 d_2 \ldots d_r = \text{g.c.d.}\ \{m_1 m_2 \ldots m_{i-1} m_{i+1} \ldots m_r,\ i = 1,2,\ldots,r\}$ as the order of G, and it is obvious that

$$\text{g. c. d.}\{m_1 \ldots m_{i-1} m_{i+1} \ldots m_r,\ i = 1,2,\ldots,r\} = \text{g. c. d.}\ \{\frac{m}{m_1}, \frac{m}{m_2}, \ldots, \frac{m}{m_r}\}$$

and thus k = order of $G = (\frac{m}{m_1}, \frac{m}{m_2}, \ldots, \frac{m}{m_r})$.

Moreover, $k = \frac{m}{[m_1, m_2, \ldots, m_r]}$ which we will use later on. ▲

2.2 Splitting of the Stabilizer Classes of a Co-Compact Fuchsian Group

Let Γ be a co-compact Fuchsian group, then for each element x of finite order there is a unique maximal finite cyclic subgroup C_Γ (x) - the centralizer of x in Γ. Since the **periods** m_i of Γ are the orders of C_Γ (x), and the latter fall in finitely many conjugacy classes called **stabilizer classes**, then they are finitely many. In fact, with presentation 1.1 they are just r numbers.

Now let $\Gamma_1 < \Gamma$ have finite index. Thus by the above argument, if C is a stabilizer class in Γ and $<x>$ a subgroup of C then $<x> \cap \Gamma_1$ is either the trivial subgroup or a maximal finite cyclic subgroup of Γ_1. By the non-trivial intersection, we obtain the stabilizer class

$$(<x> \cap \Gamma_1)^{\Gamma_1} \text{ in } \Gamma_1,$$

and hence the family

$$S_p(C) = \{(<x> \cap \Gamma_1)^{\Gamma_1}, <x> \in C : <x> \cap \Gamma_1 \neq \{1\}\} \text{ of}$$

them.

The following theorem is proved by D. Singerman (see [7]):

2.2.1 Singerman's Theorem

Suppose that the cycle lengths of the permutation $\theta(x)$ are

$$l_1 \le l_2 \le \ldots \le l_s < m = l_{s+1} = l_{s+2} = \ldots = l_{s+t}$$

where m is the order of x. Then S_p (c) consists of s stabilizer classes of periods $\frac{m}{l_1}, \frac{m}{l_2}, \ldots, \frac{m}{l_s}$.

Using Singerman's theorem in the special case that Γ_1 is a normal subgroup of finite index k in the co-compact group Γ, we get the following corollary:

2.2.2 Corollary

With the above-mentioned hypothesis and d_i the order of x_i, each stabilizer class S_p (C_i) is either empty or consists of k/d_i classes each of period m_i/d_i.

Proof

Establishing S_p (C) for each $i \in \{1,2,\ldots,r\}$ ensures that the period class $C_i = <x_i>^\Gamma$ splits in Γ_1 into S_p (C_i) and the number of classes as well as their periods can be calculated. All these splittings are disjoint and their union equals the complete set of stabilizer classes in Γ_1. Γ_1 being a normal subgroup implies that the permutations θ (x_i) are just left translations in the factor group Γ / Γ_1, so that if $x_i\Gamma_1$ has order d_i not equal to m_i, S_p (C_i) consists of k/d_i classes each of period m_i/d_i, and it is empty if $d_i = m_i$. ▲

Next we consider the commutator subgroup Γ' in the place of Γ_1 whence we will have k/l_i classes of periods m_i/l_i. Denote these by k_i and n_i, respectively, for the ease of usage. To find the genus g of Γ', we use the **Euler characteristic** χ (Γ) given as:

$$\chi(\Gamma) = 2 - 2g - \Sigma_{i=1}^{r}(1 - \frac{1}{m_i})$$

for any co-compact Fuchsian group Γ with signature (g ; m_1 , m_2 ,..., m_r) as well as the **Riemann-Hurwitz formula**

$$\chi(\Gamma_1) = k\chi(\Gamma)$$

where Γ_1 is a subgroup of finite index k in Γ. Thus we obtain

$$2 - 2g - \Sigma_{i=1}^{r} k_i(1 - \frac{1}{n_i}) = k[2 - 0 - \Sigma_{i=1}^{r}(1 - \frac{1}{m_i})],$$

and by substituting the values of k_i , n_i , l_i , and k meanwhile denoting l.c.m. {m_1 , m_2 ,..., m_r} by c, and $\frac{m_i}{c}$ [^m]. by λ_i, we find:

$$g = 1 - \frac{m}{2c}(2 - r + \Sigma_{i=1}^{r}\frac{1}{\lambda_i}).$$

Hence we have

$$\Gamma' = <t_{11}, \ldots, t_{1k_1}; t_{21}, \ldots, t_{2k_2}; \ldots; t_{r1}, \ldots, t_{rk_r};$$

$$a_1, b_1, a_2 b_2, \ldots \ldots, a_g, b_g;$$

$$t_{ij}^{n_i} = 1, i = 1,2,\ldots,r, j = 1,2,\ldots,k_i, \text{ and}$$

$$\prod_{\substack{i=1 \\ j=1,2,\dots,k_i}}^{r} t_{ij} \prod_{\nu=1}^{g} [a_\nu, b_\nu] = 1 >.$$

The SF-GP $\tilde{\Gamma}$ (§3)

Let G be the group PSL (2,R) and $p : \tilde{G} \to G$ the natural projection map from the universal covering space $\tilde{G}$ onto G. Now every transformation of G can be written uniquely in the form TR where R is an elliptic transformation fixing i and T which is defined by:

$$T(z) = az + b, \quad a,b \in R, a > 0.$$

Thus G is homeomorphic to $R^2 \times S^1$ and so $\pi_1(G) \cong \pi_1(S^1) \cong Z$ giving a short exact sequence

$$\{1\} \to Z \to \tilde{G} \xrightarrow{p} G \to \{1\}$$

where z = ker p is central in $\tilde{G}$. (See Hilton-Wylie [3] p. 268). If ζ denotes a generator of Z then ζ is represented by a homotopy class of paths $f: I \to G$ which we suppose is given by

$$f(t) = \begin{bmatrix} \cos\pi t & \sin\pi t \\ -\sin\pi t & \cos\pi t \end{bmatrix}$$

Denoting the inverse image of Γ under the map p by $\tilde{\Gamma}$ we also have a short exact sequence

$$\{1\} \to Z \to \tilde{\Gamma} \xrightarrow{p} \Gamma \to \{1\}$$

with Z again generated by ζ as above.

We now choose generators $\xi_i \in p^{-1}(x_i)$ for $\tilde{\Gamma}$. First of all, each x_i is represented by a matrix of the form $T_i M_i T_i^{-1}$ where $T_i \in SL(2,R)$ and

$$M_i = \begin{bmatrix} \cos\frac{\pi}{m_i} & +\sin\frac{\pi}{m_i} \\ \pm\sin\frac{\pi}{m_i} & \cos\frac{\pi}{m_i} \end{bmatrix}, i = 1,2,\dots,r.$$

There are two choices of signs for the top right and bottom left elements, and the same signs chosen for i = 1,2,..., r. This choice of signs corresponds to choosing x_i as clockwise or anti-clockwise rotations. However, Γ is isomorphic to a subgroup of index 2 in a group Γ_1 which contains orientation-reversing transformations (see Singerman [9]), so there is an automorphism of Γ induced by an orientation-reversing homeomorphism of H^2. Replacing the x_i by their images under this automorphism and, if necessary, we may assume that we have the + sign in the top right-hand corner of the matrix M_i, we choose ξ_i equal to the homotopy class of $[g_i]$ in $\tilde{G}$ where g_i is the path defined by

$$g_i(t) = T_i f\left(\frac{t}{m_i}\right) T_i^{-1}$$

for f as above. As G is path-connected, then any path f_1 defined by

$$f_1(t) = T_i f(t) T_i^{-1} \quad \text{for all i},$$

is homotoic to f, hence we have

$$\xi_i^{m_i} = [f_1] = \zeta.$$

From the relation $\prod_{i=1}^{r} x_i = 1$ we can deduce that

$$\xi_1 \xi_2 \dots \xi_r = \zeta^l$$

where l is an integer and is calculated as follows:

For the rotations x_i, i = 1,2,..., r with a fixed orientation choose a fixed point z in H^2 and denote by $r_z(\theta)$ the rotation through an angle θ about the point z for any real number θ. This leads to a homomorphism

$$r_z : R \to G$$

which is clearly lifted to a unique homomorphism

$$\tilde{r}_z : R \to \tilde{G},$$

since $r_z(2\pi)$ is identity then its lift $\tilde{r}_z(2\pi)$ belongs to the central subgroup C of $\tilde{G}$. Thus we can suppose that $\tilde{r}_z(2\pi) = \zeta$ is the generating element of the cyclic group C. This element is continuously dependent on z hence independent of choice of it. So the generators x_i of Γ are

$$x_i = r_{v_i}\left(2\frac{\pi}{m_i}\right), i = 1,2,\dots,r$$

where v_i are the vertices of a convex r-sided polygon A with interior angles $\alpha_i = \frac{\pi}{m_i}$. A is a fundamental domain for the group Γ_1 of isometries of H^2 which is generated by reflections σ_i, i = 1,2,..., r in the edges of A. Since $m_i \geq 2$ then $0 < \alpha_i < \pi$ and we have

$$\sigma_i^2 = 1,$$

therefore $(\sigma_1 \sigma_2)(\sigma_2\sigma_3) \ldots (\sigma_{r-1}\sigma_r)(\sigma_r\sigma_1) = 1$.

Lifting each rotation $\sigma_i\sigma_{i+1} = r_{v_i}(2\alpha_i) \in G$ to the element $\xi_i = \tilde{r}_{v_i}(2\alpha_i) \in \tilde{G}$ implies that $\Pi_{i=1}^{r}\xi_i$ belongs to C. Now if A varies continuously then $\Pi_{i=1}^{r}\xi_i$ will vary continuously. But C is a discrete group, so it implies that $\Pi_{i=1}^{r}\xi_i$ remains constant. In particular we shrink A towards the point z such that angles a_i tend towards angles β_i of some Euclidean r-sided polygon. Thus, $\xi_i = \tilde{r}_{v_i}(2\alpha_i)$ tends to the limit $\tilde{r}_z(2\beta_i)$ for each i, while $\Pi_{i=1}^{r}\xi_i$ tends to the product $\tilde{r}_z(2\beta_1 + 2\beta_2 + \ldots + 2\beta_r)$. Since $\Pi_{i=1}^{r}\xi_i$ is constant and $\Sigma_{i=1}^{r}\beta_i = (r-2)\pi$ in Euclidean polygon, then

$$\Pi_{i=1}^{r}\xi_i = \tilde{r}_z\big((r-2)\,2\pi\big) = \zeta^{r-2}.$$

Hence we get the following presentation for $\tilde{\Gamma}$:

generators: $\xi_1, \xi_2, \ldots, \xi_r$; ζ, and

relations: $\xi_i^{m_i} = \zeta, i = 1,2,\ldots,r;$

$$\Pi_{i=1}^{r}\xi_i = \zeta^{r-2};\ \zeta\xi_i\zeta^{-1} = \xi_i, \forall i.$$

Let us verify the property of the map p on co-compact groups of any genus:

3.1 Corollary

If Γ and Γ_1 are isomorphic co-compact Fuchsian groups then they have isomorphic inverse images under the map p, i.e.

$$p^{-1}(\Gamma) \cong p^{-1}(\Gamma_1).$$

Proof

Let $R_o(\Gamma,G)$ denote the space of monomorphisms r: $\Gamma \to G$ such that r (Γ) is a co-compact Fuchsian group. $R_o(\Gamma,G)$ is topologized as a subgroup of $G^{(\Gamma)}$ and it is known that it consists of the union of two disjoint connected manifolds $R_o^+(\Gamma,G)$ and $R_o^-(\Gamma,G)$, where $R_o^+(\Gamma,G)$ is the group of all isomorphisms which can be induced by orientation-preserving homeomorphisms of H^2 including the inclusion map i : $\Gamma \subset G$. (See MacBeath and Singerman [5] for more details and terminology). Γ_1 being isomorphic to Γ means that there is a map $r_1 \in R_o^+(\Gamma,G)$ such that $r_1(\Gamma) = \Gamma_1$. Define a path of monomorphisms r_t for $0 < t < 1$ with initial point $r_o = i$ and end point r_1, and set $r_t(x_i) = x_i(t)$, $r_t(a_j) = a_j(t)$, $r_t(b_j) = b_j(t)$ for i = 1,2,..., r and j = 1,2,..., g. These can be uniquely lifted to $\xi_i(t)$, $\alpha_j(t)$, and $\beta_j(t)$ in the universal covering group $\tilde{G}$ respectively, so the initial points $\xi_i(0) = \xi_i$, $\alpha_j(0) = \alpha_j$, and $\beta_j(0) = \beta_j$ are determined. Now we deduce that for the relations defining Γ and therefore $r_t(\Gamma)$ for each t, there are integers $q_o(t)$, $q_1(t)$,..., $q_r(t)$ such that the following holds:

$$\big(\xi_i(t)\big)^{m_i} = \zeta^{q_i(t)} \quad, i = 1,2,\ldots,r, \text{and}$$

$$\Pi_{i=1}^{r}\xi_i(t)\,\Pi_{j=1}^{g}[\alpha_j,\beta_j] = \zeta^{q_o(t)}$$

Since $q_i(t)$ above are continuous functions of t with integer values then they are constants and $q_i(1) = q_i(0)$ for all i, thus they establish the isomorphism between $p^{-1}(\Gamma)$ and $p^{-1}(\Gamma_1)$.

Presentation of $p^{-1}(\Gamma')$ (§4)

Suppose ϕ is the natural homomorphism which takes Γ of genus zero onto Γ/Γ'. Then the composition map ϕ^o p will take $\tilde{\Gamma}$ onto Γ/Γ', thus $p^{-1}(\Gamma')$ will be a centrally extended group of Γ' hence it will have a presentation of the following form:

generators: τ_{ij}, i = 1,2,..., r, j = 1,2,..., k_i;

α_v, β_v, v = 1,2,..., g;

ζ; and

relations: $\tau_{ij}^{n_i} = \zeta,$

$$\Pi_{\substack{i=1\\ j=1,2,\ldots,k_i}}^{r}\tau_{ij}\,\Pi_{v=1}^{g}[\alpha_v,\beta_v] = \zeta^{L},$$

and ζ commutes with every element;

where each one of the generators is chosen in the inverse image of corresponding elements of Γ' under the map p and we have:

$$p(\tau_{ij}) = t_{ij}, i = 1,2,\ldots, r, j = 1,2,\ldots, k_i,$$

$p(\alpha_\nu) = a_\nu$, $p(\beta_\nu) = b_\nu$, $\nu = 1,2,\ldots, g$, and

$p(\zeta) = 1$.

We wish to calculate the integer L. First we give some definitions and a theorem. If a group H with the following presentation:

generators: $\xi_i, \alpha_j, \beta_j, \zeta$, $i = 1,2,\ldots, r$, $j = 1,2,\ldots, g$;

relations: $\xi_i^{m_i} = \zeta^{q_i}$,

$\Pi_{i=1}^{r} \xi_i \, \Pi_{j=1}^{g} [\alpha_j, \beta_j] = \zeta^{q_o}$, and

ζ commutes with every element,

becomes a Fuchsian group when we factor out its centre $<\zeta>$, then the rational number $\left| q_o - \Sigma_{i=1}^{r} \frac{q_i}{m_i} \right|$ is called the **Euler number** of the group H and is denoted by **e(H)**. Bailey proved that this number is an invariant characteristic of a SF-GP up to isomorphism (see [1]).

4.1 Theorem

If Γ is co-compact Fuchsian group then

$$e(p^{-1}(\Gamma)) = -\chi(\Gamma)$$

where p is the projection map.

Proof

Let Γ have the following presentation:

generators: x_i, $i = 1,2,\ldots, r$,

a_j, b_j, $j = 1,2,\ldots, g$; and

relations: $x_i^{m_i} = 1$,

$$\Pi_{i=1}^{r} x_i \, \Pi_{j=1}^{g} [a_j, b_j] = 1.$$

We give the complete proof by splitting it into three cases:

Case 1: Γ has genus zero. Then its Euler characteristic $\chi(\Gamma)$ is $2 - \Sigma_{i=1}^{r}(1 - \frac{1}{m_i})$ by definition, and from the presentation of $\tilde{\Gamma} = p^{-1}(\Gamma)$, as obtained in section 3, we get $e(p^{-1}(\Gamma)) = -2 + r - \Sigma_{i=1}^{r} \frac{1}{m_i}$ and hence the relation

$$e(p^{-1}(\Gamma)) = -\chi(\Gamma)$$

holds.

Case 2: Γ has genus $g \geq 1$ and every period m_i has even multiplicity. We can suppose that Γ has period partition $(m_1, m_1, m_2, m_2, \ldots, \ldots, m_r, m_r)$. Let Γ_2 be a group with genus zero and $2g + 2 + r$ stabilizer classes such that $2g + 2$ classes have period 2, and r classes have periods $m_1, m_2, \ldots, m_r$. Then by calculating in the same way as in section 3, we find that $p^{-1}(\Gamma_2)$ has a presentation of the following form:

generators : $\lambda_1, \lambda_2, \ldots, \lambda_{2g+2}, \xi_1, \xi_2, \ldots, \xi_r, \zeta$, and

relations : $\lambda_i^2 = \xi_j^{m_j} = \zeta$,

$$\Pi_{i=1}^{2g+2} \lambda_i \, \Pi_{j=1}^{r} \xi_j = \zeta^{2g+r}.$$

Define a homomorphism $\psi: p^{-1}(\Gamma_2) \to Z_2$ by

$\psi(\lambda_i) = 1 \pmod 2$, for all i, and

$\psi(\xi_j) = \psi(\zeta) = 0 \pmod 2$, for all j.

Since $\zeta \in \ker \psi$, then there is a map $\psi^*: \Gamma_2 \to Z_2$ such that:

$$\psi = \psi^* \, o \, p.$$

Denote ker ψ^* by Γ_3. By corollary 2.2.2, each stabilizer class of $<p(\xi_j)>^{\Gamma_2}$ in Γ_2 splits into two stabilizer classes in Γ_3 each of period m_j, but the stabilizer classes $< p(\lambda_i) >$ in Γ_2 do not make any change in the period partition of Γ_3. So by the Riemann-Hurwitz formula we get the genus of Γ_3 equal to g, and hence we get $\Gamma_3 \cong \Gamma$ and by corollary 3.1, $p^{-1}(\Gamma)$ is isomorphic to $p^{-1}(\Gamma_3) = p^{-1}(\ker \psi^*)$ which is just ker ψ.

Now we calculate the ker ψ from the presentation of $p^{-1}(\Gamma_2)$. Set:

$\lambda_1^2 = \lambda'$, $\lambda_1 \lambda_i = \rho_i$, $\lambda_i \lambda_1^{-1} = \gamma_i$ for $i = 2,3,\ldots, 2g+2$, and

$\lambda_1 \xi_j \lambda_1^{-1} = \xi_j'$ for $j = 1,2,\ldots, r$,

then we get

$$\lambda' = \zeta, \gamma_i = \rho_i^{-1}\zeta, \xi_j^{m_j} = \xi'^{m_j}_j = \zeta,$$

from the short relations, and

$$\rho_2\gamma_3\rho_4\gamma_5 \cdots \rho_{2g}\gamma_{2g+1}\rho_{2g+2}\xi_1\xi_2 \cdots \xi_r = \zeta^{2g+r},$$

$$\lambda'\gamma_2\rho_3\gamma_4 \cdots \rho_{2g+1}\gamma_{2g+2}\xi'_1\xi'_2 \cdots \xi'_r = \zeta^{2g+r},$$

from the long relation and its λ_1-conjugate.

By the above short relations, we can eliminate λ', $\gamma_1, \gamma_2, \ldots, \gamma_{2g+2}$ from those long relations, and get

$$\rho_2\rho_3^{-1} \cdots \rho_{2g+1}^{-1}\rho_{2g+2}\xi_1\xi_2 \cdots \xi_r = \zeta^{g+r},$$

$$\rho_2^{-1}\rho_3 \cdots \rho_{2g+1}\rho_{2g+2}\xi'_1\xi'_2 \cdots \xi'_r = \zeta^{g+r-2}.$$

If we find the value of ρ_2 from the second relation and replace it in the first one, we obtain

$$\Pi_{i=3}^{2g+2}\rho_i^{-\varepsilon_i}\Pi_{j=1}^{r}\xi'_j\Pi_{i=3}^{2g+2}\rho_i^{\varepsilon_i}\Pi_{j=1}^{r}\xi_j = \zeta^{2g+2r-2}$$

where $\varepsilon_i = (-1)^i$. If we abelianize the group and denote the corresponding elements by a bar on top of them, we get the sum of two groups, one a free abelian group generated by $\bar{\rho}_3, \bar{\rho}_4, \ldots, \bar{\rho}_{2g+2}$ which has rank 2g, and the other a torsion subgroup with presentation:

generators: $\bar{\xi}_1, \bar{\xi}_2, \ldots, \bar{\xi}_r, \bar{\xi}'_1, \bar{\xi}'_2, \ldots, \bar{\xi}'_r, \bar{\zeta}$, and

relations: $m_i\bar{\xi}_i = m_i\bar{\xi}'_i = \bar{\zeta}, i = 1,2,\ldots, r,$

$$\Sigma_{i=1}^{r}(\bar{\xi}_i + \bar{\xi}'_i) = (2g + 2r - 2)\bar{\zeta}.$$

Then the abelianized group has the rank 2g and its torsion subgroup has the order of $m_1m_2 \ldots m_r e(p^{-1}(\Gamma))$, hence we get the result.

Case 3: Γ has genus $g \geq 1$ and periods $m_1, m_2, \ldots, m_r$. Denote this by Γ_4, and define a map $\eta: \Gamma_4 \to Z_2$ by

$$\eta(a_1) = 1 \pmod 2,$$

$\eta(x) = 0 \pmod 2$ for $x \in \Gamma_4$, $x \neq a_1$ from presentation 1.1.

Then by Singerman's theorem and the Riemann-Hurwitz formula ker η, denoted by Γ_5, has genus 2g-1 and period partition $(m_1, m_1, m_2, m_2, \ldots, m_r, m_r)$. So its Euler characteristic $\chi(\Gamma_5)$ is $-4g - 2r + 4 + 2\Sigma_{\nu=1}^{r}\frac{1}{m_\nu}$ which, by Case 2, is equal to $-e(p^{-1}(\Gamma_5))$. Seeing as $p^{-1}(\Gamma_5) = \ker(\eta \circ p)$, we can work out a presentation for it in the same way as that calculated for $p^{-1}(\Gamma)$ in section 3, thus we get the periods $m_1, m_1, m_2, m_2, \ldots, m_r, m_r$ and an even power $2l$, say, for the central element ζ in the long relation. Then with this integer l, the Euler number is

$e(p^{-1}(\Gamma_5)) = 2l - 2\Sigma_{\nu=1}^{r}\frac{1}{m_\nu}$. Comparing the two values of $e(p^{-1}(\Gamma_5))$ implies that $l = 2g + r - 2$. With this value of l we get

$$e(p^{-1}(\Gamma_4)) = -\chi(\Gamma_4).$$

Thus all three cases together ensure that the theorem is true for every co-compact Fuchsian group Γ. ▲

Now with Theorem 4.1, we compare the Euler number of $p^{-1}(\Gamma')$, (with exponent L in the long relation)

$$e(p^{-1}(\Gamma')) = L - \Sigma_{i=1}^{r}\frac{k_i}{m_i},$$

and the Euler characteristic of Γ' from its presentation obtained in section 2,

$$\chi(\Gamma') = 2 - 2g - \Sigma_{i=1}^{r}k_i(1 - \frac{1}{m_i}),$$

and we get

$$L = 2g - 2 + \Sigma_{i=1}^{r}k_i.$$

Hence we have

$$p^{-1}(\Gamma') = \langle \tau_{ij}, i = 1,2,\ldots, r, j = 1,2,\ldots, k_i; \; \alpha_\nu, \beta_\nu, \nu = 1,2,\ldots, g; \; \zeta; \; (\tau_{ij})^{n_i} = \zeta, \; \prod_{\substack{i=1 \\ j=1,2,\ldots,k_i}}^{r}\tau_{ij}\Pi_{\nu=1}^{g}[\alpha_\nu, \beta_\nu] = \zeta^{2g-2+\Sigma_{i=1}^{r}k_i} \rangle.$$

4.2 Theorem

Suppose H and H_1 are two SF-GPs where H_1 is a subgroup of H with finite index k and contains $<\zeta>$ the centre of H. If the natural homomorphism maps H_1 onto the quotient group $H/<\zeta>$, then we get

$$e(H_1) = k\, e(H).$$

Proof

In the proof of Theorem 4.1-Case 2, it was shown that the abelianized group, denoted by any letter S/S', has rank 2g if e (S) is not zero. At the same time, one could easily show that rank would be greater than 2g if e (S) were zero, whence it would be true for its subgroup $S_1/(S'\cap S_1)$, for S_1 a subgroup of finite index in S, with e (S_1) non-zero. So without loss of generality, we assume that the Euler numbers are non-zero.

Let the group H have the following presentation:

generators: $\xi_1,\xi_2, ..., \xi_r;\ \alpha_1,\beta_1,......, \alpha_g,\beta_g,\zeta$,

relations: $\xi_i^{m_i}=\zeta^{q_i}, i=1,2,..., r$,

$\prod_{i=1}^{r}\xi_i\prod_{v=1}^{g}[\alpha_v,\beta_v]=\zeta^{q_o}$, and

ζ commutes with every element.

Denote by Γ the Fuchsian projection (group) $H/<\zeta>$. Let μ be any integer divisible by m_i for every i, and define the group H^* by the following presentation:

generators: $\xi_1,\xi_2, ..., \xi_r;\ \alpha_1,\beta_1,......, \alpha_g,\beta_g,\zeta$,

relations: $\xi_i^{m_i}=\zeta^{\mu q_i}, i=1,2,..., r$,

$\prod_{i=1}^{r}\xi_i\prod_{v=1}^{g}[\alpha_v,\beta_v]=\zeta^{\mu q_o}$, and

ζ commutes with every element.

This group has the same generators and relations of group H but only the central element ζ is replaced by ζ^μ, thus the groups H and $p^{-1}(\Gamma)$, for p the projection map, are subgroups of finite index in H^* and the natural homomorphism maps both of them onto the quotient group $H^*/<\zeta>$. Let H_1^* denote the inverse image of $H_1/<\zeta>$ under the natural map of H^* onto Γ. If Γ_1 denotes the Fuchsian group $H_1/<\zeta>$, then the groups H_1 and $p^{-1}(\Gamma_1)$ are subgroups of finite index in H_1^*, and the natural homomorphism maps both of them onto the quotient group $H_1^*/<\zeta>$. Denote the index of H in H^* by ρ, then the centre $<\zeta>$ of H has index ρ in the centre of H^*. So there is an element ζ' in the centre of H^* such that $\zeta'^{\rho}=\zeta$, hence H^* has the presentation:

generators: $\xi_1,\xi_2, ..., \xi_r;\ \alpha_1,\beta_1,......, \alpha_g,\beta_g,\zeta'$,

relations: $\xi_i^{m_i}=\zeta'^{\rho q_i}, i=1,2,..., r$,

$\prod_{i=1}^{r}\xi_i\prod_{v=1}^{g}[\alpha_v,\beta_v]=\zeta'^{\rho q_o}$, and

ζ' commutes with every element.

Thus, we have

$$e(H^*)=\left|\rho q_o-\sum_{i=1}^{r}\frac{\rho q_i}{m_i}\right| = p\left|q_o-\sum_{i=1}^{r}\frac{q_i}{m_i}\right| = \rho\, e(H).$$

Similarly

$$e(H_1^*)=\rho\, e(H_1),$$

hence

$$e(H^*)\, e(H_1)=e(H_1^*)\, e(H).$$

The same argument applies to the groups H^*, H_1^*, $p^{-1}(\Gamma)$, and $p^{-1}(\Gamma_1)$.

Then we get

$$e(H^*)\, e(p^{-1}(\Gamma_1)) = e(H_1^*)\, e(p^{-1}(\Gamma)).$$

From these two relations we get

$$\frac{e(H_1)}{e(H)}=\frac{e(H_1^*)}{e(H^*)}=\frac{e\left(p^{-1}(\Gamma_1)\right)}{e\left(p^{-1}(\Gamma)\right)},$$

and by Theorem 4.1, the latter is equal to $\frac{-\chi(\Gamma_1)}{-\chi(\Gamma)}$ which is just k, the index of H_1 in H, by the Riemann-Hurwitz formula.▲

4.3 Presentation of the Derived Group $\tilde{\Gamma}'$ of the SF-Group $\tilde{\Gamma}$ -

Let τ_{ij} and τ'_{ij} be two distinct elements of the same i-th coset in the presentation of $p^{-1}(\Gamma')$. Then for each i there is an integer θ_i which establishes the relations

$$\tau'_{ij} = \tau_{ij}\zeta^{\theta_i}, j = 1,2,\ldots,k_i.$$

So, we get

$$(\tau'_{ij})^{n_i} = \tau_{ij}^{n_i}\zeta^{n_i\theta_i} = \zeta^{1+n_i\theta_i},$$

hence

$$(\tau'_{ij})^{n_i} \in \tilde{\Gamma}'.$$

We have mentioned already that the composition map $\phi \circ p$ takes $\tilde{\Gamma}$ onto the abelian group Γ/Γ'. Then the central element ζ will have the order $\tilde{k}/k$ in $\tilde{\Gamma}/\tilde{\Gamma}'$, where k is the order of Γ/Γ' and equals $\frac{m}{[m_1, m_2, \ldots, m_r]}$ by Theorem 2.1-b and $\tilde{k}$ is the order of $\tilde{\Gamma}/\tilde{\Gamma}'$ which equals the determinant of the following $(r+1) \times (r+1)$ matrix:

$$\begin{bmatrix} m_1 & & & & & -1 \\ & m_2 & & 0 & & -1 \\ & & \cdot & & & \cdot \\ & & & \cdot & & \cdot \\ 0 & & & & \cdot & \cdot \\ & & & & m_r & -1 \\ 1 & 1 & \cdot & \cdot & \cdot \quad 1 & (2-r) \end{bmatrix}$$

then

$$\tilde{k} = m\{2 - r + \Sigma_{i=1}^{r} \frac{1}{m_i}\}.$$

Let θ_o denote the number $\tilde{k}/k$, then we have

$$\theta_o = [m_1, m_2, \ldots, m_r]\{2 - r + \Sigma_{i=1}^{r} \frac{1}{m_i}\}$$

$$= \frac{m_i[\hat{m}_i]}{l_i}(2 - r - \frac{1}{m_i} + \Sigma_{j=1}^{r} \frac{1}{m_j}), j \neq i \text{ for each i}$$

$$\equiv \frac{-[\hat{m}_i]}{l_i} (\text{mod } n_i),$$

where $= [\hat{m}_i] = [m_1, m_2, \ldots, m_{i-1}, m_{i+1}, \ldots, m_r]$, and $n_i = \frac{m_i}{l_i}$.

Theorem 2.1-a implies that $\frac{[\hat{m}_i]}{l_i}$ and n_i are relatively prime, then θ_o and n_l are relatively prime for every i.

Hence we can choose θ_i in a way that we get

$$\theta_o \mid 1 + n_i\theta_i \text{ for every i},$$

in other words, there are integers e_i such that we have

$$e_i\theta_o = 1 + n_i\theta_i.$$

Denote ζ^{θ_o} by $\bar{\zeta}$, then we get

$$(\tau'_{ij})^{n_i} = (\bar{\zeta})^{e_i}$$

and the following presentation for $\tilde{\Gamma}'$:

generators: τ'_{ij}, $i = 1,2,\ldots, r$, $j = 1,2,\ldots, k_i$;

$\alpha_v, \beta_v, v = 1,2,\ldots, g$;

$\bar{\zeta}$;

relations: $(\tau'_{ij})^{n_i} = (\bar{\zeta})^{e_i}$

$$\Pi_{i=1, j=1,2,\ldots,k_i}^{r} \tau'_{ij} \Pi_{v=1}^{g}[\alpha_v, \beta_v] = (\bar{\zeta})^{e_o}, \text{ and}$$

ζ commutes with all the elements, where e_o is some integer and is calculated as follows:

$\tilde{\Gamma}'$ is a subgroup of $\tilde{\Gamma}$ and it has the conditions that the group H_1 had in Theorem 4.2 with respect to the group $\tilde{\Gamma}$ in the place of the group H. Then we have

$$e(\tilde{\Gamma}') = \tilde{k}\left(e(\tilde{\Gamma})\right).$$

But,

$$e(\tilde{\Gamma}') = \theta_0 e_0 - \Sigma_{i=1}^{r} \frac{\theta_0 e_i}{n_i}$$

from the above presentation of $\tilde{\Gamma}'$ in terms of e_0, and

$$e(\tilde{\Gamma}) = r - 2 - \Sigma_{i=1}^{r} \frac{1}{m_i}$$

from the presentation of $\tilde{\Gamma}$, section 3, and

$$\tilde{k} = m\{2 - r + \Sigma_{i=1}^{r} \frac{1}{m_i}\}.$$

So, by substituting these values in the above relation, we get

$$e_0 = \Sigma_{i=1}^{r} \frac{e_i}{n_i} - \frac{m}{[m_1, m_2, \ldots, m_r]}(2 - r + \Sigma_{i=1}^{r} \frac{1}{m_i}).$$

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