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# THE POMERON IN A MODIFIED MULTIPERIPHERAL MODEL

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**Abstract**

There have been many attempts [1-3] to produce correct pomeron by putting the n-particle production amplitude exp$\lambda$t, in the unitarity integral equation. It was found that for reasonable values of $\lambda$ one could not get the right picture of the slope of the pomeron as a function of energy $S^{\frac{1}{2}}$. We show that the interference terms in the unitary equation does not improve the slope either. However, a modified multiperipheral model predicts a correct pomeron.

## 1. Introduction

Since the original paper of Michejda, Turnau and Bialas [4] there have been many discussions of whether the multiperipheral model of particle production gives, through unitarity, the correct t-dependence of elastic scattering (Fig 1). The situation is best summed up in the article of Teper [5] where references to earlier work may be found. Teper once again finds that the predicted radius is too small in the lower energy region and rises too rapidly as a function of energy compared to experiment.

The input to the model is shown in Fig. 2, where the t-dependence arises from a factor exp($\lambda$ $t_i$) associated with each internal line. The value of $\lambda$ can be found by comparing with experimental $\vec{q}^T$ distribution in inclusive processes. The other parameter required is mass of the produced objects and their multiplicity. If we assume that they are pions then the mass is known and we can read the multiplicity directly from experiment. However, it is well known that studies of correlation require the production of cluster decaying into several pions, so it is necessary to assume a value of the cluster mass and the average multipicity of each cluster.

In the next section we calculate the average number of clusters at each energy and estimate the value of $\lambda$ from data on $\vec{q}_T^{\,2}$ distribution. This permits us to calculate the effective t-dependence of elastic scattering as a function of energy.

**Key words:** Pomeron, Multiperipheral

## 2. The Cluster Model

2.1 Determination of number of particles per cluster

Let us put the problem in a slightly different way and ask the following question: What is the adequate energy to produce n clusters? Had there been just one particle per cluster, the calculation of energy would have been easier*

$$S = A \exp((n-\alpha)/3a)$$

where a and b are the coefficients in $<n_->$ = a lnS + b and A = exp(-b/a). $\alpha$ = 2 (or 0) if initial particles have positive (or zero) overall charge. It is, however, unlikely that this situation would occur. Thus, to determine the average number of (negative) particles we introduce the following method. Using charge conservation law for, e. g. $n_{cl}$ = 3 in PP scattering case, we see that the only possible formation likely to happen is

| | CLUSTER ONE | CLUSTER TWO | CLUSTER THREE |
|---|---|---|---|
| OVERALL CHARGE | + | + | 0 |

Bearing in mind that the clusters can decay in such a way that each of them preserves the charge conservation law, one can write the following possible decay modes for the example mentioned above:

* Here we assume that there are as many neutral particles available as positive or negative ones.

| | CLUSTER ONE | CLUSTER TWO | CLUSTER THREE |
|---|---|---|---|
| OVERALL CHARGE | + | + | 0 |
| DECAY MODES | .5 ++− <br> .5 00+ | .5 ++− <br> .5 00+ | .75 +−0 <br> .25 000 |

where the numbers (= $P_i$) are to serve the weighing purpose. The calculation of energy is straightforward now:

$$S = A \exp \frac{\Sigma P_i n_i^-}{n_{cl}\, a}$$

with A as defined above. For the e rample under study, using [6]

$< n_- > = .84 \ln S - 2.14$

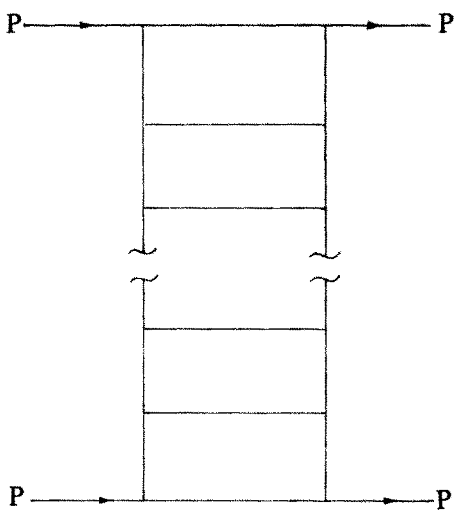


**Figure 1** Unitarity integral equation

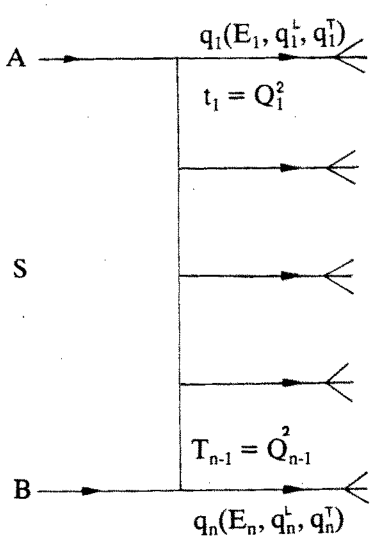


**Figure 2** Multiperipheral Model of particle production

2.2 On the value of $\lambda$

$\lambda$ is the parameter which governs the t-distribution of the amplitude, exp $\lambda$ t. We fit the data [7] namely transverse momentum distribution, for the energies calculated at 2.1. The best fits, due to error bars, give a range over $\lambda$ (Fig 3): $3.7 \leqslant \lambda \leqslant 4.4$. These values should be regarded as the lowest limits of $\lambda$. The reason is that the cutoff of $\vec{q}_T$ distribution, when dealing with particles, is sharper than that of the $\vec{q}_T$ distribution in the cluster case.

It is well known that for small momentum transfer t, one can parametrize the differential cross section as $d\sigma/dt$ exp $(-\lambda t/4)$. This, together with the optical theorem,

$$\text{Im } T^{el}(S,O) = S\, \sigma_{tot}(S)$$

gives

$$\text{Im } T^{el}(S,O) \propto \exp(-\lambda t/4).$$

Transforming the equation to the impact parameter space b, we get

$$\text{Im } T^{el}(S,O) \propto \exp(-b^2/R^2),$$

where the 'radius' R is in general a function of energy: R = R (S). See the appendix for details. The aim of the paper is to see how the radius behaves as compared to data. In Fig 4 we plot $R^2$ versus S. We see that the theory does not quite match with the experiment. Of course by choosing smaller values for $\lambda$, one can have a better slope, similar to the one in data, but with a lower lying trajectory instead.

In the next section we shall see to what extent the inclusion of the crossed diagrams would effect the radius.

## 3. The Interference Diagrams

It has usually been the case that people make the assumption that the interference terms contribute to the calulations negligibly therefore they are discarded completely. The supporting argument is that graphs with crossed lines have larger t values than the no crossed ones hence because of the sharp cutoff in t they are damped. This argument does not, however, show that even if a single crossed line graph is small, the sum of all of them would be small too; since there are so many of them. Despite this argument there have been several attempts to consider the interference diagrams in different multiperipheral models [2, 5, 8, 9].

Now that we have developed a method such that every individual diagram could be calculated separate-

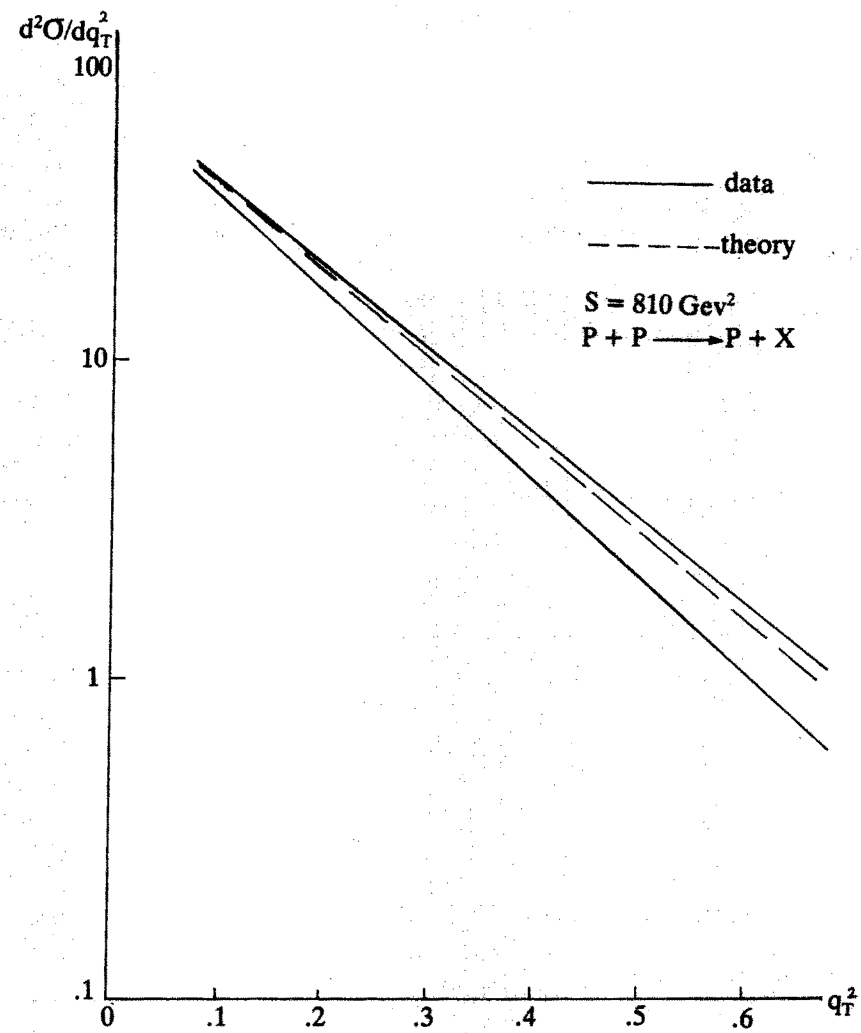


**Figure 3** Transverse momentum distribution

ly (see eqs. A7, A8 and A9), we turn to its application. To cut down the computer time and since omission of some diagrams such as those in figure 5 is justified according to the results we get from the numerical calculations, we have reduced the number of diagrams corresponding to 3, 4 and 5 intermediate states to 28 out of possible 150 ones. These diagrams are shown in Fig 6. Of course some diagrams, by way of symmetry, are representatives of two or four terms. Hence, in actual fact, we are dealing with some 65 terms.

Taking the interference diagrams into account we arrive at Fig 7, where we conclude that their contribution is so little that as far as $R^2$ is concerned, one could neglect them all.

The parameter $\lambda$ plays a crucial role in the evaluation of the importance of the crossed diagrams. This seems to be part, if not all, of the reason for the conclusion of up to 30% contribution to the slope from the crossed diagrams in reference 2.

## 4. The Modified Multiperipheral Model

The proposed link dependent model has the form

$$A(S,t_j) = \Pi_j \exp(\lambda t_j + \gamma Q_j Q_{j+2}) \qquad (1)$$

where $t_j = Q_j^2$ The parameters have already been defined in Fig 2. Equation 1 shows a correlation between every other neighbour. Terms of the form $Q_j Q_{j+1}$ introduce nothing new [10]:

$$(Q_j - Q_{j+1})^2 = q_{j+1}^2$$

$$-2Q_j Q_{j+1} + Q_j^2 + Q_{j+1}^2 = S_O$$

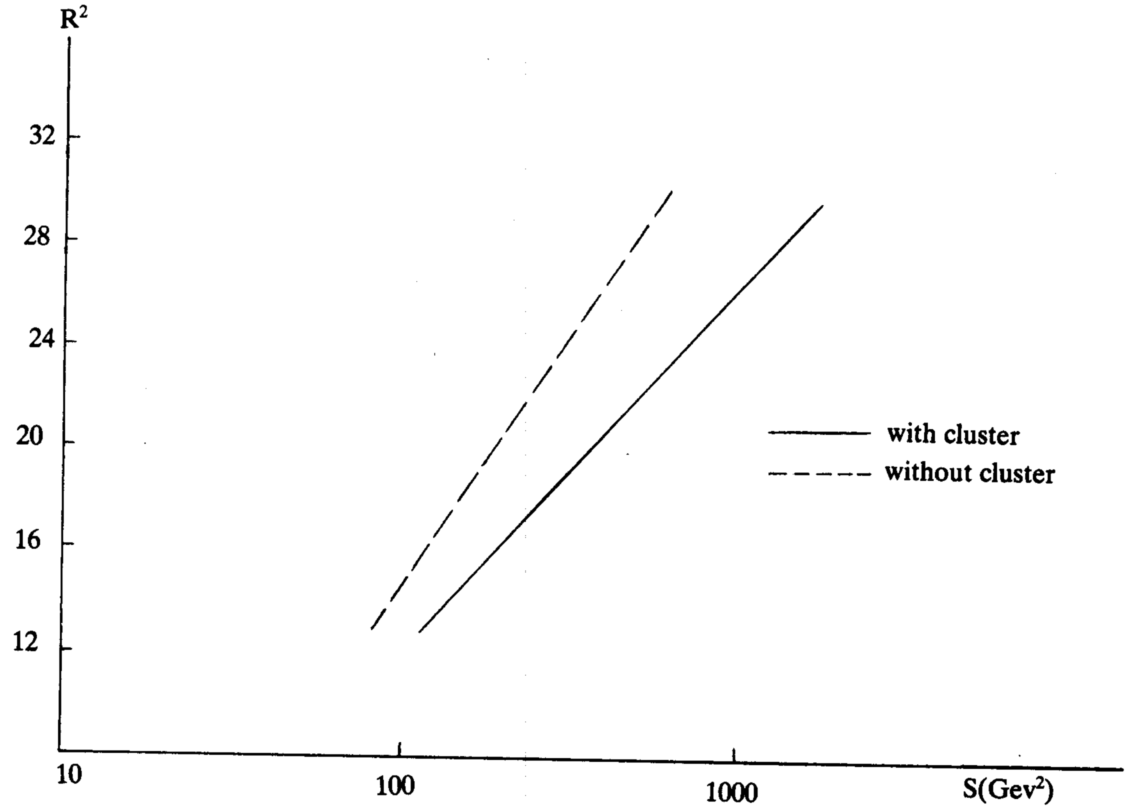


**Figure 4** Dependence of R on S

where $S_O$, being a constant, is the prong mass squared. Hence

$$Q_jQ_{j+1} = \tfrac{1}{2}(Q_j^2 + Q_{j+1}^2 - S_O).$$

On the grounds that $Q_j^2$ type terms have already been considered (see section 3) $Q_jQ_{j+1}$ can be discarded. Therefore we arrive at $Q_jQ_{j+2}$. Let us write $Q_{j+2}$ in terms of $Q_j$

$$Q_{j+2} = Q_j - (q_{j+1} + q_{j+2}). \qquad (2)$$

But

$$q_{j+1} + q_{j+2} = [(q_{j+1}^+ + q_{j+2}^+)(q_{j+1}^- + q_{j+2}^-) - (\vec{q}_{j+1}^{\,T} + \vec{q}_{j+2}^{\,T})^2]^{1/2} = [S_O(2 + \frac{X_{j+1}}{X_{j+2}} + \frac{X_{j+2}}{X_{j+1}}) - (\vec{q}_{j+1}^{\,T} + \vec{q}_{j+2}^{\,T})^2]^{1/2}.$$

Since $X_{j+1}/X_j = S^{1/(1-n)}$, hence

$$q_{j+1} + q_{j+2} \simeq \alpha - \frac{}{2\alpha}(\vec{q}_{j+1}^{\,T} + \vec{q}_{j+2}^{\,T})^2 \qquad (3)$$

where

$\alpha = \sqrt{S_O}\,(2 + S^{1/n-1} + S^{1/1-n})^{1/2}$

and $Q_j$ is the square root of eq A3. The exponent in eq 1 now becomes

$$F_j \equiv \lambda t_j + \gamma Q_jQ_{j+2}$$

$$= \{-(\lambda - \gamma)S_O\,\xi_j - \gamma S_O\,\xi_j^{1/2}(2 + S^{1/n-1} + S^{1/1-n})^{1/2}\}$$

$$-\mu_j(\sum_{r=1}^{j} \vec{q}_r^{\,T})^2 + \frac{\gamma\,\xi^{1/2}}{2(2 + S^{1/n-1} + S^{1/1-n})}(\vec{q}_{j+1}^{\,T} + \vec{q}_{j+2}^{\,T})^2 =$$

$$F_j^L + F_j^T \qquad (4)$$

where L and T stand for longitudinal and transverse parts of the exponent respectively and

$$\xi_j = (\sum_{j+1}^{n} X_r)(\sum_{1}^{J} X_r^{-1})$$

$$\mu_j = \lambda + \gamma - \frac{\gamma(2 + S^{1/n-1} + S^{1/1-n})^{1/2}}{2\,\xi_j^{1/2}}$$

The (extra) terms indicated in brackets in eq. 4 must be added to eqs A4 and A14. The rest of the calculations will be the same and in fact the end result will look like eq. A18.

To compare the radius with data, we must assign first a value to $\lambda$ and $\gamma$. This is done in Fig 8. where we have

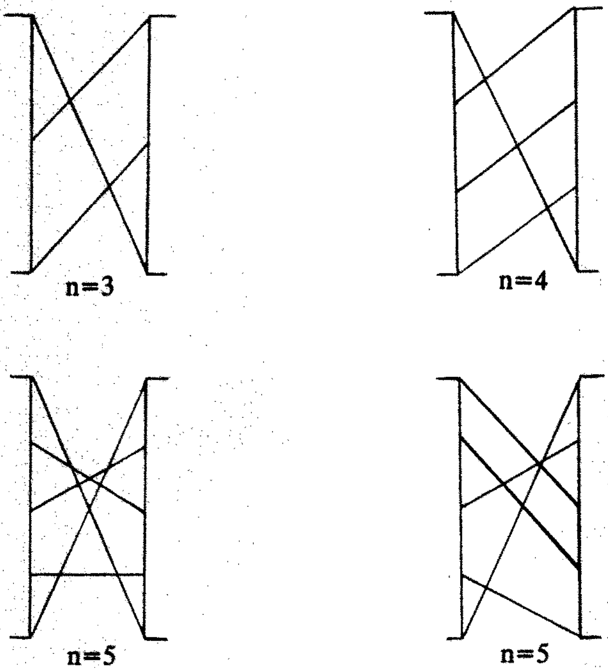


**Figure 5** Different interference diagrams

plotted $d^2\sigma/d\vec{q}_T^2$ versus $\vec{q}_T^2$. We find $\lambda = 2.6$ and $\gamma = -2$. Figure 9 shows the matching of the theory with experiment, where the variation of $R^2$, or alternatively the slope of the pomeron, against S is shown.

In conclusion, the agreement of the energy dependence of the radius with experiment over a big range of energy is a support to the proposed model.

## Acknowlegment

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## Appendix

Consider the process $A + B \longrightarrow 1 + 2, \ldots, n$. The outgoing particles will have energies $E_j$, longitudinal momenta $q_j^L$ and transverse momenta $q_j^T$. Following ref. 3, we consider the corresponding impact parameters of the outgoing particles, $\mathbf{b}_j$, as the conjugate variables of the transverse momenta $\mathbf{q}_j^T$, with

$$\tilde{A}(\vec{b}_j, q_j^L) = \int \prod_{j=1}^{n} [d^2\vec{q}_j^{\,T} e^{i(\vec{q}_j^{\,T}.\vec{b}_j)}] \, A(\vec{q}_j^{\,T}, q_j^L) \, \delta^{(2)}(\sum_{i=1}^{n} \vec{q}_i^{\,T}) \qquad (A1)$$

which differs from an ordinary Fourier transformation just by a momentum conservation delta function. Separation of transverse kinematics from longitudinal ones, together with the unitarity equation for the elastic amplitude results on

$$\mathcal{J}m \tilde{A}_{el}(\vec{b}) = \int \prod_{j=1}^{n} [d^2\vec{b}_j d Y_j] \, \delta(\Sigma q_j^L) \, \delta(\Sigma E_j - \sqrt{S}) \, \delta^{(2)}(\Sigma \vec{b}_j)$$
$$\delta^{(2)}(\vec{b} - \Sigma \frac{q_i^L}{p} \vec{b}_j) \, [\tilde{A}(\vec{b}_j, q_j^L)]^2 \qquad (A2)$$

where the rapidity is defined by

$$Y_j = \tfrac{1}{2} \ln \frac{E_j + q_j^L}{E_j - q_j^L},$$

and P is the centre of mass momentum ($S \approx 4P^2$). As mentioned earlier, the amplitude will be supposed to have the following form,

$$A(\vec{q}_j^{\,T}, q_j^L) = \exp(\lambda \sum_{j=1}^{n-1} t_j),$$

where

$$t_j = (P_a - \sum_{i=1}^{j} q_i)^2 = (P_a^+ - \sum_{i=1}^{j} q_i^+)(P_a^- - \sum_{i=1}^{j} q_i^-) - (\sum_{i=1}^{j} \vec{q}_{Ti})^2$$

We have defined the longitudinal variables $q_i^{\pm}$ as

$$q_i^{\pm} = E_i \pm q_i^L$$

Which are related to the cluster mass, $S_O$, by

$$q_i^+ q_i^- = S_o .$$

Using the momentum conservation,

$$t_j = (\sum_{i=j+1}^{n} q_i^+)(q_a^- - \sum_{i=1}^{j} q_i^-) - (\sum_{i=1}^{j} \vec{q}_i^{\,T})^2 \qquad (A3)$$

An important property of the components $q^{\pm}$ is that under Lorentz transformation along the longitudinal directions they transform like

$$q^{\pm} \longrightarrow e^{\pm\emptyset} q^{\pm}$$

so that their ratios will be invariant under these transformations.

A1.a The longitudinal calculation

We shall not consider the transverse part of the equation A3 for the time being and shall deal with it in A1.b. There, Eq. A3, $q_a^-(= E_a - q_a^L)$ is small so that with a very good approximation we can write it as

$$t_j = -S_o (\sum_{i=j+1}^{n} X_i)(\sum_{i=1}^{j} X_i)^{-1} \qquad (A4)$$

where we have defined $X_i = q_i^+$. We shall define the matrix $M_{ij}^1$ by the following equation,

$$\sum_{j=1}^{n-1} t_j = -S_o M_{ij}^1 X_i X_j \, ; \, (i,j = 1, \ldots, n) \qquad (A5)$$

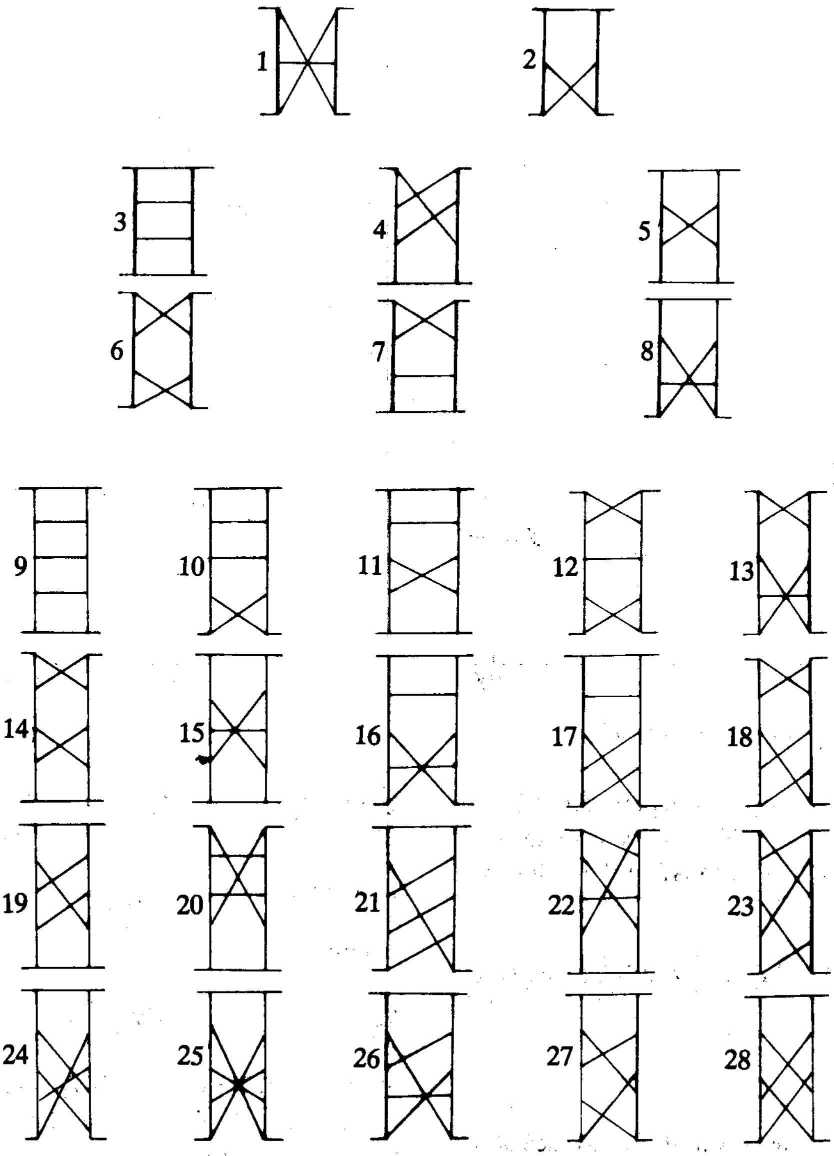


**Figure 6** Different interference diagrams

$M_{ij}$ is an n by n matrix whose diagonal is zero and can easily be calculated

$$M^{1}_{ij} = \begin{pmatrix} 00\ldots0 \\ 10\ldots0 \\ 210..0 \\ . \\ . \\ n\text{-}1..0 \end{pmatrix}$$

As the imaginary part of the amplitude depends on the square of the amplitude, there shall be another t-dependence in the exponent which may conveniently be written as

$$\sum_{j=1}^{n-1} t'_j = - S_o \, M^{2}_{ij} X_i X_j^{-1} ; (i,j = 1,\ldots,n) \quad \text{(A6}$$

where $M^2$ of A6 in general will be a permutation of $M^1$ i A5,

$$M^2 = \mathcal{T} M^1 \quad \text{(A7}$$

We define a new matrix M as the sum of these tw matrices,

$$M_{ij} = M^{1}_{ij} + M^{2}_{ij} \quad \text{(A8}$$

Therefore

$$\lambda\,(\Sigma\, t_k + \Sigma\, t'_k) = - \lambda\, S_o \, M_{ij} X_i X_j^{-1} \quad \text{(A9}$$

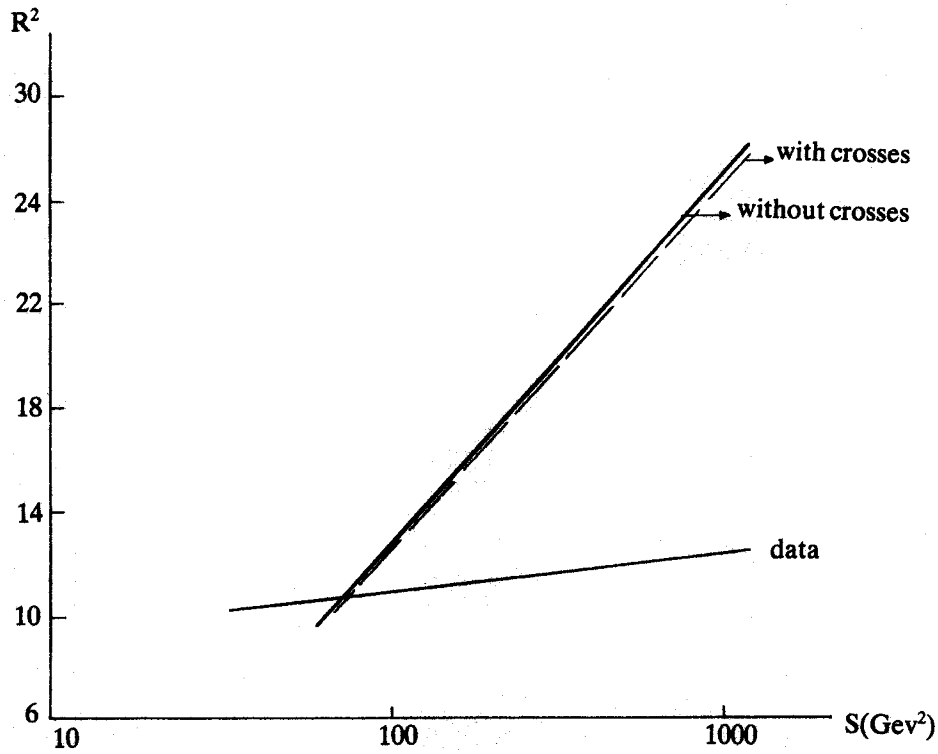


**Figure 7** Variation of $R^2$ with S; interferences included

is the exponential of the amplitude squared in A2. Realizing that $dY_i = dX_i/X_i$, we can write the longitudinal part of the equation A2 in the following form

$$\mathcal{L}=\int_{-\infty}^{\infty}\prod_{i=1}^{n}[\frac{dx_i}{X_i}]\exp(-\lambda S_o M_{ij}X_iX_j^{-1})\delta(\sqrt{S}-\sum_{J=1}^{n}X_j)\delta(\sqrt{s}-\sum_{j=1}^{n}\frac{S_o}{X_j})$$

Defining a new dimensionless variable $z_i$ in place of $X_i$ by $z_i = X_i/S_O^{1/2}$, we get

$$\mathcal{L}=\int_{-\infty}^{\infty}\prod_{i=1}^{n}[\frac{dZ_i}{Z_i}]\exp(-\lambda S_O M_{ij}Z_iZ_j^{-1})\delta(\sqrt{\frac{S}{S_o}}-\sum Z_j)\delta(\sqrt{\frac{S}{S_o}}-\sum Z_j^{-1}) \qquad (A10)$$

Unfortunately there seems to be no exact analytic way of doing this integral. So to proceed, we maximize the exponential by those $z_i$, say $z_i^o$, which minimize $M_{ij}z_iz_j^{-1}$

$$z_i = z_i^o + \varepsilon_i$$

and keep only the terms which are of the order of $\varepsilon$ or $\varepsilon^2$. $z_i^o$ are subject to the two constraints which are imposed by the two delta functions of A10. Hence,

$$\mathcal{L}=\prod_{i=1}^{n}[Z_i^o]^{-1}\exp(-\lambda S_o M_{ij}Z_i^oZ_j^{o,-1})\int\prod_{i=1}^{n}[d\varepsilon_i]\exp(-\lambda S_O Z_{ij}\varepsilon_i\varepsilon_j)\delta(\sum\varepsilon_j)\delta(\sum\frac{\varepsilon_j}{Z_j^{o2}}),$$

where

$$Z_{ij}\varepsilon_i\varepsilon_j = M_{ij}Z_i^oZ_j^{o-3}\varepsilon_j^2 - M_{ij}Z_j^{o-2}\varepsilon_i\varepsilon_j\,.$$

Note that the terms linear in $\varepsilon_i$, because of the choice of the $z_i^o$, cancel. Writing the delta functions in their integral form, we get

$$\mathcal{L}=\prod_{i=1}^{n}[Z_i^o]^{-1}\exp(-\lambda S_o M_{ij}Z_i^oZ_j^{o,-1})\int dk\,dl\int\prod[d\varepsilon_i]\exp[i(k+\frac{l}{Z_i^{o2}})\varepsilon_i-\lambda S_o Z_{ij}\varepsilon_i\varepsilon_j]. \qquad (A11)$$

The $\varepsilon_i$ integration gives*

$$\mathcal{L}=\frac{\pi^{n/2}\exp(-\lambda S_o M_{ij}Z_i^oZ_j^{-1})}{(\lambda S_o)^{n/2}\sqrt{\det Z}\prod[Z_i^o]}\int dk\,dl\exp[-\frac{1}{4\lambda S_o}Z_{ij}^{-1}(k+\frac{l}{Z_i^{o2}})(k+\frac{l}{Z_j^{o2}})] \qquad (A12)$$

* Proceeding from A11 to A12 we have assumed that Z is not a singular matrix. If it were singular, there would have been a modification in the denominator of the second fraction in A13.

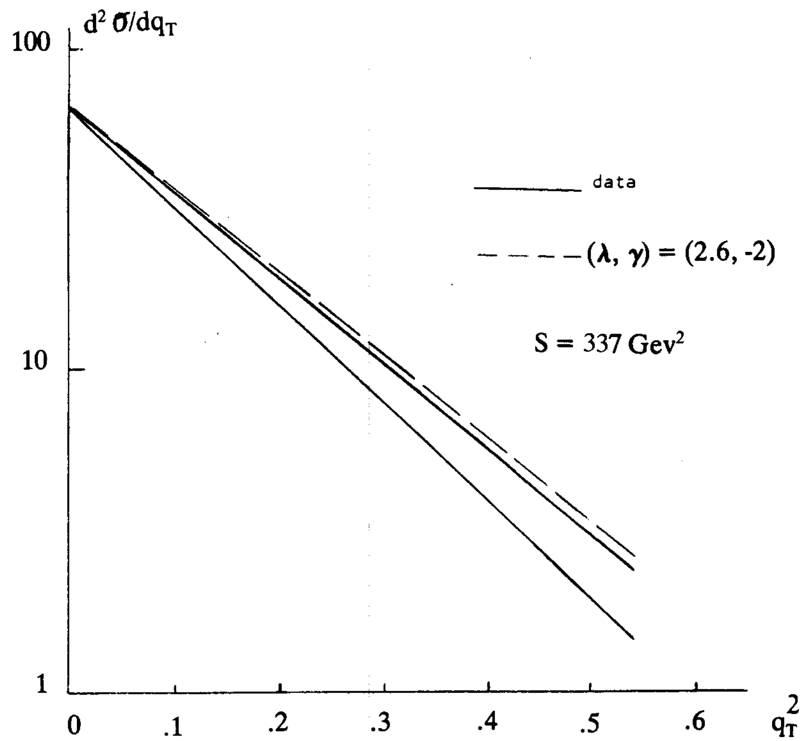


**Figure 8** Plot of $d^2\sigma/dq_T^2$ versus $q_T^2$ in order to assign values for

Now doing the k and l integration one concludes that

$$\mathcal{L} = \frac{2\pi \exp(-\lambda S_o M_{ij} \mathring{Z}_i \mathring{Z}_j^{-1})}{(\lambda S_o \pi^{-1})^n \sqrt{\det Z}\, \prod [\mathring{Z}_i]} \cdot \frac{1}{(4\Gamma_1\Gamma_2 - \Gamma_3^2)^{1/2}} \quad \text{(A13)}$$

where

$$\Gamma_1 = -\frac{1}{4\lambda S_o} \Sigma Z_{ij}^{-1}$$

$$\Gamma_2 = \frac{1}{4\lambda S_o} \Sigma Z_{ij}^{-1} \mathring{Z}_i^{-2} \mathring{Z}_j^{-2}$$

$$\Gamma_3 = \frac{1}{\lambda S_o} \Sigma Z_{ij}^{-1} (\mathring{Z}_i^{-2} + \mathring{Z}_j^{-2})$$

L weighs different interference diagrams and as we shall see later, it appears as a coefficient in the term which includes the radius.

A. 1. b The radius

Having dealt with the longitudinal part of the amplitude we shall discuss the transverse part here. The procedure will be more or less as before. We are mainly concerned with the term which we suppressed in A3, namely, $t_j^T = -(\sum_{i=1}^{j} \vec{q}_i^T)^2$, which corresponds to the transverse part of the amplitude,

$$A(\vec{q}_j^T) = \prod_{j=1}^{n} \exp[-\lambda(\sum_{i=1}^{j} \vec{q}_i^T)^2]. \quad \text{(A 14)}$$

As before let us define an n by n and symmetric matrix $L_{ij}^1$ by the following equation,

$$L_{ij}^1\, \vec{q}_i^T \vec{q}_j^T = \sum_{j=1}^{n} (\sum_{i=1}^{j} \vec{q}_i^n)^2.$$

The amplitude, now, looks like,

$$A(\vec{q}_j^T) = \exp(-\lambda L_{ij}^1\, \vec{q}_i^T q_j^T).$$

One can transform this to the $\vec{b}_i$ space using equation A1

$$\tilde{A}(\vec{b}_j) = \frac{4\pi^2}{(\lambda/\pi)^{n-1} \det L^1 (\sum_{ij=1}^{n} L^{1-1}_{ij})} \exp(-\frac{1}{\lambda} B_{ij}^1 \vec{b}_i \cdot \vec{b}_j)$$

where

$$B_{ij}^1 = -\frac{1}{4} [L_{ij}^{1^{-1}} - \frac{1}{\Sigma L_{ij}^{1^{-1}}} L_{ir}^{1^{-1}} \mathcal{J}_{rs} L_{sj}^{-1}] \quad \text{A15}$$

and

$$\mathcal{J}_{rs} = 1, \quad \forall r,s = 1,\ldots,n.$$

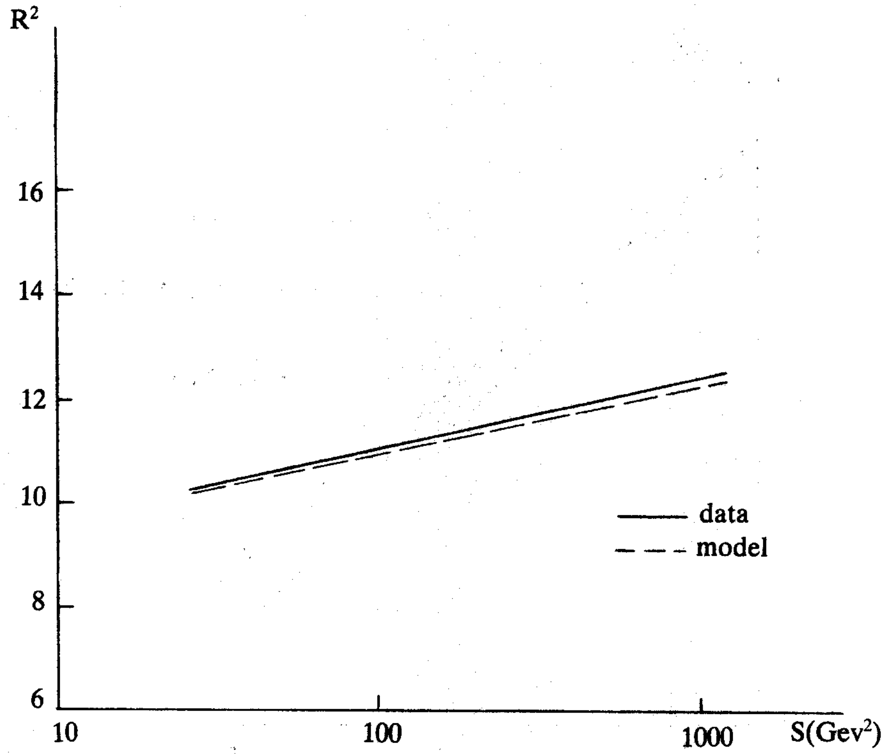


**Figure 9** Variation of $R^2$ with S as predicted from the modified model

Matrix $B^1$ is n by n and symmetric.
Let us rewrite the transverse part of equation A2,

$$T(\vec{b}) = \int_{-\infty}^{\infty} \prod_{j=1}^{n} [d^2\vec{b}_j]\, \delta^{(2)}\Big(\sum_{j=1}^{n} \vec{b}_j\Big)\, \delta^{(2)}\Big(\vec{b} - \Sigma X_j \vec{b}_j\Big) |\tilde{A}(b_j)|^2 . \qquad A16$$

Once more we need to permute A 14

$B_{ij}^2 = \mathcal{P}(B^1_{ij})$
and
$B_{ij} = B^1_{ij} + B^2_{ij}$.
Hence,
$|\tilde{A}(\vec{b}_i)|^2 = h_1 \exp(-\lambda^{-1} B_{ij}\, \vec{b}_i\, \vec{b}_j)$,
where,

$$h_1 = \frac{16\pi^4}{(\lambda/\pi)^{2(n-1)}} [(\Sigma_{i,j} L^{1\,-1}_{ij})^2 \det L^1 \det L^2]^{-1}$$

$B_{ij}$ is an n by n symmetric matrix. It could be, and in most cases is, singular. In what follows we shall assume that it is singular. The nonsingular case could be proceeded likewise. The fact that $B_{ij}$ is singular does not mean that the integral A16 will be divergent at all. The reason for this is the existence of the delta functions. This makes it possible to get rid of the zero eigenvalues corresponding to matrix B as explained below. Writing the delta function of A16 in its integral form we get,

$$T(\vec{b}) = h_1 \int_{-\infty}^{\infty} d^2x\, d^2y \exp(i\vec{y}.\vec{b}) \int \prod^{n} [d^2 b_i \exp(i(\vec{x} - X_i.\vec{Y}).\vec{b}_i)] \exp(-\lambda^{-1} B_{ij}\, \vec{b}_i \vec{b}_j)$$

$$T(\vec{b}) = h_2 \int d^2x\, d^2y \exp(i\vec{y}.\vec{b})\, \delta^{(2)}\Big(\vec{X} - \frac{\Sigma\, Cin\, X_i}{\Sigma\, Cin} \vec{Y}\Big)$$

$$\exp[-\lambda F_{ij} (\vec{X} - X_i\vec{Y})(\vec{X} - X_j\vec{Y})],$$

where

$$h_2 = h_1 (\pi\lambda)^{n-1} [\prod_{i=1}^{n-1} [\lambda_i] (\Sigma^{n}_{i=i} C_{jn})]^{-1},$$

and $C_{ij}$ is the matrix which diagonalizes $B_{ij}$ to produce the eigenvalues $\lambda_i$. The details of integration over $\vec{b}_i$ is at the end of this section where $F_{ii}$ is also defined. Using the delta function one can easily do the integration over $\vec{x}$,

$$T(b) = h_2 \int d^2Y \exp(i\vec{y}.\vec{b}) \exp(-\lambda\, \theta\, \vec{Y})^2,$$

where

$$\theta = \sum_{ij=1}^{n} F_{ij} (\gamma - X_i)(\gamma - X_j),$$

and

$$\gamma = \Big(\sum_{i=1}^{n} C_{in} X_i\Big) / \Big(\sum_{i=1}^{n} C_{in}\Big).$$

One, therefore, finally ends up with

$$\Upsilon(\vec{b}) = \frac{\pi h_2}{} \exp\left(-\frac{\vec{b}^2}{4\lambda\theta}\right). \qquad A17$$

The radius is customarily defined according to

$$R^2 = 4\lambda\theta \qquad A18$$

* * *

In this part we would like to evaluate the following integral,

$$I = \int_{-\infty}^{\infty} \prod_{i=1}^{n} [d^2\vec{b}_i \exp(i\vec{\alpha}_i . \vec{b}_i)] \exp(-B_{ij}\vec{b}_i\vec{b}_j),$$

where $B_{ij}$ is a symmetric and singular matrix. Let C be the matrix which transforms B to D, where D is diagonal,

$$D_{ij} = (C^T B C)_{ij} .$$

Put

$$\tilde{\vec{b}}_i = \sum_j C_{ji}\vec{b}_j \qquad \text{i.e. } (\tilde{\vec{b}}_i = C\vec{b}_i)$$

and

$$\tilde{\vec{\alpha}}_i = C\vec{\alpha}_i .$$

Since

$$B_{ij}\vec{b}_i\vec{b}_j = D_{ij}\tilde{\vec{b}}_i\tilde{\vec{b}}_j$$

and

$$\sum_i \vec{\alpha}_i . \vec{b}_i = \sum \tilde{\vec{\alpha}}_i . \tilde{\vec{b}}_i .$$

then we can write I in the following way

$$I = \int_{-\infty}^{\infty} \prod_{i=1}^{n} [d^2\tilde{\vec{b}}_i \exp(i\tilde{\vec{\alpha}}_i . \tilde{\vec{b}}_i)] \exp(-D_{ij}\tilde{\vec{b}}_i\tilde{\vec{b}}_j).$$

The singularity of matrix B implies that one of the entries of matrix D,

$$D_{ij} = \begin{pmatrix} \lambda_1 & & O \\ & \lambda_2 & \\ O & & \ddots \; \lambda_n \end{pmatrix}$$

say $\lambda_n$, must be zero. Therefore,

$$I = \int_{-\infty}^{\infty} d^2\tilde{\vec{b}}_n \exp(i\tilde{\vec{\alpha}}_n . \tilde{\vec{b}}_n) \int_{-\infty}^{\infty} \prod_{i=1}^{n-1} [d^2\tilde{\vec{b}}_i \exp(i\tilde{\vec{\alpha}}_i . \tilde{\vec{b}}_i)] \exp(-D\!\!\!/_{ij}\tilde{\vec{b}}_i^{\,2}),$$

where

$$D\!\!\!/_{ij} = \begin{pmatrix} \lambda_1 & & O \\ & \lambda_2 & \\ O & & \ddots \; \lambda_{n-1} \end{pmatrix}$$

The evaluation of first and second integral is easy enough,

$$I = \frac{\pi^{n-1}}{\det \phi} \delta^{(2)}(\tilde{\vec{\alpha}}_n) \exp\left[-\frac{1}{4} D\!\!\!/^{-1}_{ij} \tilde{\vec{\alpha}}_i^{\,2}\right].$$

Putting back $\tilde{\vec{\alpha}}_i$ in $\vec{\alpha}_i$ form, we get

$$I = \frac{\pi^{n-1}}{\det \phi} \delta^{(2)}\left(\sum_i C_{in}\vec{\alpha}_i\right) \exp\left[-\frac{1}{4}\sum_{i=1}^{n-1} \frac{1}{\lambda_i}\left(\sum_{j=1}^{n} C_{ji}\vec{\alpha}_j\right)^2\right]$$

This is the result, but to make it look more elegant let us define a new n by n symmetric matrix, $F_{ij}$, by

$$F_{ij}\vec{\alpha}_i\vec{\alpha}_j = \frac{1}{4}\sum_{i=1}^{n-1}\frac{1}{\lambda_i}\left(\sum_{j=1}^{n} C_{ji}\vec{\alpha}_j\right)^2$$

then

$$I = \frac{\pi^{n-1}}{\prod_{i=1}^{n-1}\lambda_i} \delta^{(2)}\left(\sum_i C_{in}\vec{\alpha}_i\right) \exp[-F_{ij}\vec{\alpha}_i\vec{\alpha}_j].$$

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