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THE ALMOST SURE CONVERGENCE OF WEIGHTED SUMS OF NEGATIVELY DEPENDENT RANDOM VARIABLES

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Abstract

In this paper we study the almost universal convergence of weighted sums $\sum_{k=1}^{\infty} a_{nk} X_k$ for sequence $\{X_n, n\geq 1\}$ of negatively dependent (ND) uniformly bounded random variables, where a_{nk} $n\geq 1$, $k\geq 1$ is an array of nonnegative real numbers such that $\sum_{j=k}^{\infty} a_{nj}^2 = O(k^{-\beta})$ for every $\beta > 0$ and $E|X_n | F_{n-1}| = 0$, $F_n = \sigma(X_1,\ldots,X_n)$ for every $n>1$.

Introduction

In many stochastic models the assumption of independent random variables is not plausible. In fact, increases in some random variables are often related to decreases in other random variables and the assumption of negative dependent is more appropriate than an independent assumption. Lehmann (1966) investigated various concepts of positive and negative dependence in the bivariate case. Strong concepts of bivariate positive negative dependence were introduced by Esary and Proschem (1972). Also Esary, Proschen and Walkup (1967) introduced a concept of association which implied a strong form of positive dependence. Their concept has been very useful in reliability theory and applications. Multivariate generalizations of these concepts of dependence were initiated by Harris (1970) and Brindley and Thompson (1972) and later developed by Ebrahimi and Ghosh (1981), Karlin (1980), Block and Ting (1981), Block, Savits and Shaked (1982). Moreover, Matula (1992) studied the almost sure convergence of sums of ND random variables, and Bozorgnia, Patterson and Taylor (1996) studied limit theorems for dependent random variables. Let $\{X_n, n\geq 1\}$ be a sequence of ND uniformly bounded random variables where $a_{nk}, n\geq 1, k\geq 1$ is an array of nonnegative real numbers such that $\sum_{j=k}^{\infty} a_{nj}^2 = O(k^{-\beta})$ for every $\beta>0$ and $E|X_n|F_{n-1}| = 0$, $F_n = \sigma(X_1,\ldots,X_n)$ for every n≥1. Some convergence theorems for $T_n = \sum_{k=1}^{\infty} a_{nk}^2 X_k$ have been studied by Chow (1966) for the case where $\{X_n, n\geq 1\}$ is an independent sequence, the case of m-dependent has been discussed by Ouy (1967). Here we study the strong law of large numbers for the weighted sums $T_n = \sum_{k=1}^{\infty} a_{nk}^2 X_k$ under certain uniformly bounded conditions on the negatively dependent random variables.

Keywords: Weighted sums; Convergence theorems; Strong law of large numbers; Exponential bounds; Martingale

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Definition

The random variables $X_1,\ldots,X_n$ are said to be ND if we have

$$P\left[\cap_{j=1}^{n} (X_j \leq x_j)\right] \leq \Pi_{j=1}^{n} P\left[X_j \leq x_j\right] \qquad (1)$$

and

$$P\left[\cap_{j=1}^{n} (X_j \leq x_j)\right] \leq \Pi_{j=1}^{n} P\left[X_j \leq x_j\right] \qquad (2)$$

for all $x_1,\ldots,x_n \in$ R. Conditions (1) and (2) are equivalent for n=2. However, Ebrahimi and Ghosh (1981) show that these definitions do not agree for $n\geq 3$. An infinite sequence $\{X_n, n\geq 1\}$ is said to be ND if every finite subset $X_1,\ldots,X_n$ is ND. The following example shows that the sum and the absolute value of ND random variables may not be a ND variable.

Example

i) Suppose X_1,X_2,X_3 are given by a joint probability distribution $f(0,0,0)=f(1,0,1)=0$, $f(0,0,1)=f(0,1,0)=.2$, $f(0,1,1)=f(1,1,0)=f(1,1,1)=.1$, $f(1,0,0)=.3$

a) Then X_1,X_2,X_3 are ND random variables, Ebrahimi and Ghosh (1981).

b) $Y=X_1+X_2$ and X_3 are not ND random variables. Because for every $1\leq y<2$, $0\leq x<1$ we have

$$P[Y\leq y, X_3\leq x]=5/10>P[Y\leq y]\,P[X_3\leq x]=48/100$$

ii) Suppose X_1,X_2 are random variables with p.d.f.

$X_2\backslash X_1$	-1	0	1
-1	0	$\frac{1}{9}$	$\frac{2}{9}$
0	$\frac{1}{9}$	$\frac{1}{9}$	0
1	$\frac{2}{9}$	$\frac{1}{9}$	$\frac{1}{9}$

Then X_1,X_2, are ND random variables but $|X_1|$ and $|X_2|$ are not, since for every $0\leq x<1$ and $0\leq y<1$ we have

$$P[|X_1|\leq x, |X_2|\leq y]=1/9>P[|X_1|\leq x]\,P[|X_2|\leq y]=6/81.\square$$

The following Lemmas are listed for reference in obtaining the main result in the next section. Detailed proofs can be founded in the Bozorgnia, Patterson and Taylor (1996).

Lemma 1. Let $\{X_n, n\geq 1\}$ be a sequence of ND random variables and $\{f_n, n\geq 1\}$ be a sequence of Borel functions all of which are monotone increasing (or all are monotone decreasing). Then $\{f_n(X_n), n\geq 1\}$ is a sequence of ND random variables.

Lemma 2. Let $X_1,\ldots,X_n$ be a finite sequence of ND random variables and $t_1,\ldots t_n$ be all nonnegative (or nonpositive) then:

$$E\left[e^{\sum_i^n t_i X_i}\right]\leq \prod_{i=1}^{n} Ee^{t_i X_i}.$$

Lemma 3. Let X be a random variable with $E(X)=0$ and $|X|\leq c<\infty$ a.e. then for every real number t:

$$Ee^{tX}\leq e^{t^2c^2} \quad \text{and} \quad Ee^{t|X|}\leq 2e^{t^2c^2}$$

Proof. For $c=1$ Chow (1966). For general c, apply the $c=1$ result with X replaced by X/c.

Results

In this section, we first obtain exponential bounds for probabilities $P[max_{j\leq m}|T_{nj}|\geq x]$ and $P[|T_n|>\varepsilon]$, then we prove that convergence a.e. and in probability for the sequence $\{T_{nm}, m\geq 1\}$ are equivalent. Where:

$T_n=\sum_{k=1}^{\infty} a_{nk}X_k$, $T_{nm}=\sum_{k=1}^{m} a_{nk}X_k$ and

$A_n=\sum_{j=1}^{\infty} a_{nj}^2$ with $E[X_n|F_{n-1}]=0$ and $F_n=\sigma(X_1,\ldots,X_n)$ for every $n\geq 1$.

Theorem 1. Let $\{X_n, n\geq 1\}$ be a sequence of ND random variables such that $|X_k|\leq c<\infty$ a.e. for $k\geq 1$ then for every $x>0$:

$$P\left[\max_{j\leq m}|T_{nj}|\leq x\right]\leq 2exp\left[-\frac{x^2}{4c^2A_n}\right].$$

Proof. By Lemmas 1, 2 and 3 for every $h\geq 0$ we have:

$$E\,e^{h|T_{nm}|}\leq E\,e^{hT_{nm}}+E\,e^{-hT_{nm}}\leq$$

$$\prod_{k=1}^{m}E\,[e^{ha_{nk}X_k}]+\prod_{k=1}^{m}E\,[e^{-ha_{nk}X_k}]\leq$$

$$2exp\,[h^2c^2\sum_{k=1}^{m}a_{nk}^2]\leq 2exp\,[h^2c^2A_n].$$

Since $\{T_{nm}, F_m, m\geq 1\}$ is martingale and $\{|T_{nm}|, F_m\}$ is submartingale and $\varphi(t)=e^{th}$ for each $h\geq 0$ is increasing and convex function, then by submartingale inequality we have:

$$P\left[\max_{j\leq m}|T_{nj}|\geq x\right]=P\left[\max_{j\leq m}\varphi(|T_{nj}|\geq\varphi(x)\right]\leq$$

$$\frac{E\left[\varphi(|T_{nm}|)\right]}{\varphi(x)}\leq 2exp\,[-hx+h^2c^2A_n],$$

and for $h = \frac{x}{2c^2 A_n}$, we have:

$$P\left[\max_{j \le m} |T_{nj}| \ge x\right] \le 2exp\left[-\frac{x^2}{4c^2 A_n}\right].\square$$

Lemma 4. Let $\{X_n, n\ge 1\}$ be a sequence of ND random variables with $|X_n| \le c < \infty$ a.e. for every n. Then for every $\varepsilon > 0$ and $h > 0$, we have:

$$Ee^{hT_n} \le exp\ [h^2 c^2 A_n],$$

and

$$P\ [|T_n| > \varepsilon] \le 2exp\ [-\frac{\varepsilon^2}{4c^2 A_n}].$$

Proof. By Fatou's Lemma and Lemmas 1,2 and 3 we have:

$$Ee^{hT_n} \le exp\ [h^2 c^2 A_n].$$

Now following the proof of Theorem 1 we have:

$$P\ [|T_n| > \varepsilon] \le 2exp\ [-\frac{\varepsilon^2}{4c^2 A_n}].\square$$

Theorem 2. Under assumptions of Theorem 1

i) If $\{T_{nm}, m \ge 1\}$ converges in probability for every n then it converges a.e.

ii) $T_n = \sum_{k=1}^{\infty} a_{nk} X_k$ converges a.e. for each n.

Proof. i) Let $T_{nm} \to l_n$ in probability for every n then there exists a subsequence $\{m_k, k \ge 1\}$ such that $T_{nm_k} \to l_n$ *a.e.*. We define:

$$S_{nk} = \max_{m_k < m \le m_{k+1}} |T_{nm} - T_{nm_k}|$$

by Theorem 1 we have:

$$P\ [S_{nk} > \varepsilon] \le 2exp\ [-\frac{\varepsilon^2}{4c^2 \sum_{j=m_{k+1}}^{\infty} a_{nj}^2}].$$

Hence, by Borel Cantelli's Lemma $S_{nk} \to 0$ a.e. $k \to \infty$.

Thus

$$|T_{nm} - l_n| \le S_{nk} + |T_{nm_k} - l_n| \to 0.\quad a.e.$$

The sequences on the right side converge to zero and the proof is complete.

ii) For every $N > m$ by Lemma 4 and part (i) we have:

$$P\ [|T_{nN} - T_{nm}| > \varepsilon\ ] \le 2exp\ [-\frac{\varepsilon^2}{4c^2 \sum_{j=m+1}^{\infty} a_{nj}^2}].$$

If $m \to \infty$, the left hand side of the above inequality tends to zero, hence T_{nm} converges in probability by cauchy criterion. Now part (i) shows that T_n converges *a.e.*

Theorem 3. Let $\{X_{nk}, k \ge 1\}$ be an array of rowwise ND random variables with $\sup_k |X_{nk}| \le c < \infty$ for every n and $E[X_{nm}|F_{n(m-1)})] = 0$, $m\ge 1$, where $F_{nm} = \sigma(X_{n1},\dots, X_{nm})$. Then

i) $T_n = \sum_{k=1}^{\infty} a_{nk} X_{nk}$ converges *a.e.* for each n.

ii) If $\sum_{n=1}^{\infty} exp\ [\frac{\varepsilon^2}{4c^2 A_n}] < \infty$ for every $\varepsilon>0$ then:

$$\lim_{n\to\infty} \sum_{k=1}^{n} a_{nk} X_{nk} = 0.\ a.e.$$

Proof. (i) follows by Theorem 1 and 2.
ii) By Lemma 4 we have:

$$\sum_{n=1}^{\infty} P\ [|\sum_{k=1}^{n} a_{nk} X_{nk}| > \varepsilon] \le 2 \sum_{n=1}^{\infty} exp\ \{-\frac{\varepsilon^2}{4c^2 A_n}] < \infty.\square$$

Theorem 4. Let $\{X_n, n\ge 1\}$ be a sequence of ND random variables with $|X_n| \le c < \infty$ *a.e.* Then

i) If $\lim_{n\to\infty} \sum_{k=1}^{n} a_{nk}^2 = l$, $(0<l<\infty)$. Then for every $\beta>0$

$$\lim_{n\to\infty} n^{-\beta} \sum_{k=1}^{n} a_{nk} X_k = 0\ .\ a.e. \qquad (3)$$

ii) If $0<a_{nk}\le Bn^{-\beta}$, $k\le n$ for some $0<B<\infty$ and $\beta>1/2$. Then

$$\lim_{n\to\infty} \sum_{k=1}^{n} a_{nk} X_k = 0\ .\ a.e. \qquad (4)$$

Proof. By Lemma 4 for every $\varepsilon > 0$ we have:

$$\sum_{n=1}^{\infty} P\ [|n^{-\beta} \sum_{k=1}^{n} a_{nk} X_k| > \varepsilon] \le 2 \sum_{n=1}^{\infty} exp\ [-\frac{n^{2\beta}\varepsilon^2}{4c^2 \sum_{k=1}^{n} a_{nk}^2}] < \infty.$$

and

$$\sum_{n=1}^{\infty} P\,[|\sum_{k=1}^{n} a_{nk} X_k| > \varepsilon] \leq \sum_{n=1}^{\infty} 2exp\,[-\frac{\varepsilon^2}{4c^2\sum_{k=1}^{n} a_{nk}^2}]$$

$$\leq \sum_{n=1}^{\infty} 2exp\,[-\frac{\varepsilon^2 n^{2\beta-1}}{4c^2B^2}] \leq \infty.$$

Now (3) and (4) are true by Borel Cantelli's Lemma.□

Corollary 2. Let $\{X_n, n\geq 1\}$ be a sequence of ND random variables with $|X_n| \leq c < \infty$ *a.e.* for each n. Then

i) If $\sum_{k=1}^{n} exp\,[-\frac{\varepsilon^2}{4c^2A_n}] < \infty$ for every $\varepsilon > 0$ or $A_n = 0\,(l_{n(n)}^{-1})$ then:

$$lim_{n\to\infty} \sum_{k=1}^{n} a_{nk}X_k = 0. \;\; a.e. \qquad (5)$$

ii) If $S_n = \sum_{k=1}^{n} X_k$, then for some $\alpha > 0$

$$lim_{n\to\infty}\, n^{-1/2}(ln^{-(1+\alpha)/2}(n))S_n = 0 \;\; a.e. \qquad (6)$$

iii) If $\sum_{k=1}^{n} a_{nk}^2 = O\,(n^{-\alpha})$ for some $\alpha > 0$ then:

$$lim_{n\to\infty} \sum_{k=1}^{n} a_{nk}X_k = 0. \;\; a.e. \qquad (7)$$

Proof. By Lemma 4 and part (ii) of Theorem 3 we obtain (5) and (7).

To prove (ii) see Chow (1966).□

For uniformly bounded sequence of ND random variables we have the following example.

Example. Let $\{X_n, n\geq 1\}$ be a sequence of ND random variables with $P[X_n = 1] = 1 - P[X_n = -1]$ for $n\geq 1$. Then for some $\beta > (1/2)$

$$\lim_{n\to\infty} \frac{1}{n^{\beta}} \sum_{k=1}^{n} X_k = 0. \;\; a.e.\square$$

Theorem 5. Let $\{X_n\ n\geq 1\}$ and $\{U_n, n\geq 1\}$ be two independent sequences and $\{X_n n\geq 1\}$ be a sequence of ND random variables with $|X_n| \leq c$ a.e., $E[X_n|\, F_{n-1}] = 0$ for every $n\geq 1$ and $\{U_n, n\geq 1\}$ is a sequence of independent random variables. Let a_{nk} be an array of non-negative real numbers such that $\sum_{j=k}^{\infty} a_{nj}^2 = O\,(k^{-\beta})$ for every $\beta > 0$. Then $\sum_{k=1}^{\infty} a_{nk} X_k f\,(U_k)$ converges a.e., when $f(x)$ is a nonnegative, monoton function and bounded by M.

Proof. Let $Y_n = X_n f(U_n)$, we know that by Theorem 1 in Lehmann (1966) if X and Y are ND and U and V are independent random variables and independent of X and Y then for every measurable, nondecreasing funtions h and g, $h(X, U)$ and $g(Y, V)$ are ND random variables. Thus $\{Y_n, n\geq 1\}$ is a sequence of ND random variables and we have:

$$Eexp[hY_n] = E[E[exp[hX_n f\,(U_n))]\;|U_n|] \leq$$

$$Eexp[h2c2M^2] = exp[h^2c^2M^2].$$

Thus by Lemma 4 for $\varepsilon > 0$

$$P\,[|\sum_{k=1}^{\infty} a_{nk} X_k f(U_k)| > \varepsilon] \leq 2exp\,[-\frac{\varepsilon^2}{4M^2c^2A_n}],$$

hence Theorem 2 completes the proof.□

Conclusion

Let $\{X_n, n\geq 1\}$ be a sequence of independent uniformly bounded random variables, then the assumption $E\,[X_n|\, F_{n-1}] = 0$ can be replaced by $E(X_n) = 0$ and all the above results will be true in this case.

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