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BAYES ESTIMATION USING A LINEX LOSS FUNCTION

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Abstract

This paper considers estimation of normal mean θ when the variance is unknown, using the LINEX loss function. The unique Bayes estimate of θ is obtained when the precision parameter has an Inverse Gaussian prior density.

1. Introduction:

Let $X_1,\ldots,X_n$ be a random sample from a normal distribution with p. d. f.

$$f(x|\theta,\sigma^2) = (2\pi\sigma^2)^{-1/2}\exp\{-(x-\theta)^2/2\sigma^2\}, -\infty < X < \infty$$

where $\theta \in (-\infty, \infty)$ and $\sigma \in (0,\infty)$. This paper considers estimation of θ when (i) σ^2 is known and (ii) σ^2 is a nuisance parameter, using the LINEX loss function, which is defined as

$$L(\theta, \hat{\theta}) = b\{e^{a(\hat{\theta}-\theta)} - a(\hat{\theta}-\theta) - 1\}$$

where $b > 0$ is the scale parameter and $a \neq 0$ is the shape parameter.

The LINEX loss function was introducted by Varian [5] and extensively discussed by Zellner [6]. It is useful when a given positive overestimation error is regarded as more serious than a negative underestimation error of the same magnitude ($a > 0$) or vice versa ($a < 0$). As an example, in dam construction an underestimate of the peak water leveal is usually much more serious than an overestimation. Another example is the case of real state assessments, Varian [5] found that the use of the asymetric LINEX loss function may be more appropriate than the squared error loss function. A full discussion of the properties of the LINEX loss function may be found in Zellner [6].

When σ^2 is known, Zellner [6], showed that using the LINEX loss, $\bar{X} - a\sigma^2/2n$ dominates $\bar{X}$ and Rojo [3] considered the admissibility of the linear estimator $c\bar{X} + d$ relative to the LINEX loss function. He showed that $c\bar{X} + d$ is admissible for θ with respect to the LINEX loss function whenever $0 \leq c < 1$ or $c = 1$ & $d = -a\sigma^2/2n$ and otherwise it is inadmissible. Also, see Sadooghi - Alvandi & Nematollahi [4]. An important consideration is when X_i's have a common unknown variance, i.e., σ^2 is unknown. Zellner [6] suggested replacement of σ^2 by $S^2 = \sum_{i=1}^{n}(X_i-\bar{X})^2/(n-1)$ in $\bar{X} - a\sigma^2/2n$ and showed that the resulting estimator uniformly dominates the sample mean in terms of risk relative to the LINEX loss. However, we do not know any other optimal property of the proposed estimator.

In section 2 of this paper we show that the estimator $\bar{X} - a\sigma^2/2n$, when σ^2 is known, is the only minimax and admissible estimators of θ in the class of linear estimators of the form $c\bar{X} + d$ and in section 3, we obtain a unique Bayes estimate of θ when σ^2 is a nuisance parameter using the LINEX loss function.

2. Minimaxity:

To show minimaxity of $\bar{X} - a\sigma^2/2n$ under LINEX loss, when σ^2 is known, it is easy to verify that the risk of estimators of the form $c\bar{X} + d$ is (see Rojo [3])

$$R(\theta, c\bar{X} + d) = b\{\exp[a^2\sigma^2c^2/2n + a(d + \theta(c-1))] - a(d + \theta(c-1)) - 1\} \quad (1)$$

and so

$$R(\theta, \bar{X} - a\sigma^2/2n) = ba^2\sigma^2/2n, \text{ (which is free from } \theta).$$

Now since $\bar{X} - a\sigma^2/2n$ is admissible and has constant

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risk, it follows from Lemma 3.3 of Lehmann [2] that it is minimax.

Now, we verify that $\bar{X} - a\sigma^2/2n$ is the only minimax and admissible estimator of θ in the class of estimators of the form $c\bar{X} + d$ when σ^2 is known, using the LINEX loss function. To see this, note that $c\bar{X} + d$ is admissible only if $0 \le c < 1$ or $c = 1, d = -a\sigma^2/2n$, and since $R(\theta, c\bar{X} + d)$ is given by (1), using the fact that $\exp(X) \ge 1 + x + x^2/2$ for all $x \ge 0$, it follows that there is $\theta^* < 0$, if $a > 0$ and $\theta^* > 0$ if $a < 0$ such that

$$R(\theta^*, c\bar{X} + d) \ge \sup_{\theta} R(\theta, \bar{X} - a\sigma^2/2n) = ba^2\sigma^2/2n.$$

Hence $c\bar{X} + d$ can not be minimax and admissible when $c \ne 1$ and $d \ne -a\sigma^2/2n$.

Remark 1.1: The class of estimators in section 2 is restricted to the linear estimators. Note that the class of all Bayes estimators, using normal priors, and generalized Bayes estimators, using vague prior, of θ when σ^2 is known are in the form $c\bar{X} + d$ which is including the usual estimator of θ.

3. Bayes Estimate:

Let $r = 1/\sigma^2$ be the precision which is unknown. Suppose that conditional on r, θ has a normal distribution with mean μ and variance $1/\lambda r$, where μ, λ (>0) are both known constants, i.e.

$$\theta|r \sim N(\mu, 1/\lambda r),$$

and r has an Inverse Gaussian distribution with known parameters β and α, i.e.,

$$\pi(r) = (\alpha/2\pi)^{1/2} r^{-3/2} \exp\{-\alpha/2\beta^2 r)(r-\beta)^2\}.$$

The reason for the choice of Inverse Gaussian prior are: (i) certain integral to become convergent, (ii) the joint prior of (θ, r) becomes as a conjugate prior. (See Remarks 3.1 and 3.2).

Hence, the kernel of joint prior density of θ and r is

$$\pi(\theta, r) \propto r^{-1} \exp\{-(\lambda r/2)(\theta-\mu)^2 - (\alpha r/2\beta^2 + \alpha/2r)\}.$$

To work out the Bayes estimate of θ, note that the kernel of likelihood function is

$$Z(\theta, r) \propto r^{n/2} \exp\{-\frac{r}{2}[(\theta-\bar{X})^2 + ns^2]\},$$

where $ns^2 = \sum_{i=1}^{n}(X_i - \bar{X})^2$. Combining the likelihood with the joint prior density of (θ, r), we obtain the joint posterior density $\pi^*(\theta, r|\underset{\sim}{X}) \propto r^{n/2-1} \exp\{-\frac{r}{2}[(\theta-\bar{X})^2 + ns^2] - (\lambda r/2)(\theta-\mu)^2 - \alpha r/2\beta^2 - \alpha/2r\}$. (2)

One can easily verify that

$$n(\theta-\bar{X})^2 + \lambda(\theta-m)^2 = (n+\lambda)(\theta-\eta)^2 + \frac{n\lambda}{n+\lambda}(\bar{X}-\mu)^2$$

and

$$\{\dots\} = -\frac{(n+\lambda)r}{2}(\theta-\eta)^2 - \gamma r - \frac{\alpha}{2r},$$

with

$$\eta = (n\bar{x} + \lambda\mu)/(n+\lambda)$$

and

$$2\gamma = ns^2 + \frac{n\lambda}{n+\lambda}(\bar{x}-\mu)^2 + \frac{\alpha}{\beta^2}.$$

Hence, (2) reduces to

$$\pi^*(\theta, r|x) \propto r^{n/2-1} e^{-\frac{\alpha}{2r}-\gamma r} \exp[-\frac{(n+\lambda)r}{2}(\theta-\eta)^2] \quad (3)$$

From (3), it is clear that

$$\theta|\underset{\sim}{x}, r \sim N(\eta, 1/(n+\lambda)r)$$

and

$$\pi^*(r|\underset{\sim}{X}) \propto r^{(n-1)/2} e^{-\frac{\alpha}{2r}-\gamma r}$$

Using the integral representation (cf. Gradshteyn and Ryzhik, p 340, formula 9),

$$\int_0^\infty x^{v-1} e^{-\frac{\zeta}{x} - Px} dx = 2(\frac{\zeta}{p})^{v} K_v(2\sqrt{p\zeta}).(\mathrm{Re}\,\zeta > 0, \quad (4)$$

$\mathrm{Re}\, p > 0)$

where $k_v(.)$ is the modified Bessel function of the third kind, we obtain (with $\upsilon = (n+1)/2, \zeta = \alpha/2, p = \gamma$)

$$\pi^*(r|\underset{\sim}{x}) = \frac{2^{(n-1)/2}}{(\alpha/\gamma)^{(n+1)/2} K_{(n+1)/2}(\sqrt{2\gamma\alpha})} r^{(n-1)/2} e^{\frac{\alpha}{2r}-\gamma r}$$

Now, one can easily verify that (see Zellner [6]) the Bayes estimate of θ using the LINEX loss is

$$\delta_B(\underset{\sim}{x}) = -\frac{1}{a}\log M_{\theta|\underset{\sim}{x}}(-a),$$

where $M_{\theta|\underset{\sim}{x}}(.)$ denotes the moment generating function of θ $\underset{\sim}{x}$.

To obtain the Bayes estimate of θ for our problem, note that

$$\begin{aligned} M_{\theta}|_{\underset{\sim}{x}}(t) &= E[e^{t\theta}|\underset{\sim}{x}] \\ &= E\{E[e^{t\theta}|\underset{\sim}{x}, r]\} \\ &= E\{\exp[\eta t + t^2/2(n+\lambda)r]\} \\ &= e^{\eta t} E\{\exp[\frac{t^2}{2(n+\lambda)}/r]\} \\ &= e^{\eta t}\int_0^\infty e^{\frac{t^2}{2(n+\lambda)}/r}\pi^*(r|\underset{\sim}{X})dr. \end{aligned}$$

Using once again the formula (4), with $\upsilon = (n+1)/2$, $\zeta = \frac{\alpha}{2} - \frac{t^2}{2(n+\lambda)}$ and $p = \gamma$, we obtain

$$M_\theta|_{\underset{\sim}{x}}(t) = e^{\eta t}[1-t^2/\alpha(n+\lambda)]^{(n+1)/2}\frac{k_{(n+1)/2}(\sqrt{2\gamma(\alpha-\frac{t^2}{n+\lambda})})}{k_{(n+1)/2}(\sqrt{2\gamma\alpha})}$$

provided $\alpha > t^2 / (n + \lambda)$.
Now using (5) and (6) we obtain

$$\delta_B(\underset{\sim}{x}) = \frac{n\bar{x}+\lambda\mu}{n+\lambda} - \frac{1}{a}\log\{[1-\frac{a^2}{(n+\lambda)\alpha}]^{(n+1)/2}$$

$$\frac{k_{(n+1)/2}(\sqrt{2\gamma(\alpha-\frac{a^2}{n+\lambda})})}{k_{(n+1)/2}(\sqrt{2\gamma\alpha})}$$

provided $\alpha > a^2 / (n + \lambda)$ (note that γ is a function of the sample variance). And since $\delta_B(\underset{\sim}{x})$ is the unique Bayes estimate of θ w. r. t. joint normal - Inverse Gaussian prior using the LINEX loss function, hence it is admissible.

As a special case, taking $\alpha = 2a^2 / (n + \lambda)$, $\delta_B(\underset{\sim}{X})$ becomes

$$\delta_B(x) = \frac{n\bar{x}+\lambda\mu}{n+\lambda} - \frac{1}{a}\log\{2-(n+1)/$$

$$2\frac{K_{(n+1)/2}(\sqrt{2\gamma a^2/(n+\lambda)})}{K_{(n+1)/2}(\sqrt{4\gamma a^2/(n+\lambda)})}\}$$

Remark 3.1: Note that for the usual joint Normal - Gamma conjugate prior, the Bayes estimate of θ using LINEX loss function does not exist.

Remark 3.2: It is easily seen that the joint prior, $\pi(\theta, r)$ is a conjugate prior by considering that -

$\pi(r)\alpha r^{-3/2}e^{\frac{\alpha}{2r}-\frac{\alpha}{2\beta^2}r}$ and $\pi \quad | r)\ \mu(r)$ and $\pi^*(\theta|\underset{\sim}{x}, r)\ \pi^*(r|\underset{\sim}{x})$ belongs to the same class of density functions.

Remark 3.3: The result presented in section 3 also relates to estimation of regression coefficients and to prediction problems as pointed out on p.448 of Zellner [6].

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Author's Note:
Re: On Modality and Divisibility of *Poisson* and Binomial Mixtures pg. 202 - 207 Vol. 1 No. 3 Spring 1990
Added in the proof:
The portion of example 3, showing that a Poisson - mixture with strongly unimodal continuous mixing distribution is not correct. Indeed, as revealed to the author by D. N. Shanbhag (private communication), from a result in Karlin's book on total positivity follows that the «if» part of theorem 2 for this property also remains valid.
