AFFINE SUBGROUPS OF THE CLASSICAL GROUPS AND THEIR CHARACTER DEGREES

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Abstract

In this paper we describe how the degrees of the irreducible characters of the affine subgroups of the classical groups under consideration can be found inductively. In [4] Gow obtained certain character degrees for all of the affine subgroups of the classical groups. We apply the method of Fischer to the above groups and, in addition to the character degrees given in [4], we obtain some new character degrees for these groups.

1. Introduction

Let G be one of the symplectic, unitary or orthogonal groups defined over V= V(n, q), the vector space of dimension n over the field with q elements. The subgroups of G which fix a certain non-zero vector of V are called affine subgroups of G. Let us call one of these groups A. In [4] certain irreducible characters of A are found. In [4] Gow also found certain irreducible characters of the affine subgroups of the general linear group. These characters have also been studied in [10] for the symplectic case. Our aim in this paper is to employ the powerful and interesting method of Fischer described in [2] and [3] to obtain all the irreducible characters of A inductively in the cases that Fischer's method is applicable. This method has already been applied to the general linear groups and the symplectic groups in [1], certain wreath products in [7] and certain extensions of the symmetric groups in [8].

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We refer the reader to the books [5] and [6] for notations concerning the above groups and character theory. All the characters concerned are over the complex field C.

Let H be a group and $V \trianglelefteq H$. Let Irr(V) denote the set of all the irreducible characters of V. Then H acts on Irr(V) and for $\chi \in \mathrm{Irr}(V)$ the stabilizer of χ in H, denoted by I_χ, is called the inertia group of χ in H. Clearly $V \trianglelefteq I_\chi$, and the factor group $\bar{I}_\chi = I_\chi/V$ is called the inertia factor of I_χ. We say that $\chi \in \mathrm{Irr}(V)$ is extendible to an irreducible character φ of its inertia group in H if $\varphi\downarrow_V = \chi$. In [2] a method is presented to calculate the character table of any group extension H = V.G provided that every irreducible character of V extends to an irreducible character of its inertia group in H. This is the case if $I_\chi = V.\bar{I}_\chi$ is a split extension and χ is a linear character of V (see problem 6.18 [6]). Since for all the affine groups concerned in this paper I_χ, where χ is a linear character, is in the above form therefore we are able to use Fischer's method.

If χ is an irreducible character of V and $\hat{\chi}$ denotes its extension to I_χ, then by Clifford's theorem [6] every irreducible character of H is of the form $(\hat{\chi}.\beta)\uparrow^H$ where β is an irreducible character of I_χ with the property that V

$\subseteq ker\beta$. Here the corresponding irreducible character of $\bar{I}_\chi$ is also denoted by β. Evaluation of $(\hat{\chi}.\beta)\uparrow^H$ on an element h which maps to a conjugacy class κ of G involves a matrix which is denoted by F_χ^κ and is called the Fischer matrix of χ at the class κ. The precise description of this matrix is given in [7], [8] and [9]. Moreover if $\chi_1,\ldots,\chi_s$ are representatives of the orbits of H acting on the irreducible characters of V and M = $\{v \mid \bar{I}_{\chi_v}$ contains an element in $\kappa\}$ = $\{v_1,\ldots,v_l\}$, then the Fischer matrix of H at h is

$$F^\kappa = \begin{bmatrix} F^\kappa_{\chi_{v_1}} \\ \vdots \\ F^\kappa_{\chi_{v_1}} \end{bmatrix}$$

Moreover if $C^\kappa_{\chi_i}$ is the part of the character table of $\bar{I}_{\chi_i}$ consisting of the columns corresponding to the classes of $\bar{I}_{\chi_i}$ which fuse to κ in G, then the character table of H at the classes $\kappa_1,\ldots,\kappa_t$ is given by the matrix product $C^\kappa_{\chi_i}.F^\kappa_{\chi_i}$, $1 \le i \le s$, where $\kappa_1,\ldots,\kappa_t$ are all the conjugacy classes of H which map to κ.

2. The Affine Symplectic Group

In this section we follow [4] for a description of the affine symplectic group. Let V(2n,q) be the vector space of dimension 2n defined over GF(q), q a power of the prime p. Let $\{e_1,e_2,\ldots,e_{2n}\}$ be a basis for V(2n,q) and let f be the non-degenerate symplectic form on V(2n,q) defined by $f(e_i, e_j) = \delta(i, 2n+1-j)$, $1 \le i < j \le 2n$ and $f(x,x) = 0$ for all x $\in$ V(2n,q). Then the subgroup of GL(2n,q) leaving f invariant is G(n) = SP(2n,q). The group G(n) acts transitively on the non-zero vectors of V(2n,q) and the stabilizer of e_1 under G(n) is the group A(n). In [4] Gow has shown that A(n) is a split extension of a p-group P(n) of order q^{2n-1} by a group isomorphic to SP(2n-2,q).

These groups may be described as follows. If again f denotes the restriction of the above form to V(2n-2,q) generated by $\{e_2,\ldots e_{2n-1}\}$, then P(n) is isomorphic to the group consisting of [v, a], $v \in$ V(2n-2,q), $a \in$ GF(q), with multiplication $[v, a]\,[u, b] = [v+u, a+b+f(v,u)]$. A($n$) is isomorphic to the group P(n).G(n-1), where G(n-1) consists of the matrices in GL(2n-2,q) leaving f invariant. The action of G(n-1) on P(n) is as follows:
$[v, a]^A = [A^{-1}v, a]$ where $v \in$ V(2n-2,q), $a \in$ GF(q), A $\in$ G(n-1). Furthermore the action of A(n) on P(n) is as follows:
$[v, a]^{[u,b]A} = [A^{-1}v, a+2f(v, u)]$ where $u, v \in$ V(2n-2, q), a,b $\in$ GF(q), A $\in$ G(n-1).

If q is even, then P(n) is an elementary abelian 2-group and when q is odd, then P(n) is a special p-group. We apply Fischer's method for linear characters of P(n).

Theorem 1. Let G(n) = SP(2n, q), q odd, and let A(n) denote the stabilizer of a non-zero vector. Then degrees of some of the irreducible characters of A(n) are as follows: the degrees of the irreducible characters of SP(2n-2,q), the degrees of the irreducible characters of A(n-1) multiplied by q^{2n-2}-1.

Proof. Since A(n) is a split extension of P(n) by G(n-1), then for any linear character $\chi \in$ Irr(V), $\bar{I}_\chi$ is a split extension of P(n) by $\bar{I}_\chi$. Hence we will only consider linear characters of P(n). Since P(n)′ = Z(P(n)) = $\{[0, a] \mid a \in$ GF(q)$\}$ is a group of order q, therefore P(n) has q^{2n-2} linear characters. These linear characters may be described as follows. Let GF(p) be the prime subfield of GF(q) and tr : GF(q) $\to$ GF(p) be the trace map. Let ε be a primitive p^{th} root of unity in C. It is easy to verify that for each $u \in$ V(2n-2, q), the function χ_u: P(n) $\to C$ given by $\chi_u([v, a]) = \varepsilon^{tr(f(u,v))}$ is a linear character of P(n) and that all the q^{2n-2} linear characters of P(n) arise in this way.

Now if χ_u is invariant under A $\in$ G(n-1), then $\chi_u^A([v, a]) = \chi_u([v, a])$ for all $v \in$ V(2n-2,q), $a \in$ GF(q). Considering the action of G(n-1) on P(n) we obtain $tr(f(Au-u,v)) = 0$, for all $v \in$ V(2n-2,q). Suppose Au-u = w $\neq 0$. Since f is non-degenerate, the linear functional ε_w: V$\to$GF(q) given by $\varepsilon_w(v) = f(w,v)$ is onto. Therefore all the elements of GF(q) are of the form $f(w,v)$ for some $v \in$ V(2n-2,q). But this implies that the trace function is identically zero which is a contradiction. Hence Au = u. If u = 0, then $\{\chi_0\}$ is an orbit with inertia factor group G(n-1), and if $u \neq 0$, then $\{\chi_u \mid u \in$ V(2n-2, q)$^\#\}$ is another orbit with inertia factor group A(n-1).

Now we find degrees of some of the irreducible characters of the group A(n) and for this we must find the first entry of the matrices $F^{id}_{\chi_i}$, i = 1,2, where χ_i is a representative from each of the two orbits of linear characters of P(n). But the first entry of each of the above matrices is the orbit size in the case of linear characters which are 1, q^{2n-2}-1 respectively. Since some part of the first column of the character table of the group A(n) equals the first column of the matrix product $C^{id}_{\chi_i}.F^{id}_{\chi_i}$, i = 1,2, hence the theorem follows.■

The above theorem shows that if we know the character degrees of G(k), $k<n$, then some character degrees of A(n) can be obtained using induction on n. Using only the existence of the identity character in G(k) we obtain the following Corollary.

Corollary 2. The group A(n) has irreducible characters of degree $(q^{2n-2}-1)\ldots(q^{2n-2k}-1)$.

We now consider the case when q is even. In this case P(n) is an elementary abelian 2-group and A(n) has $2q$ orbits on P(n). If V is a vector space of even dimension over GF(q), q even, then there are two non-equivalent quadratic forms defined over V which are denoted by Q^+ and Q^-. If dim V= $2m$, then the subgroup of GL($2m$, q), leaving these forms invariant, are denoted by $O^+(2m, q)$ and $O^-(2m,q)$ respectively.

Theorem 3. Let G(n) = SP ($2n$, q), q even, and let A(n) denote the stabilizer of a non-zero vector. Then the degrees of the irreducible characters of A(n) are as follows: the degrees of the irreducible characters of SP($2n$-2,q), the degrees of the irreducible characters of A(n-1) multiplied by $q^{2n-2}-1$, the degrees of the irreducible characters of $O^+(2n-2, q)$ multiplied by $\frac{1}{2}q^{n-1}(q^{n-1}+1)$ and the degrees of the irreducible characters of $O^-(2n-2,q)$ multiplied by $\frac{1}{2}q^{n-1}(q^{n-1}-1)$. Moreover the number of characters in each of the last two cases is q-1.

Proof. As we remarked earlier in this case, P(n) is an elementary abelian 2-group of order q^{2n-1} and is isomorphic to the group consisting of $[v,a], v\in V(2n-2,q), a\in GF(q)$, with multiplication $[v,a]\,[u,b]=[v+u, a+b+f(v,u)]$. We also know that the action of A(n) on P(n) is as follows:

$$[v,a]^{[u,b]A}=[A^{-1}v, a], A\in SP(2n-2, q); [v,a], [u,b]\in P(n).$$

Therefore A(n) has 2q orbits on P(n). The number of the orbits of A(n) on Irr(P(n)) is also 2q which will be described as follows.

Similar to the proof of Theorem 1, for every vector u in V($2n$-2,q) the function $\chi_u: P(n)\to C$ given by $\chi_u([v,a]) = (-1)^{tr(f(u,v))}$ is an irreducible character of P(n). Here of course tr maps GF(q) onto GF(2). The group G(n-1) acting on the set of characters of this type produces two orbits of sizes 1 and $q^{2n-2}-1$ with inertia factor groups G(n-1) and A(n-1) respectively.

Let $Q^\varepsilon, \varepsilon=\pm1$, be the representatives of the two classes of quadratic forms with associated bilinear form f. Then it is easy to check that for any field element $b\in GF(q)$ the function $\chi_b: P(n)\to C$ given by $\chi_b([v,a])=(-1)^{tr(b(Q^\varepsilon(v)+a))}$ is a linear character of P(n). If b and c are distinct elements of GF(q), then χ_b and χ_c lie in different orbits under G(n-1). Because if $\chi_b^A=\chi_c$ for some A $\in$ G(n-1), then $\chi_b^A([0, a])=\chi_c([0,a])$ for all $a\in$ GF(q) which implies that $tr((b-c)a)=0$ and since $b\neq c$ this implies that the trace function is identically zero. Therefore if b runs through the q-1 non-zero elements of GF(q) we have 2(q-1) orbit representatives χ_b for the orbits of G(n-1) on Irr(P(n)). The orbit sizes are $\frac{1}{2}q^{n-1}(q^{n-1}+\varepsilon)$ where $\varepsilon=\pm1$. Now as in [4] it can be shown that the stabilizer of χ_b in G(n-1), $0\neq b\in$ GF(q), is a group isomorphic to $O^\varepsilon(2n-2,q)$ if χ_b is taken from an orbit of length $\frac{1}{2}q^{n-1}(q^{n-1}+\varepsilon)$. Now a similar argument as that given at the end of the proof of Theorem 1 gives the result.■

Again considering the above theorem and the identity character we obtain the following Corollary, in addition to the degrees which are obtained by Gow in [4].

Corollary 4. The group A(n) has q-1 irreducible characters of degree $\frac{1}{2}q^{n-1}(q^{n-1}+\varepsilon)$ and for each k, $1\le k\le n-2$ it has q-1 irreducible characters of degree $\frac{1}{2}q^{n-k-1}(q^{n-k-1}+\varepsilon)$ $(q^{2n-2}-1)\ldots(q^{2n-2k}-1)$ where $\varepsilon=\pm1$.

3. The Affine Unitary Group

Let $V(n, q^2)$ denote the vector space of dimension n over the Galois field equipped with a non-degenerate Hermitian form f. Then the unitary group defined on $V(n,q^2)$ is denoted by $U(n,q^2)$. This group acts transitively on the set of non-zero isotopic vectors of $V(n, q^2)$ and the affine unitary group is defined to be the stabilizer of a non-zero isotropic vector under the action of $U(n, q^2)$. We set G(n)= $U(n,q^2)$ and denote the affine unitary group by A(n). Therefore we have $[G(n): A(n)]=(q^n-(-1)^n)(q^{n-1}+(-1)^n)$ and according to [4] the group A(n) is a split extension of a special p-group P(n) of order q^{2n-3} by a subgroup isomorphic to G(n-2).

Let $\{e_1,e_2,\ldots,e_n\}$ be a basis for $V(n, q^2)$ and let f be the non-degenerate Hermitian form on $V(n, q^2)$ defined by $f(e_i, e_j)=\delta(i,n+1-j)$, $1\le i\le j\le n$. In this case e_1 is an isotropic vector and we let A(n) to be the stabilizer of e_1 under the action of G(n). According to [4] the group P(n) is isomorphic to the group P= $\{[v,a]\mid v\in V(n-2, q^2), a\in GF(q^2), tr(a)+f(v,v)=0\}$ where tr is the trace function from GF(q^2) to GF(q) and where the multiplication in P is given as follows:

$$[v, a]\,[u, b]=[v+u, a+b-f(v, u)).$$

The action of A(n) on P(n) is as follows: $[v, a]^{[u,b]A}=[A^{-1}v, a+f(u, v)-\overline{f(u,v)}]$, where the bar denotes the involuntary automorphism of GF(q^2) sending each element to its q^{th} power. Therefore we see that A(n) on P(n) has 2q orbits and hence the number of orbits of A(n) acting on the set of the irreducible characters of P(n) is also 2q. We will find all orbits on the set of linear characters of P(n) in the following theorem.

Theorem 5. Let $G(n) = U(n, q^2)$ and let A(n) denote the stabilizer of a non-zero isotropic vector. Then degrees of some of the irreducible characters of A(n) are as follows: the degrees of the irreducible characters of $U(n\text{-}2, q^2)$; the degrees of the irreducible characters of A(n-2) multiplied by $(q^{n-2}-(-1)^n)(q^{n-3}+(-1)^n)$; the degrees of the irreducible characters of $U(n\text{-}3, q^2)$ multiplied by $q^{n-3}(q^{n-2}-(-1)^n)$. Furthermore there are q-1 irreducible characters of the latter type.

Proof. It is easy to see that the center of the group P(n) is equal to the set $\{[0,a] \mid a \in GF(q^2), tr(a)=0\}$ and therefore P(n) has q^{2n-4} linear characters. These q^{2n-4} linear characters may be described as follows. Let p be the characteristic of GF(q) and tr denote the trace function from GF(q) to GF(p). If ε denotes a primitive p^{th} root of unity in C, then for each vector $u \in V(n\text{-}2, q^2)$ the function χ_u: P(n) $\to$ C given by $\chi_u([v, a]) = \varepsilon^{tr(f(u,v))}$ is a linear character of P(n). All the q^{2n-4} linear characters of P(n) are of the above forms. Now if χ_u is fixed by some $A \in G(n\text{-}2)$, then as in the proof of Theorem 1 we get $Au = u$. If $u = 0$, then we have one orbit of size 1 with the inertia factor group G(n-2). If $u \neq 0$ and $f(u, u) = 0$, then since G(n-2) is transitive on the set of non-zero isotropic vectors of $V(n\text{-}2,q^2)$ we get another orbit of size $(q^{n-2}-(-1)^n(q^{n-3}+(-1)^n)$ with inertia factor group isomorphic to A(n-2). Since for a given $0 \neq c \in GF(q)$, the group G(n-2) acts transitively on the set of vectors $u \in V(n\text{-}2,q^2)$ such that $f(u,u)=c$, therefore we obtain q-1 orbits each of size $q^{n-3}(q^{n-2}-(-1)^n)$ with inertia factor groups isomorphic to G(n-3).

So far we have obtained q+1 orbits of G(n-2) on the set of linear characters of P(n). Now considering the Fischer matrices and the inertia factor groups we obtain the result.■

Corollary 6. For integers n, k set $\varphi(n,k) = (q^{n-2}-(-1)^n)(q^{n-3}+(-1)^n)\ldots(q^{n-2k}-(-1)^n)(q^{n-2k-1}+(-1)^n)$. Then for each k, $1 \leq k \leq \left\lceil \frac{n-2}{2} \right\rceil$, the group A($n$) has irreducible characters of degree $\varphi(n,k)$. It also has q-1 character degrees as follows: $q^{n-3}(q^{n-2}-(-1)^n)$, and $\varphi(n,k)q^{n-2k-3}(q^{n-2k-2}-(-1)^n)$ where $1 \leq k \leq \left\lceil \frac{n-4}{2} \right\rceil$.

4. The Affine Orthogonal Group

In this section first we consider the orthogonal groups in odd characteristics and odd dimensions. Therefore let $V = V(2n+1, q)$ be a $(2n+1)$-dimensional vector space over GF(q) with basis $\{e_1, e_2, \ldots, e_{2n+1}\}$ and let f be the non-degenerate symmetric bilinear form defined by $f(e_i, e_j) = \delta(i, 2n+2-j)$, $1 \leq i \leq j \leq 2n+1$, on V. We let $G(n) = O(2n+1, q)$ to be the group of invertible linear transformations of V leaving f invariant. In this case $Q(v) = \frac{1}{2} f(v, v)$ defines a quadratic form and G(n) acts transitively on the set of all the non-zero isotropic vectors. We let the affine group A(n) in this case be the stabilizer of a non-zero isotropic vector namely e_1. We have $[G(n): A(n)] = q^{2n}-1$.

Lemma 7. A(n) is the semi-direct product of an abelian group of order q^{2n-1} and a group isomorphic to G(n-1).

Proof. Let J_{2n+1} be the matrix of f relative to the basis $\{e_1, e_2, \ldots, e_{2n+1}\}$. If a $(2n+1) \times (2n+1)$ matrix x fixes e_1 it must have the form $x = \begin{bmatrix} 1 & u & a \\ 0 & A & v \\ 0 & w & b \end{bmatrix}$ where u and w are row vectors in dimension $2n$-1, v is a column vector in dimension $2n$-1, A is a $(2n\text{-}1) \times (2n\text{-}1)$ matrix and $a,b \in GF(q)$. Now since x must leave the form f invariant we obtain $b = 1$, $w = 0$, $A^t J_{2n-1} A = J_{2n-1}$, $2a + v^t J_{2n-1} v = 0$ and $u = -v^t J_{2n-1} A$. Therefore the general form of an element of A(n) is $x = \begin{bmatrix} 1 & -v^t J_{2n-1} A & a \\ 0 & A & v \\ 0 & 0 & 1 \end{bmatrix}$ where $2a + v^t J_{2n-1} v = 0$, $A^t J_{2n-1} A = J_{2n-1}$. We set $G = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 1 \end{bmatrix} \mid A^t J_{2n-1} A = J_{2n-1} \right\}$ and

$$P = \left\{ \begin{bmatrix} 1 & -v^t J_{2n-1} & a \\ 0 & I_{2n-1} & v \\ 0 & 0 & 1 \end{bmatrix} \mid 2a + v^t J_{2n-1} v = 0 \right\}$$

Since x can be written as $x = \begin{bmatrix} 1 & -v^t J_{2n-1} & a \\ 0 & I_{2n-1} & v \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 1 \end{bmatrix}$, therefore A($n$) is the semi-direct product of a p-group P of order q^{2n-1} by a group G which is isomorphic to $O(2n\text{-}1, q) = G(n\text{-}1)$. If the restriction of f to the $(2n\text{-}1)$-dimensional subspace generated by $\{e_2, \ldots, e_{2n}\}$ is again denoted by f, then P is isomorphic to the group $P(n) = \{[v, a] \mid v \in V(2n\text{-}1, q), a \in GF(q), 2a + f(v, v) = 0\}$, where multiplication in

P(n) is as follows: $[v, a]\,[u, b] = [v + u, a+b-f(v,u)]$. Since f is symmetric therefore P(n) is an abelian group and the lemma is proved.■

When q is odd, then $Q(v) = \frac{1}{2}f(v, v)$ defines a quadratic form on V($2n$+1,q). Therefore P(n) may be written as P(n)= $\{[v, -Q(v)] \mid v \in V(2n-1, q)\}$. The multiplication of elements of P(n) is given by $[v, -Q(v)]\,[u, -Q(u)] = [v+u, -Q(v+u)]$. The action of A($n$) on P($n$) is as follows:

$[v, -Q(v)]^{[u, -Q(u)]A} = [A^{-1}v, -Q(v)]$ where A is identified with the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and u, v are vectors of the ($2n$-1)-dimensional vector space $V(2n-1, q)$. Now we consider the orbits of A(n) on P(n). The number of vectors v for which $Q(v) = 0$ is q^{2n-2} and A(n) has two orbits on the set of these vectors, namely the zero vector and the set of non-zero isotropic vectors with stabilizers isomorphic to P(n).G(n-1) = A(n) and P(n).A(n-1) respectively. From the action of A(n) on P(n) we see that for the vectors v, where $Q(v) = a$ is a fixed non-zero element of GF(q), the set $\{[v, -Q(v)] \mid v \in V(2n-1, q)\}$ is an orbit of A(n). If a runs through the square elements of GF(q), then we produce $\frac{q-1}{2}$ orbits each of size $q^{2(n-1)}+q^{n-1}$ with stabilizers isomorphic to P(n).$O^+(2n-2,q)$, and if a runs through the non-square elements in GF(q) then we obtain $\frac{q-1}{2}$ orbits each of size $q^{2(n-1)}-q^{n-1}$ with stabilizers isomorphic to P(n).$O^-(2n-2,q)$. Here $O^{\pm}$ represents the two classes of orthogonal groups in even dimensions. Therefore we see that A(n) on P(n) has q+1 orbits. We use this information in the following theorem to find the degrees of the irreducible characters of A(n).

Theorem 8. Let G(n) = O($2n$+1,q), q odd, and A(n) denote the stabilizer of a non-zero isotropic vector. Then the degrees of the irreducible characters of A(n) are as follows: the degrees of the irreducible characters of G(n-1) = O($2n$-1,q); the degrees of the irreducible characters of A(n-1) multiplied by $q^{2n-2}-1$; the degrees of the irreducible characters of $O^+(2n-2, q)$ multiplied by $q^{n-1}(q^{n-1}+1)$; and the degrees of the irreducible characters of $O^-(2n-2, q)$ multiplied by $q^{n-1}(q^{n-1}-1)$. Furthermore there are $\frac{q-1}{2}$ irreducible characters for each of the two latter families.

Proof. Since A(n) on P(n) has q+1 orbits, therefore A(n) has q+1 orbits on Irr(P(n)). Suppose u is a fixed vector of V($2n$-1,q) and tr is the trace map from GF(q) to GF(p) and ε is a primitive p^{th} root of unity in C. Then it is easy to see that the function χ_u: P(n) $\to C$ defined by $\chi_u([v, -Q(v)]) = \varepsilon^{tr(f(u,v))}$ is a linear character of P(n) and that all the q^{2n-1} irreducible characters of P(n) arise in this way as u runs in V($2n$-1, q). As before, we can prove that A $\in$ G(n-1) fixes χ_u if and only if Au = u. Now from the description of the orbits of A(n) on P(n), the orbits of A(n) on Irr(P(n)) are as follows: $\{\chi_0\}$, $\{\chi_u \mid u$ is a non-zero isotropic vector$\}$, $\frac{q-1}{2}$ orbits of the form $\{\chi_u \mid$ Q(u) is a fixed non-zero square in GF(q)$\}$ and $\frac{q-1}{2}$ orbits of the form $\{\chi_u \mid$ Q(u) is a fixed non-square in GF(q)$\}$. The inertia factor groups are isomorphic to the groups G(n-1), A(n-1), $O^+(2n-2,q)$ and $O^-(2n-2, q)$ respectively. Now by considering the orbit sizes and inertia factor groups we obtain the results.■

Corollary 9. For each k, $1 \le k \le n-1$, the group A(n) has characters of degree $(q^{2n-2}-1)\ldots(q^{2n-2k}-1)$. It has also $\frac{q-1}{2}$ character degrees as follows: $q^{n-1}(q^{n-1}+\varepsilon)$ and $(q^{2n-2}-1)\ldots(q^{2n-2k}-1)\,q^{n-k-1}(q^{n-k-1}+\varepsilon)$ where $1 \le k \le n-2$ and $\varepsilon = \pm 1$.

We now consider the orthogonal groups in odd characteristics and even dimensions. Therefore let V= V($2n$, q) be a $2n$-dimensional vector space over GF(q), q odd with a basis $\{e_1, e_2, \ldots, e_{2n}\}$. In this case there are two equivalence classes of symmetric bilinear forms defined on V. We denote these forms by f^+ and f^- and they may be given as: $f^+(e_i, e_j) = \delta(i, 2n+1-j)$, $1 \le i \le j \le 2n$ and $f^-(e_i, e_j) = \delta(i, 2n+1-j)$, for $1 \le i \le j \le 2n$, $(i,j) \ne (n, n), (n+1, n+1)$ and $f^-(e_n, e_n) = 1$, $f^-(e_{n+1}, e_{n+1}) = -a$, where a is a non-square field element. The group of $2n \times 2n$ invertible matrices over GF(q) leaving invariant $f^{\pm}$ is denoted by G(n) = $O^{\pm}(2n, q) = O^{\varepsilon}(2n, q)$ where $\varepsilon = \pm$ in this setting but later on in formulae involving ε we assume $\varepsilon = \pm 1$. There are $(q^n-\varepsilon)(q^{n-1}+\varepsilon)$ non-zero isotropic vectors in V on which G(n) acts transitively. We let the affine orthogonal group in this case be the stabilizer of a vector of this type namely e_1 and denote this group by A(n). We have [G(n): A(n)]= $(q^n-\varepsilon)(q^{n-1}+\varepsilon)$. As in Lemma 7 we can prove that A(n) is a semi-direct product of an abelian group of order q^{2n-2} with a group isomorphic to G(n-1). The group P(n) is given by P(n) = $\{[v, -Q^{\pm}(v)] \mid v \in V(2n-2, q)\}$ where $Q^{\pm}(v) = \frac{1}{2}f^{\pm}(v, v)$ is the quadratic form associated with $f^{\pm}$. The product of elements of P(n) and the action of A(n) on P(n) is the same as in the case n odd and q odd. Again we have q+1 orbits of A(n) on P(n) and also on Irr(P(n)). The inertia factor groups are G(n-1), A(n-1) and q-1 groups isomorphic to O(2n-3, q). We state the following theorem for the degrees of the irreducible characters of A(n).

Theorem 10. Let G(n)= O(2n, q), q odd, and A(n) denote the stabilizer of a non-zero isotropic vector. Then the degrees of the irreducible characters of the group A(n) are as follows: the degrees of $O^{\varepsilon}(2n\text{-}2, q)$, the degrees of A($n$-1) multiplied by $(q^{n-1}-\varepsilon)(q^{n-2}+\varepsilon)$, the degrees of O(2$n$-3, q) multiplied by $q^{n-2}(q^{n-1}-\varepsilon)$. Furthermore there are q-1 characters of the mentioned latter degrees.

Corollary 11. For each k, $1 \leq k \leq n\text{-}2$, the group A(n) has characters of degrees $(q^{n-1}-\varepsilon)(q^{n-2}+\varepsilon)\ldots(q^{n-k}-\varepsilon)(q^{n-k-1}+\varepsilon)$, where $\varepsilon = \pm 1$. It has also q-1 character degrees as follows: $q^{n-2}(q^{n-1}-\varepsilon)$ and $(q^{n-1}-\varepsilon)(q^{n-2}+\varepsilon)\ldots(q^{n-k}-\varepsilon)(q^{n-k-1}+\varepsilon)q^{n-k-2}(q^{n-k-1}-\varepsilon)$ where $1 \leq k \leq n\text{-}3$ and $\varepsilon = \pm 1$.

Finally we consider the orthogonal groups in characteristic two. If the dimension of the underlying space is odd, then the orthogonal group in this case is isomorphic to some symplectic group. Therefore we assume that V= V(2n, q) is a vector space of dimension 2n over the field GF(q), q a power of two, with a basis $\{e_1, e_2, \ldots e_{2n}\}$. The orthogonal groups in this case are defined with respect to a non-degenerate quadratic form. Let Q be a quadratic form defined over V. We have $Q(u+v) = Q(u) + Q(v) + f(u, v)$ where f is the bilinear form associated with Q. The form f is alternating and we assume it is non-degenerate. It is known that in this case there are two equivalence classes of non-degenerate quadratic forms defined on V. They are denoted by $Q^{\pm}$ and the group of invertible 2n×2n matrices fixing $Q^{\pm}$ is denoted by G(n)= $O^{\pm}(2n, q) = O^{\varepsilon}(2n, q)$. An isotropic vector in this case is a vector v such that $Q^{\varepsilon}(v) = 0$. It is easy to show that there are $(q^{n-1}+\varepsilon)(q^{n}-\varepsilon)$ non-zero isotropic vectors on which G(n) acts transitively. The stabilizer of such a vector is the orthogonal affine group in this case and is denoted by A(n). We have $[G(n): A(n)] = (q^{n-1}+\varepsilon)(q^{n}-\varepsilon)$. Using similar techniques as before it can be shown that A(n) is the semi-direct product of an abelian group of order q^{2n-2} with a group isomorphic to G(n-1). The group P(n) consists of the pairs $[v, Q^{\varepsilon}(v)]$, $v \in$ V(2n-2, q), where multiplication is as follows: $[v, Q^{\varepsilon}(v)][u, Q^{\varepsilon}(u)] = [v+u, Q^{\varepsilon}(v+u)]$. The group A($n$) acts on P($n$) in the following manner: $[v, Q^{\varepsilon}(v)]^{[u,b]A} = [A^{-1}v, Q^{\varepsilon}(v)]$.

Therefore A(n) has q+1 orbits on P(n) as well as on Irr(P(n)). The orbits are as follows: {0}, $\{v \in V(2n\text{-}2, q)^{\#} \mid Q^{\varepsilon}(v) = 0\}$ and q-1 orbits of the type $\{v \in V(2n\text{-}2, q) \mid Q^{\varepsilon}(v)$ is a fixed non-zero field element$\}$, with sizes 1, $(q^{n-2}+\varepsilon)(q^{n-1}-\varepsilon)$ and q-1 orbits of sizes $q^{n-2}(q^{n-1}-\varepsilon)$ respectively. The inertia factor groups are: G(n-1), A(n-1) and q-1 groups isomorphic to O(2n-3,q) respectively. Therefore in this case we have exactly the same statement as in Theorem 10 and Corollary 11 for the degrees of the irreducible characters of the group A(n).

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