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A RESEARCH NOTE ON THE SECOND ORDER DIFFERENTIAL EQUATION

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Abstract

Let U(t, λ) be solution of the Dirichlet problem

$$y''+(\lambda t-q(t))y=0 \qquad -1\leq t\leq 1$$
$$y(-1)=0=y(x),$$

with variable t on (-1, x), for fixed x, which satisfies the initial condition

$$U(-1,\lambda)=0, \qquad \frac{\partial U}{\partial t}(-1,\lambda)=1.$$

In this paper, the asymptotic representation of the corresponding eigenfunctions of the eigen values has been investigated. Furthermore, the leading term of the asymptotic formula for $\frac{\partial U}{\partial \lambda}(x,\lambda_n(x))$, $\lambda'_n(x)$ and $\int_{-1}^{x} vU^2(v,\lambda_n)dv$ is obtained where $\lambda_n(x)$ is a negative eigenvalue of the Dirichlet problem on $[-1,x]$ with fixed $x<0$.

1. Introduction

[1]The asymptotic nature of the approximation solution of the differential equation,

$$y''+(\lambda t-q(t))y=0 \qquad -1\leq t\leq 1 \tag{1}$$

subject to boundary conditions

$$y(-1)=0=y(x).$$

has been investigated in [1], where $q(t)$ is a continuous function in the interval [-1, 1], $x\in[-1,1]$, is fixed and λ is a real parameter. Let $U(t,\lambda)$ solve the initial value problem (1) with initial condition

$$U(-1,\lambda)=0, \qquad \frac{\partial U}{\partial t}(-1,\lambda)=1. \tag{2}$$

Keywords: Turning points; Asymptotic solutions

By Halvorson's result [3], $U(x,\lambda)$ is an entire function of order $\frac{1}{2}$ for each x. The function $U(x,\lambda)$ has a zero set for each x, say $\{\lambda_n(x)\}$, so that $U(x,\lambda_n(x))=0$, which corresponds to eigenvalues of the Dirichlet problem for equation (1) on the closed interval $[-1,x]$. Note that $\lambda_n(x)\neq 0$ for any x by Sturm's comparison theorem since we assume that $0\leq q(t)$.

Indeed, *each non-negative continuous function* $q(x)$ defines $U(x,\lambda,q)$, which is C^2 in x and $\lambda_n(x,q)$ solves $U(x,\lambda_n(x,q))=0$. It is known that for a non-negative continuous function $q(x)$, the eigenvalues of the Dirichlet problem for (1) on $[-1,x]$, are real and simple (See[4] §10.61), hence,

$$\frac{\partial U}{\partial \lambda}(x,\lambda_n(x))\neq 0$$

for each fixed x. The Dirichlet problem corresponding to

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equation(1) on $[-1, x]$ where $x<0$ is fixed, has an infinite number of negative eigenvalues $\{\lambda_n(x)\}$. The asymptotic distribution of each function $\lambda_n(x)$ is of the form

$$\sqrt{-\lambda_n(x)}=\frac{n\pi}{\int_{-1}^{x}\sqrt{-t}dt}+O(\frac{1}{n}),\ x<0$$

and

$$\lim_{x\to -1}\lambda_n(x)=-\infty \qquad \lambda_1(x)>\lambda_2(x)>\ldots$$

For more details see [2].

For $x\in(0,1]$, fixed, the Dirichlet problem for (1) on $[-1, x]$ has an infinite number of positive and negative eigenvalues which we denote by $\{u_n(x)\}$, $\{r_n(x)\}$, respectively.

The positive eigenvalues $u_n(x)$ admit the asymptotic representation

$$\sqrt{u_n(x)}=\frac{n\pi-\pi/4}{\int_{0}^{x}\sqrt{t}dt}+\frac{1}{2n\pi}T_1+O(\frac{1}{n^2}) \tag{4}$$

where

$$T_1=\frac{5}{72\int_{0}^{x}\sqrt{t}dt}+\frac{1}{2}\int_{0}^{x}\frac{q(t)}{\sqrt{t}}\,dt$$

Similarly, the negative eigenvalues, $r_n(x)$, admit the asymptotic representation of the form

$$\sqrt{-r_n(x)}=\frac{n\pi-\pi/4}{\int_{-1}^{0}\sqrt{-t}dt}+\frac{1}{2n\pi}T_2+O(\frac{1}{n^2}) \tag{5}$$

where

$$T_2=\frac{5}{72\int_{-1}^{0}\sqrt{-t}dt}+\frac{1}{2}\int_{-1}^{0}\frac{q(t)}{\sqrt{-t}}dt.$$

In [1], it was shown that

$$U(t,\lambda)=\begin{cases}\frac{1}{(-t)^{1/4}\sqrt{\lambda}}(1+O(\frac{1}{\sqrt{\lambda}})\sinh(p(t)\sqrt{\lambda}) & \text{if } -1\le t<0\\ \frac{\pi^{1/2}Ai(o)}{\lambda^{5/12}}\{e^{\frac{2}{3}\sqrt{\lambda}}-\frac{\sqrt{3}}{2}e^{-\frac{2}{3}\sqrt{\lambda}}\}\{1+O(\frac{1}{\sqrt{\lambda}})\} & \text{if } t=0\\ \frac{1}{t^{1/4}\sqrt{\lambda}}\{e^{\frac{2}{3}\sqrt{\lambda}}\cos(\frac{2}{3}t^{3/2}\sqrt{\lambda}-\pi/4)+ & \\ e^{-\frac{2}{3}\sqrt{\lambda}}\frac{1}{2}\sin(\frac{2}{3}t^{3/2}\sqrt{\lambda}-\pi/4)\}\{1+O(\frac{1}{\sqrt{\lambda}})\} & \text{if } 0<t\end{cases} \tag{6}$$

and

$$\frac{\partial U}{\partial t}(t\lambda)=\begin{cases}(-t)^{1/4}\{1+O(\lambda^{-1/2})\}\cosh(p(t)\sqrt{\lambda}) & \text{if } -1\le t<0\\ \pi^{1/2}\lambda^{-1/12}Bi'(o)\{\frac{1}{2}e^{-\frac{2}{3}\sqrt{\lambda}}+\frac{1}{\sqrt{3}}e^{\frac{2}{3}\sqrt{\lambda}}\}\{1+O(\frac{1}{\sqrt{\lambda}})\} & t=0\\ t^{1/4}\{\frac{1}{2}e^{-\frac{2}{3}\sqrt{\lambda}}\cos(\frac{2}{3}t^{3/2}\sqrt{\lambda}-\pi/4)- & \\ e^{\frac{2}{3}\sqrt{\lambda}}\sin(\frac{2}{3}t^{3/2}\sqrt{\lambda}-\pi/4)\}\{1+O(\frac{1}{\sqrt{\lambda}})\} & \text{if } 0<t\le 1\end{cases} \tag{7}$$

where

$$p(t)=\int_{-1}^{t}\sqrt{v}\,dv$$
$$Ai(0)=\frac{1}{3^{2/3}\Gamma(2/3)},\ Bi'(0)=-\frac{\sqrt{3}}{3^{1/3}\Gamma(1/3)} \tag{8}$$

2. Eigenfunctions

We now state a theorem which gives asymptotic approximation for the eigenfunctions of the Sturm-Liouville equation in one turning point case.

Theorem 1. Let $U(t,\lambda)$ be the solution of the Dirichlet problem (1) with variable t on $(-1, x)$, for fixed x which satisfies the initial condition(2). Then

a) For $x\in[-1, 0)$ fixed, the corresponding eigenfunctions of the negative eigenvalues $\lambda_n(x)$, has asymptotic representation,

$$U(t,\lambda_n(x))=\frac{p(x)\sin\frac{n\pi p(t)}{p(x)}}{(-t)^{\frac{1}{4}}n\pi}\{1+O(\frac{1}{n})\}$$

and

$$\frac{\partial U}{\partial t}(t,\lambda_n(x))=(-t)^{\frac{1}{4}}\cos\frac{n\pi p(t)}{p(x)}\{1+O(\frac{1}{n})\}$$

where $p(x)$ is defined in (8).

b) For $x\in(0, 1]$ fixed, the corresponding eigenfunctions of the positive eigenvalues, $u_n(x)$, admit the asymptotic representation,

$$U(t,u_n(x)) = \frac{e^{\frac{2}{3}\frac{n\pi-\frac{\pi}{4}}{f(x)}} \cos[\frac{2}{3} t^{\frac{3}{2}}(\frac{n\pi-\frac{\pi}{4}}{f(x)}) - \frac{\pi}{4}]}{t^{\frac{1}{4}}(\frac{n\pi-\frac{\pi}{4}}{f(x)})}\{1+O(\frac{1}{n})\}$$

and

$$\frac{\partial U}{\partial t}(x,u_n(x)) = -t^{\frac{1}{4}} e^{\frac{2}{3}(\frac{n\pi-\frac{\pi}{4}}{f(x)})} \sin[\frac{2}{3} t^{\frac{3}{2}}(\frac{n\pi-\frac{\pi}{4}}{f(x)}) - \frac{\pi}{4}]\{1+O(\frac{1}{n})\}$$

where

$$f(x) = \int_0^x \sqrt{v}\,dv \qquad x>0$$

c) For $x\in(0, 1]$ fixed, the corresponding eigenfunctions of the negative eigenvalues, $r_n(x)$, admit the asymptotic representation,

$$U(t,r_n(x)) = \frac{2}{t^{1/4}3n\pi} e^{-\pi(n-1/4)} \cos[t^{\frac{3}{4}}\pi(1/4-ni) - \pi/4]\{1+O(1/n)\}$$

$$\frac{\partial U}{\partial t}(x,r_n(x)) = -t^{1/4} e^{-\pi(n-1/4)} \sin(-\frac{1}{2} t^{\frac{3}{2}}\pi(n-1/4))\{1+O(1/n)\}$$

Proof. a) In this case the eigenvalues are negative. Substituting the asymptotic form (3) in (6) and noting that

$$\sqrt{\lambda} = i\sqrt{-\lambda_n(x)}$$

we can get

$$U(t,\lambda_n(x)) = \frac{\sinh(ip(t)\sqrt{-\lambda_n(x)})}{(-t)^{\frac{1}{4}}(i\sqrt{-\lambda_n(x)})}\{1+O(\frac{1}{n})\}$$

$$= \frac{\sin(ip(t)\sqrt{-\lambda_n(x)})}{(-t)^{\frac{1}{4}}(i\sqrt{-\lambda_n(x)})}\{1+O(\frac{1}{n})\}$$

$$= \frac{\sin\frac{p(t)n\pi}{p(x)}\{\cos O(\frac{1}{n}) + \cos\frac{p(t)n\pi}{p(x)}\sin O(\frac{1}{n})}{(-t)^{\frac{1}{4}}[\frac{n\pi}{p(x)} + O(\frac{1}{n})]}\{1+O(\frac{1}{n})\}$$

$$= \frac{\sin\frac{p(t)n\pi}{p(x)}\{1+O(\frac{1}{n^2})\}+O(\frac{1}{n})}{(-t)^{\frac{1}{4}}\frac{n\pi}{p(x)}[1+O(\frac{1}{n^2})]}\{1+O(\frac{1}{n})\}$$

$$= \frac{p(x)\sin\frac{n\pi p(t)}{p(x)}}{(-t)^{\frac{1}{4}}n\pi}\{1+O(\frac{1}{n})\}$$

and similarly inserting the asymptotic form (3) in (7) we obtain

$$\frac{\partial U}{\partial t}(t,\lambda_n(x)) = \cosh(ip(t)\sqrt{-\lambda_n(x)})(-t)^{\frac{1}{4}}\{1+O(\frac{1}{n})\}$$

$$= \cos(p(t)\sqrt{-\lambda_n(x)})(-t)^{\frac{1}{4}}\{1+O(\frac{1}{n})\}$$

$$= \cos(\frac{n\pi p(t)}{p(x)} + O(\frac{1}{n}))(-t)^{\frac{1}{4}}\{1+O(\frac{1}{n})\}$$

$$= \{\cos(\frac{n\pi p(t)}{p(x)}\cos O(\frac{1}{n}) - \sin(\frac{n\pi p(t)}{p(x)}\sin O(\frac{1}{n})\}(-t)^{\frac{1}{4}}\{1+O(\frac{1}{n})\}$$

$$= \{\cos(\frac{n\pi p(t)}{p(x)})(1+O(\frac{1}{n^2}) - \sin(\frac{n\pi p(t)}{p(x)})O(\frac{1}{n})\}(-t)^{\frac{1}{4}}\{1+O(\frac{1}{n})\}$$

$$= (-t)^{\frac{1}{4}}\cos(\frac{n\pi p(t)}{p(x)})\{1+O(\frac{1}{n})\}$$

Notice that in the above equation a long but straight forward calculation, we used the following facts for large n.

$$\cos O(\frac{1}{n}) = 1+O(\frac{1}{n^2})$$

$$\sin O(\frac{1}{n}) = O(\frac{1}{n})$$

b) By inserting the asymptotic form (4) in (6), we get

$$U(t,u_n(x)) = \frac{e^{\frac{2}{3}[\frac{n\pi-\frac{\pi}{4}}{f(x)}+O(\frac{1}{n})]} \cos\{\frac{2}{3} t^{\frac{3}{2}}[\frac{n\pi-\frac{\pi}{4}}{f(x)} + O(\frac{1}{n}) - \frac{\pi}{4}]\}}{t^{\frac{1}{4}}[\frac{n\pi-\frac{\pi}{4}}{f(x)} + O(\frac{1}{n})]}\{1+O(\frac{1}{n})\}$$

$$= \frac{e^{\frac{2}{3}[\frac{n\pi-\frac{\pi}{4}}{f(x)}+O(\frac{1}{n})]}[1+O(\frac{1}{n})]\{\cos\frac{2}{3}t^{\frac{3}{2}}[\frac{n\pi-\frac{\pi}{4}}{f(x)}-\frac{\pi}{4}]\}\cos O(\frac{1}{n}) + \sin\{\frac{2}{3}t^{\frac{3}{2}}[\frac{n\pi-\frac{\pi}{4}}{f(x)}-\frac{\pi}{4}]\}\sin O(\frac{1}{n})\}}{t^{\frac{1}{4}}[\frac{n\pi-\frac{\pi}{4}}{f(x)}+O(\frac{1}{n})]}$$

$$\{1+O(\frac{1}{n})\}$$

$$= \frac{e^{\frac{2}{3}(\frac{n\pi-\frac{\pi}{4}}{f(x)})} \cos[\frac{2}{3}t^{\frac{3}{2}}(\frac{n\pi-\frac{\pi}{4}}{f(x)}-\frac{\pi}{4}]}{t^{\frac{1}{4}}[\frac{n\pi-\frac{\pi}{4}}{f(x)}]}\{1+O(\frac{1}{n})\}$$

Similarly, inserting the distribution of positive eigenvalues $U_n(x)$ in (7), we obtain

$$\frac{\partial U}{\partial t}(x,u_n(x)) = -t^{\frac{1}{4}} e^{\frac{2}{3}\left[\frac{n\pi-\frac{\pi}{4}}{f(x)}+O(\frac{1}{n})\right]} \sin\left\{\frac{2}{3}t^{\frac{3}{2}}\left[\frac{n\pi-\frac{\pi}{4}}{f(x)}+O(\frac{1}{n})\right]-\frac{\pi}{4}\right\}\left\{1+O(\frac{1}{n})\right\}$$

$$= -t^{\frac{1}{4}} e^{\frac{2}{3}\left[\frac{n\pi-\frac{\pi}{4}}{f(x)}\right]\left[1+O(\frac{1}{n^2})\right]} \sin\left\{\frac{2}{3}t^{\frac{3}{2}}\left[\frac{n\pi-\frac{\pi}{4}}{f(x)}-\frac{\pi}{4}\right]\cos O(\frac{1}{n})+\cos\left[\frac{2}{3}t^{\frac{3}{2}}\left[\frac{n\pi-\frac{\pi}{4}}{f(x)}-\frac{\pi}{4}\right]\right.\right.$$
$$\left.\sin O(\frac{1}{n})\right\}\left\{1+O(\frac{1}{n})\right\}$$

$$= -t^{\frac{1}{4}} e^{\frac{2}{3}\left(\frac{n\pi-\frac{\pi}{4}}{f(x)}\right)} \sin\left[\frac{2}{3}t^{\frac{3}{2}}\left(\frac{n\pi-\frac{\pi}{4}}{f(x)}\right)-\frac{\pi}{4}\right]\left\{1+O(\frac{1}{n})\right\}$$

c) By noting that $r_n(x)$ for large enough n is negative. Hence, inserting the asymptotic form of $\lambda = r_n(x)$ in (6) and (7) we get the result.

3. Example in the classical Case

In the classical case there are some examples which provide easy formulas for the $\lambda_n(x)$.

For instance, let us consider the Dirichlet problem for

$$y'' + (\lambda t - q(t))\,y = 0$$

on $[-1, x]$, for fixed $x < 0$, where

$$q(t) = \frac{5}{16t^2} - \frac{t}{4}.$$

The two linearly independent solutions of this equation are of the form

$$y(t) = (-t)^{-1/4} e^{\frac{1}{3}(-t)^{3/2}(1-2d)} \qquad -1 \le t < 0$$

where $d = \frac{1}{2} \pm \sqrt{\lambda + 1/4}$. The solution which satisfies the initial conditions

$$U(-1,\lambda) = 0 \qquad \frac{\partial U}{\partial t}(-1,\lambda) = 1$$

is

$$U(t,\lambda) = \frac{(-t)^{-1/4}}{\sqrt{\lambda+1/4}} \sinh p(t)\sqrt{\lambda+1/4} \qquad t<0, \tag{9}$$

with $p(x) = \int_{-1}^{x} \sqrt{-v}\,dv$. The eigenvalues of this problem are

$$\lambda_n(x) = \frac{n^2\pi^2}{p^2(x)} - 1/4 \qquad x<0. \tag{10}$$

Consequently,

$$\sqrt{\lambda_n(x)+1/4} = i\frac{n\pi}{p(x)}$$

Inserting in (9) we get

$$U(t,\lambda_n(x)) = \frac{(-t)^{-1/4}}{i\frac{n\pi}{p(x)}} \sinh p(t)\, i\frac{n\pi}{p(x)}$$
$$= \frac{p(x)}{(-t)^{1/4} n\pi} \sin\frac{n\pi p(t)}{p(x)}$$

We see that the leading term is in agreement with our results.

4. Some Properties of the Eigenfunctions in the Classical Case

Now we study some results in connection with the eigenfunctions corresponding to the eigenvalues in the classical case.

The infinite number of negative eigenvalues, $\lambda_n(x)$ are the zeros of $U(x, \lambda)$. Since $U(x, \lambda)$ is an entire functions of order $\frac{1}{2}$ for each x, it therefore holds that by Hadamard's theorem (see [5], page 24), the product formula

$$U(x,\lambda) = c\prod_{n=1}^{\infty}\left(1-\frac{\lambda}{\lambda_n(x)}\right)$$

where c is a constant independent of λ but may depend on x because the genus of U is zero. In order to estimate c, we rewrite the infinite product as

$$U(x,\lambda) = c\prod\left(1-\frac{\lambda}{\lambda_n(x)}\right)$$
$$= c\prod\frac{\lambda_n(x)-\lambda}{\lambda_n(x)}$$
$$= c_1\prod\frac{\lambda-\lambda_n(x)}{z_n^2} \tag{11}$$

with

$$c_1 = c\prod\frac{-z_n^2}{\lambda_n(x)}$$

where $Z_n = \frac{n\pi}{p(x)}$, $p(x)$ is defined in (8). Note that since

$$\frac{-z_m^2}{\lambda_m(x)} = 1 + O(1/m^2),$$

the infinite product $\prod \frac{-z_m^2}{\lambda_m(x)}$ is absolutely convergent on any compact subinterval of (-1, 0) by the following theorem (2). The function $\frac{-z_m^2}{\lambda_m(x)}$ is continuous and so the O-term is uniformly bounded in x.
Now we will first approximate the infinite products, then by using the asymptotic form of $U(x, \lambda)$, we will determine c_1. The following theorems play an important role in estimating the infinite product.

Thorem 2. $\prod_0^\infty (1+p_n)$ converges absolutely if, and only if $\sum_0^\infty p_n$ converges absolutely where the p_n are arbitrary complex constants.

Proof. See [5] § 18. 13.

Theorem 3. If $p_n(z)$ is analytic in a simply connected domain D and if $\sum_0^\infty [p_n(z)]$ converges uniformly in every closed region R of D, then

$$\prod_0^\infty (1+p_n(z))$$

converges uniformly to $f(z)$ in every such R and $f(z)$ is analytic in D.

Proof. See [5] § 18. 19.

Theorem 4. (a) Suppose A_{mn}, M, $N>1$ are complex numbers satisfying

$$|a_{mn}| = O\left(\frac{1}{|m^2-n^2|}\right) \qquad m \neq n$$

then, for each $1 \leq n$,

$$\prod_{1\leq m, m\neq n} (1+a_{mn}) = 1+O\left(\frac{\log n}{n}\right)$$

(b) In addition, if b_n, $1 \leq n$ is a square summable sequence of complex numbers, then

$$\prod_{m,n>1, m\neq n} (1+a_{mn}b_n) < \infty$$

Proof. See [6], p. 165.

Theorem 5. Let $z_m = \frac{m\pi}{p(x)}$ and $\lambda_m(x)$, $1 \leq m$ be a sequence of continuous functions such that for each x

$$\lambda_m(x) = -\frac{m^2\pi^2}{p^2(x)} + O(1) \qquad -1<x\leq a<0$$

where $p(x) = \int_{-1}^{x} \sqrt{-t}\, dt$. Then the infinite product

$$\prod_{1\leq m} \frac{\lambda - \lambda_m(x)}{z_m^2}$$

is an entire function of λ for fixed x in (-1, 0) whose roots are precisely $\lambda_m(x)$, $1 \leq m$. Moreover,

$$\prod_{1\leq m} \left(\frac{\lambda - \lambda_m(x)}{z_m^2}\right) = \frac{\sinh p(x)\sqrt{\lambda}}{p(x)\sqrt{\lambda}}\left(1+O\left(\frac{\log n}{n}\right)\right)$$

uniformly on the circles $|\lambda| = \frac{(n+1/2)^{2\pi^2}}{p^2(x)}$.

Proof. Let x be fixed. By the uniform boundedness of $\lambda_m(x) + \frac{m^2\pi^2}{p^2(x)}$ for $1 \leq m$,

$$\sum_{1\leq m} \left|\frac{\lambda - \lambda_m(x)}{z_m^2} - 1\right| = \sum_{1\leq m} \left|\frac{\lambda - O(1)}{z_m^2(x)}\right|$$

converges uniformly on bounded subsets of the complex plane. Therefore by Theorem (3), the infinite product converges to an entire function of λ whose roots are precisely $\lambda_m(x)$, $1<m$.
Now, since the infinite product form of sinh z is

$$\sinh z = z \prod \left(1+\frac{z^2}{m^2\pi^2}\right)$$

see [7], Therefore,

$$\begin{aligned} \sinh c\sqrt{z} &= c\sqrt{z} \prod \left(1+\frac{c^2 z}{m^2\pi^2}\right) \\ &= c\sqrt{z} \prod \left(1+\frac{z}{z_m^2}\right) \end{aligned} \tag{12}$$

(13)

Consequently,

$$\frac{\sqrt{\lambda} p(x)}{p(x)\sqrt{\lambda}} = \prod \left(1+\frac{\lambda}{z_m^2}\right)$$

thus, the quotient of the infinite products of (11) and (12) is

$$\frac{\prod \frac{\lambda - \lambda_m(x)}{z_m^2}}{\prod \left(1+\frac{\lambda}{z_m^2}\right)} = \prod_{1\leq m} \left(\frac{\lambda - \lambda_m}{\lambda + z_m^2}\right).$$

Furthermore

$$\left|\frac{\lambda - \lambda_m}{\lambda + z_m^2} - 1\right| = \left|\frac{-\lambda_m - z_m^2}{\lambda + z_m^2}\right| \leq \frac{|O(1)|}{\left||\lambda| - \frac{m^2\pi^2}{p^2}\right|}$$

Therefore, on the circles $|\lambda|=\frac{(n+1/2)^2\pi^2}{p^2(x)}$, the uniform estimates

$$\frac{-\lambda_m+\lambda}{z_m^2+\lambda}=\begin{cases}1+O(1/n) & \text{if } m=n\\ 1+O\left(\frac{1}{|m^2-n^2|}\right) & m\neq n\end{cases}$$

hold. By Theorem (4),

$$\prod_{1\le m}\frac{-\lambda_m+\lambda}{\lambda+z_m^2}=(1+O(\tfrac{1}{n}))(1+O(\tfrac{\log n}{n}))=\{1+O(\tfrac{\log n}{n})\}$$

uniformly on these circles. Therefore

$$\prod_{1\le m}\frac{\lambda+\lambda_m(x)}{z_m^2}=\frac{\sinh p(x)\sqrt{\lambda}}{p(x)\sqrt{\lambda}}(1+O(\tfrac{\log n}{n}))$$

Theorem 6. For $-1\le x<0$,

$$U(x,\lambda)=\frac{p(x)}{(-x)^{1/4}}\prod\frac{\lambda-\lambda_k(x)}{z_k^2}$$

where $p(x)=\int_{-1}^{x}\sqrt{-t}\,dt$ and $U(t,\lambda)$ is a solution of the initial value problem (2) for the Equation (1) and $\{\lambda_k(x)\}$ is the sequence of eigenvalues for the Dirichlet problem associated with (1) on $[-1,x]$, i.e.,

$$y(-1,\lambda)=0=y(x,\lambda)$$

here $z_m=\frac{m\pi}{p(x)}$, as usual.

Proof. From (11) and (6). we have

$$U(x,\lambda)=c_1\prod\frac{\lambda-\lambda_k(x)}{z_k^2}$$

$$=\frac{1}{(-x)^{1/4}\sqrt{\lambda}}(1+O(\tfrac{1}{\sqrt{\lambda}}))\sinh(p(x)\sqrt{\lambda})\qquad \lambda\to\infty$$

where $p(x)$ is defined in (8).

From Theorem (5), uniformly on the circles $|\lambda|=\frac{(n+1/2)^2\pi^2}{p^2(x)}$, we have

$$\prod\frac{\lambda-\lambda_k(x)}{z_k^2}=\frac{\sinh(p(x)\sqrt{\lambda})}{p(x)\sqrt{\lambda}}(1+O(\tfrac{\log n}{n}))$$

where on the circles $|\lambda|=\frac{(n+1/2)^2\pi^2}{p^2(x)}$.

$$c_1=\frac{U(x,\lambda)}{\prod\frac{\lambda-\lambda_k(x)}{z_k^2}}=\frac{p(x)}{(-x)^{1/4}}(1+O(\tfrac{\log n}{n}))$$

we get

$$c_1=\frac{p(x)}{(-x)^{1/4}}.$$

We will often use the abbreviated notation $\dot{U}\equiv\frac{\partial U}{\partial\lambda}$.

Theorem 7. Let $U(t,\lambda)$ be the solution of boundary value problem

$$y''+(\lambda t-q)y=0\qquad -1\le t\le d$$

where d is arbitrary but fixed and

$$y(-1)=0=y(d)\qquad \frac{\partial y}{\partial t}(-1)=1.$$

Then

$$\dot{U}(d,\lambda)U'(d,\lambda)=\int_{-1}^{d}tU^2dt.$$

Proof. Differentiating the equation with respect to λ yields

$$\dot{U}''+tU+(\lambda t-q)\dot{U}=0$$

Multiplying this equation by U, the original equation by $\dot{U}$ and taking the difference we obtain

$$\dot{U}''U-\dot{U}U''+tU^2=0,$$

hence

$$\int_{-1}^{d}tU^2dt=\dot{U}(d,\lambda)\,U'(d,\lambda)$$

since $\dot{U}(-1,\lambda)=0$ by (2).

Theorem 8. Let $U(t,\lambda)$ solve the initial value problem (1) with initial condition (2) for $-1\le t<0$.

Then

$$\frac{\partial U}{\partial\lambda}(x,\lambda_n(x))=\frac{p^3(x)\,(-1)^{n+1}}{2n^2\pi^2(-x)^{1/4}}\{1+O(\tfrac{\log n}{n})\}$$

where $p(x)$ is defined in (8) and $\lambda_n=\lambda_n(x)$ is the sequence of eigenvalues of the Dirichlet problem (1) on $[-1,x]$ for

$x<0$.

Proof. From Theorem (5)

$$\sinh p(x)\sqrt{\lambda} = p(x)\sqrt{\lambda}\prod (1+\frac{\lambda}{z_k^2})$$

where $z_k = \frac{k\pi}{p(x)}$, $k \in N$, and p(x) is defined in (8), we have

$$\frac{d}{d\lambda}\{\frac{\sinh p(x)\sqrt{\lambda}}{p(x)\sqrt{\lambda}}\}\Big|_{\lambda=-z_n^2} = \frac{p^2(x)}{n^2\pi^2}\prod_{1\le k, n\ne k}(1-\frac{z_n^2}{z_k^2})$$

where

$$\prod_{k\ne n}(1-\frac{z_n^2}{z_k^2}) = \frac{(-1)^{n+1}}{2}$$

For fixed x, from Theorem (6), we have

$$U(x,\lambda) = \frac{p(x)}{(-x)^{1/4}}\prod\frac{\lambda-\lambda_n(x)}{z_n^2}$$

therefore,

$$\frac{\partial U}{\partial\lambda}(x,\lambda_n) = \{\frac{p^3(x)}{(-x)^{1/4}n^2\pi^2}\}\prod_{1\le k, k\ne n}\frac{\lambda_n-\lambda_k(x)}{z_k^2}$$

where

$$\frac{\frac{\partial U}{\partial\lambda}(x,\lambda_n)}{\frac{(-1)^{n+1}p^2(x)}{2n^2\pi^2}} = \frac{p(x)}{(-x)^{1/4}}\prod_{1\le k, k\ne n}\frac{\lambda_n-\lambda_k(x)}{z_k^2-z_n^2}$$

Since

$$\frac{\lambda_n-\lambda_k}{z_k^2-z_n^2} = 1+O(\frac{1}{|k^2-n^2|}) \qquad k\ne n$$

therefore by Theorem (4), we get

$$\frac{\partial U}{\partial\lambda}(x,\lambda_n) = \frac{p^3(x)(-1)^{n+1}}{2n^2\pi^2(-x)^{1/4}}\{1+O(\frac{\log n}{n})\}$$

Theorem 9. For fixed $x<0$, let $\lambda_n(x)$ be the sequence of negative eigenvalues of the equation (1) for the Dirichlet problem on [-1, x] where $0\le q(t)$, so that U(x, $\lambda_n(x)$)= 0. Then we have

a) $\lambda_n(x)$ is twice continously differentiable and

$$\lambda'_n(x) = \frac{2(-x)^{1/2}n^2\pi^2}{p^3(x)}\{1+O(\frac{\log n}{n})\} \qquad n\to\infty$$

b)

$$\lim_{k\to\infty}\frac{d}{dx}\log\lambda_k(x) = \frac{2z'_k}{z_k},$$

c) The series

$$\sum_{n\ne k, 1\le k}\frac{-\lambda_n\lambda'_k(x)}{(\lambda_k(x)-\lambda_n(x))\lambda_k(x)},$$

is uniformly convergent on any compact subset of (-1,0),

d)

$$\int_{-1}^{x} vU^2(v,\lambda_n)\,dv = \frac{p^3(x)}{2n^2\pi^2}\{1+O(\frac{\log n}{n})\} \qquad n\to\infty$$

where $z_k(x) = \frac{n\pi}{p(x)}$ and $p(x) = \int_{-1}^{x}\sqrt{-t}dt$.

Proof. a) It is known that for a non-negative continuous function $q(x)$, the eigenvalues of the Dirichlet problem for (1) on [-1, x] are real and simple (see [4], §10.61), i.e.,

$$\frac{\partial U}{\partial\lambda}(x,\lambda_n(x))\ne 0$$

for each $x \in (-1, 0)$. It follows from the implicit function theorem that $\lambda_n(x, q)$ is C^2 in x and

$$\lambda'_n = -\left(\frac{\frac{\partial U}{\partial x}(x,\lambda)}{\frac{\partial U}{\partial\lambda}(x,\lambda)}\right)_{\lambda=\lambda_n(x)}.$$

From Theorems (1) and (6), we have

$$\lambda'_n(x) = \frac{2(-x)^{1/2}n^2\pi^2\cos\{p(x)\sqrt{-\lambda_n}\}}{p^3(x)(-1)^n}\{1+O(\frac{\log n}{n})\} \qquad n\to\infty$$

By inserting

$$\sqrt{-\lambda_n} = \frac{n\pi}{p(x)}+O(\frac{1}{n})$$

in the above formula, we get

$$\lambda'_n(x) = \frac{2(-x)^{1/2}n^2\pi^2\cos\{n\pi+O(\frac{1}{n})\}}{p^3(x)(-1)^n}\{1+O(\frac{\log n}{n})\} \qquad n\to\infty$$

By the mean value theorem we have

$$\cos O(\frac{1}{n}) = 1 + O(\frac{1}{n^2})$$

therefore,

$$\lambda'_n(x) = \frac{2(-x)^{1/2} n^2\pi^2}{p^3(x)}\{1 + O(\frac{\log n}{n})\} \qquad n \to \infty$$

(b) From (a) and the distribution of $\sqrt{-\lambda_n(x)}$, we immediately obtain (b).

(c) By (a) and (b), the sequence

$$\frac{\lambda'_k(x)}{\lambda_k(x)}$$

is uniformly bounded on any compact subset of (-1, 0). Thus, the above series is uniformly convergent by the *M*-test.

d) From Theorem (7),

$$\dot{U}(x,\lambda_n)U'(x,\lambda_n) = \int_{-1}^{x} vU^2 dv$$

Substiting the asymptotic form of $\dot{U}(x, \lambda_n)$ and $U'(x, \lambda_n)$ from Theorem (6) and theorem (7), respectively and using the mean value theorem for cos ($n\pi$+O(1/n)), we finally obtain the result.

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