The Asymptotic Form of Eigenvalues for a Class of Sturm-Liouville Problem with One Simple Turning Point

A. Jodayree Akbarfam[*] and H. Kheiri

Faculty of Mathematical Sciences, Tabriz University, Tabriz, Islamic Republic of Iran

Abstract

The purpose of this paper is to study the higher order asymptotic distributions of the eigenvalues associated with a class of Sturm-Liouville problem with equation of the form $w'' = (\lambda^2 f(x) - R(x))w$ (1), on $[a,b]$, where λ is a real parameter and $f(x)$ is a real valued function in $C^2(a,b)$ which has a single zero (so called turning point) at point $x = x_0$ and $R(x)$ is a continuously differentiable function. We prove that, as a classical case, the asymptotic form of eigenvalues of (1) with periodic boundary condition $w(a) = w(b)$, $w'(a) = w'(b)$ as well as with Semi-periodic boundary condition $w(a) = -w(b)$, $w'(a) = -w'(b)$ are the same as Dirichlet boundary condition $w(a) = 0 = w(b)$. We also study the asymptotic formula for the eigenvalues of (1) with boundary condition $w'(a) = 0 = w'(b)$, as well as $w(a) = 0 = w'(b)$ and $w'(a) = 0 = w(b)$.

Keywords: Asymptotic distribution; Turning point; Sturm-Liouville problem; Non-definite

1. Introduction

Many differential equations occurring in mathematical physics are reducible to the form

$$w'' = (\lambda^2 f(x) - R(x))w , \qquad (1)$$

where
(i) $f, R : [a,b] \mapsto \mathbb{R}$ are n times continuously differentiable function, $2 \le n$
(ii) $f(x_0) = 0$, and $\frac{f(x)}{x - x_0} = h(x)$ is positive and twice continuously differentiable within (a,b), where x_0 is an interior point of (a,b)

(iii) λ is a real parameter.
By using Longer's transformation ([8], 10.4.115)

$$\xi(x) = -\left\{ \int_x^{x_0} (f(t))^{\frac{1}{2}} dt \right\}^{\frac{2}{3}} \quad x \le x_0$$

$$\xi(x) = \left\{ \int_{x_0}^x (-f(t))^{\frac{1}{2}} dt \right\}^{\frac{2}{3}} \quad x_0 \le x$$

and new dependent variable $y(\xi) = \tilde{f}^{\frac{1}{4}} w$ the Equation (1) reduces the form

$$y'' = (u''\xi - q(\xi))y , \qquad (2)$$

where

$$q(\xi) = \frac{R}{\tilde{f}} - \tilde{f}^{-\frac{3}{4}} \frac{d^2}{dx^2}(\tilde{f}^{-\frac{1}{4}})$$

and

$$u^2 = \frac{9}{4}\lambda^2, \qquad \tilde{f}(x) = (\frac{d\xi}{dx})^2 = \frac{4f(x)}{9\xi(x)}.$$

Under this transformation, let $\xi = c, d$ correspond to the end points $x = a, b$ respectively, where $c < 0 < d$.

In [1], it was derived the higher order asymptotic formula for the positive eigenvalues (2) with Dirichlet boundary condition $y(c) = 0 = y(d)$. In [3], [5], and [8] the distribution of eigenvalues for the classical case of Sturm-Liouville problems with periodic boundary condition is derived. It means that in these papers the weight function $f(x)$ in (1) dose not change sign in [a,b]. In ([9], chapter 1.4.3) it has been proved that the distribution of the semi-periodic problem has the same asymptotic expansion as the Dirichlet problem for equation (1) whenever $f(x)$ does not have a turning point.

In [10] Langer has expressed the solution of the given equation in terms of the solution of a simple, so-called "standard" equation which gives an adequate description of the behavior of solution of the original equation. Note that firstly, the asymptotic behavior of these solutions simply described theory but rather cumbersome to apply for getting the higher order distribution of eigenvalues associated with periodic and semi-periodic boundary conditions, so in this paper the method of establishing the estimation is based on the formal solution of (2) constructed by Olver [2].

2. Periodic Boundary Condition

Let $y_1 = y_1(\xi, \lambda), y_2 = y_2(\xi, \lambda)$ be two linear independent solutions of (2). The eigenvalues of equation (2) with periodic boundary conditions

$$y(a) = y(b), y'(a) = y'(b)$$

are the roots of

$$\Delta_n(u) = W(y_1, y_2)(c) - y_1(c)y'_2(d) - y_1(d)y'_2(c)$$
$$+ y_2(c)y'_1(d) + y_2(d)y'_1(c) + W(y_1, y_2)(d)$$
$$= 0$$

where W denotes the wronskian of y_1, y_2. Since by Liouville-Abel formula, we have

$$W(y_1, y_2)(c) = W(y_1, y_2)(d) = W(y_1, y_2)(0) = W(0)$$

hence

$$\Delta_n(u) = 2W(0) - y_1(c)y'_2(d) - y_1(d)y'_2(c)$$
$$+ y_2(c)y'_1(d) + y_2(d)y'_1(c) \qquad (3)$$
$$= 0$$

In studying the asymptotic behavior of the eigenvalues, it is very convenient to use the asymptotic solutions given in ([2], chapter 11, §7.2) as follows:

Theorem 1. The differential Equation (2) has, for each value of u and each nonnegative integer n, a pair of infinitely differentiable solutions $W_1(u, \xi)$ and $W_2(u, \xi)$ given by their approximations $W_{2n+1,1}(u, \xi)$, $W_{2n+1,2}(u, \xi)$ where

$$W_{2n+1,1}(u, \xi) = Bi(u^{2/3}\xi) \sum_{s=0}^{n} \frac{A_s(\xi)}{u^{2s}}$$
$$+ \frac{Bi'(u^{2/3}\xi)}{u^{4/3}} \sum_{s=0}^{n-1} \frac{B_s(\xi)}{u^{2s}} + \varepsilon_{2n+1,1}(u, \xi),$$

$$W_{2n+1,2}(u, \xi) = Ai(u^{2/3}\xi) \sum_{s=0}^{n} \frac{A_s(\xi)}{u^{2s}}$$
$$+ \frac{Ai'(u^{2/3}\xi)}{u^{4/3}} \sum_{s=0}^{n-1} \frac{B_s(\xi)}{u^{2s}} + \varepsilon_{2n+1,2}(u, \xi)$$

where

$$A_0(\xi) = 1,$$

$$B_s(\xi) = \frac{1}{2\xi^{1/2}} \int_0^\xi (R(v)A_s(v) - A_s''(v)) \frac{dv}{v^{1/2}} \quad 0 < \xi,$$

$$B_s(\xi) = \frac{1}{2(-\xi)^{1/2}} \int_\xi^0 (R(v)A_s(v) - A_s''(v)) \frac{dv}{(-v)^{1/2}} \quad 0 > \xi,$$

$$A_{s+1}(\xi) = -\frac{1}{2}B'_s(\xi) + \frac{1}{2} \int_0^\xi R(v)B_s(v)dv,$$

$\varepsilon_{2n+1,i}(u, \xi), i = 1, 2$, are error bounds, and $Ai(u^{2/3}\xi)$, $Bi(u^{2/3}\xi)$ are two independent solutions of

$$\frac{d^2W}{d\xi^2} = u^2\xi W(\xi).$$

called Airy functions.

Proof. See ([2], chapter 11, §7.2).
Note that $A_s(\xi)$ and $B_s(\xi)$ are infinitely differentiable. The error bounds for large u are uniform with respect to ξ. The asymptotic form of $Ai(u^{2/3}\xi)$

and $Bi(u^{2/3}\xi)$ for $\xi > 0$ is not the same as for $\xi < 0$. Therefore, the solutions $W_{2n+1,1}(u,\xi)$ or $W_{2n+1,2}(u,\xi)$ have different asymptotic forms for $\xi > 0$ and $\xi < 0$. In the rest of this paper we shall use the symbols $W^{+}(u,\xi)$, $W^{-}(u,\xi)$ to signify the asymptotic form of $W(u,\xi)$ for $\xi > 0$, $\xi < 0$, respectively, as $u \to \infty$.

Lemma 1. The asymptotic forms of $Ai(u^{2/3}\xi)$, $Ai'(u^{2/3}\xi)$, $Bi(u^{2/3}\xi)$ and $Bi'(u^{2/3}\xi)$ are given by (for $u \to \infty$),

$$Ai(u^{2/3}\xi) \square \frac{e^{-\frac{2}{3}u\xi^{3/2}}}{2\pi^{1/2}u^{1/6}\xi^{1/4}} \sum_{s=0}^{\infty} (-1)^s \frac{u_s}{(\frac{2}{3}u\xi^{3/2})^s}$$

for $\xi > 0$,

$$Ai'(u^{2/3}\xi) \square -\frac{u^{1/6}\xi^{1/4}e^{-\frac{2}{3}u\xi^{3/2}}}{2\pi^{1/2}} \sum_{s=0}^{\infty} (-1)^s \frac{v_s}{(\frac{2}{3}u\xi^{3/2})^s}$$

for $\xi > 0$

$$Ai(u^{2/3}\xi) \square$$

$$\frac{1}{\pi^{1/2}u^{1/6}(-\xi)^{1/4}} \{\cos(\frac{2}{3}u(-\xi)^{3/2}$$

$$-\frac{\pi}{4}) \sum_{s=0}^{\infty} (-1)^s \frac{u_{2s}}{(\frac{2}{3}u(-\xi)^{3/2})^{2s}}$$

$$+\sin(\frac{2}{3}u(-\xi)^{3/2}$$

$$-\frac{\pi}{4}) \sum_{s=0}^{\infty} (-1)^s \frac{u_{2s+1}}{(\frac{2}{3}u(-\xi)^{3/2})^{2s+1}}\}$$

for $\xi < 0$,

$$Ai'(u^{2/3}\xi) \square$$

$$\frac{u^{1/6}(-\xi)^{1/4}}{\pi^{1/2}} \{\sin(\frac{2}{3}u(-\xi)^{3/2}$$

$$-\frac{\pi}{4}) \sum_{s=0}^{\infty} (-1)^s \frac{v_{2s}}{(\frac{2}{3}u(-\xi)^{3/2})^{2s}}$$

$$-\cos(\frac{2}{3}u(-\xi)^{3/2}$$

$$-\frac{\pi}{4}) \sum_{s=0}^{\infty} (-1)^s \frac{v_{2s+1}}{(\frac{2}{3}u(-\xi)^{3/2})^{2s+1}}\}$$

for $\xi < 0$,

$$Bi(u^{2/3}\xi) \square \frac{e^{\frac{2}{3}u\xi^{3/2}}}{\pi^{1/2}u^{1/6}\xi^{1/4}} \sum_{s=0}^{\infty} \frac{u_s}{(\frac{2}{3}u\xi^{3/2})^s}$$

for $\xi > 0$,

$$Bi'(u^{2/3}\xi) \square \frac{u^{1/6}\xi^{1/4}e^{\frac{2}{3}u\xi^{3/2}}}{\pi^{1/2}} \sum_{s=0}^{\infty} \frac{v_s}{(\frac{2}{3}u\xi^{3/2})^s}$$

for $\xi > 0$,

$$Bi(u^{2/3}\xi) \square$$

$$\frac{1}{\pi^{1/2}u^{1/6}(-\xi)^{1/4}} \{-\sin(\frac{2}{3}u(-\xi)^{3/2}$$

$$-\frac{\pi}{4}) \sum_{s=0}^{\infty} (-1)^s \frac{u_{2s}}{(\frac{2}{3}u(-\xi)^{3/2})^{2s}}$$

$$+\cos(\frac{2}{3}u(-\xi)^{3/2}$$

$$-\frac{\pi}{4}) \sum_{s=0}^{\infty} (-1)^s \frac{u_{2s+1}}{(\frac{2}{3}u(-\xi)^{3/2})^{2s+1}}\}$$

for $\xi < 0$,

$$Bi'(u^{2/3}\xi) \square \frac{u^{1/6}(-\xi)^{1/4}}{\pi^{1/2}} \{\cos(\frac{2}{3}u(-\xi)^{3/2}$$

$$-\frac{\pi}{4}) \sum_{s=0}^{\infty} (-1)^s \frac{v_{2s}}{(\frac{2}{3}u(-\xi)^{3/2})^{2s}}$$

$$+\sin(\frac{2}{3}u(-\xi)^{3/2}$$

$$-\frac{\pi}{4}) \sum_{s=0}^{\infty} (-1)^s \frac{v_{2s+1}}{(\frac{2}{3}u(-\xi)^{3/2})^{2s+1}}\}$$

for $\xi < 0$,

and

$$Ai(0) = \frac{1}{3^{2/3}\Gamma(2/3)} = \frac{Bi(0)}{\sqrt{3}}, \quad Ai'(0) = -\frac{1}{3^{1/3}\Gamma(1/3)} = -\frac{Bi'(0)}{\sqrt{3}}$$

where $u_0 = 1 = v_0$

$$u_s = \frac{(2s+1)(2s+3)(2s+5)...(6s-1)}{(216)^s s!}$$

$$v_s = -\frac{6s+1}{6s-1}u_s \quad s \geq 1$$

Proof. See ([2], chapter 11, §1.1 and §1.2)

In order to compute the asymptotic of these

eigenvalues we use the flowing notations:

$$A(\xi) = \sum_{s=0}^{n} \frac{A_s(\xi)}{u^{2s}}, \quad B(\xi) = \sum_{s=0}^{n-1} \frac{B_s(\xi)}{u^{2s}},$$

$$M_1(\xi) = \sum_{s=0}^{n} \frac{u_s}{(\frac{2}{3}u\xi^{3/2})^s}, \quad M_2(\xi) = \sum_{s=0}^{n} (-1)^s \frac{u_s}{(\frac{2}{3}u\xi^{3/2})^s},$$

$$P_1(\xi) = \sum_{s=0}^{n} \frac{v_s}{(\frac{2}{3}u\xi^{3/2})^s}, \quad P_2(\xi) = \sum_{s=0}^{n} (-1)^s \frac{v_s}{(\frac{2}{3}u\xi^{3/2})^s},$$

$$R_1(\xi) = \sum_{s=0}^{n} \frac{u_{2s}}{(\frac{2}{3}u\xi^{3/2})^{2s}}, \quad R_2(\xi) = \sum_{s=0}^{n} (-1)^s \frac{u_{2s}}{(\frac{2}{3}u\xi^{3/2})^{2s}},$$

$$K_1(\xi) = \sum_{s=0}^{n} \frac{v_{2s}}{(\frac{2}{3}u\xi^{3/2})^{2s}}, \quad K_2(\xi) = \sum_{s=0}^{n} (-1)^s \frac{v_{2s}}{(\frac{2}{3}u\xi^{3/2})^{2s}},$$

$$T_1(\xi) = \sum_{s=0}^{n} \frac{u_{2s+1}}{(\frac{2}{3}u\xi^{3/2})^{2s+1}}, \quad T_2(\xi) = \sum_{s=0}^{n} (-1)^s \frac{u_{2s+1}}{(\frac{2}{3}u\xi^{3/2})^{2s+1}},$$

$$L_1(\xi) = \sum_{s=0}^{n} \frac{v_{2s+1}}{(\frac{2}{3}u\xi^{3/2})^{2s+1}}, \quad L_2(\xi) = \sum_{s=0}^{n} (-1)^s \frac{v_{2s+1}}{(\frac{2}{3}u\xi^{3/2})^{2s+1}},$$

$$\omega(\xi) = \frac{2}{3}u\xi^{3/2} - \frac{\pi}{4}$$

By theorem ([2], chapter 11, §7.2) using the above notation we write,

$$y_1(c,u) =: W^-_{2n+1,1}(c,u) =$$

$$\frac{A(c)}{\pi^{1/2}u^{1/6}(-c)^{1/4}}\{(-R_2(-c)\sin\omega(-c)$$

$$+T_2(-c)\cos\omega(-c)\}$$

$$+\frac{(-c)^{1/4}B(c)}{u^{7/6}\pi^{1/2}}\{(K_2(-c)\cos\omega(-c)$$

$$+L_2(-c)\sin\omega(-c)\} + O\left(\frac{1}{u^{4n+19/6}}\right), \qquad (4)$$

$$y'_1(c,u) =: \frac{d}{d\xi}W^-_{2n+1,1}(c,u) =$$

$$\frac{A'(c)+u^{4/3}B(c)}{\pi^{1/2}u^{1/6}(-c)^{1/4}}\{(-R_2(-c)\sin\omega(-c)$$

$$+T_2(-c)\cos\omega(-c)\}$$

$$+\frac{(-c)^{1/4}(A(c)u^2+B'(c))}{u^{7/6}\pi^{1/2}}\{(K_2(-c)\cos\omega(-c)$$

$$+L_2(-c)\sin\omega(-c)\} + O\left(\frac{1}{u^{4n+2/3}}\right), \qquad (5)$$

$$y_2(c,u) =: W^-_{2n+1,2}(c,u) =$$

$$+\frac{A(c)}{\pi^{1/2}u^{1/6}(-c)^{1/4}}\{R_2(-c)\cos\omega(-c)$$

$$+T_2(-c)\sin\omega(-c)\}$$

$$+\frac{(-c)^{1/4}B(c)}{u^{7/6}\pi^{1/2}}\{K_2(-c)\sin\omega(-c)$$

$$-L_2(-c)\cos\omega(-c)\} + O\left(\frac{1}{u^{4n+19/6}}\right), \qquad (6)$$

$$y'_2(c,u) =: \frac{d}{d\xi}W^-_{2n+1,2}(u,c) =$$

$$+\frac{A'(c)+u^{4/3}B(c)}{\pi^{1/2}u^{1/6}(-c)^{1/4}}\{R_2(-c)\cos\omega(-c)$$

$$+T_2(-c)\sin\omega(-c)\}$$

$$+\frac{(-c)^{1/4}(A(c)u^2+B'(c))}{u^{7/6}\pi^{1/2}}\{K_2(-c)\sin\omega(-c)$$

$$-L_2(-c)\cos\omega(-c)\} + O\left(\frac{1}{u^{4n+5/6}}\right), \qquad (7)$$

$$y_1(d,u) =: W^+_{2n+1,1}(d,u) =$$

$$\frac{e^{\frac{2}{3}ud^{3/2}}}{\pi^{1/2}u^{1/6}d^{1/4}}\{(M_1(d)A(d)+O(\frac{1}{u^{n+1}})\} \qquad (8)$$

$$+\frac{d^{1/4}e^{\frac{2}{3}ud^{3/2}}}{\pi^{1/2}u^{7/6}}\{P_1(d)B(d)+O(\frac{1}{u^{n+1}})\},$$

$$y'_1(d,u) =: \frac{d}{d\xi}W^+_{2n+1,1}(d,u) =$$

$$\frac{e^{\frac{2}{3}ud^{3/2}}}{\pi^{1/2}u^{1/6}d^{1/4}}\{(M_1(d)(A'(d)$$

$$+u^{4/3}B(d))+O(\frac{1}{u^{3n-1/3}})\} \qquad (9)$$

$$+\frac{d^{1/4}e^{\frac{2}{3}ud^{3/2}}}{\pi^{1/2}u^{7/6}}\{P_1(d)(A(d)u^2$$

$$+B'(d)u^{2/3})+O(\frac{1}{u^{3n+8/3}})\},$$

$$y_2(d,u) =: W^+_{2n+1,2}(d,u) =$$

$$\frac{e^{-\frac{2}{3}ud^{3/2}}}{2\pi^{1/2}u^{1/6}d^{1/4}}\{M_2(d)A(d)+O(\frac{1}{u^{n+1}})\} \qquad (10)$$

$$-\frac{d^{1/4}e^{-\frac{2}{3}ud^{3/2}}}{2\pi^{1/2}u^{7/6}}\{P_2(d)B(d)+O(\frac{1}{u^{n+1}})\},$$

$$y'_2(d,u) =: \frac{d}{d\xi} W^+_{2n+1,2}(d,u) =$$

$$\frac{e^{-\frac{2}{3}ud^{3/2}}}{2\pi^{1/2}u^{1/6}d^{1/4}}\{M_2(d)(A'(d)$$

$$+u^{4/3}B(d))+O(\frac{1}{u^{3n-1/3}})\} \qquad (11)$$

$$-\frac{d^{1/4}e^{-\frac{2}{3}ud^{3/2}}}{2\pi^{1/2}u^{7/6}}\{P_2(d)(u^2A(d)$$

$$+B'(d))+O(\frac{1}{u^{3n+7/3}})\},$$

hence

$$y_2(d,u)y'_1(c,u)=e^{-\frac{2}{3}ud^{3/2}}O(u), \; y'_2(d,u)y_1(c,u)$$
$$=e^{-\frac{2}{3}ud^{3/2}}O(u) \qquad (12)$$

as $u \to \infty$. And

$$m(d,u) =: \frac{y'_1(d,u)}{y_1(d,u)}=u^{(4/3)}dB_0+O(u),$$

$$p(d,u) =: \frac{y'_2(c,u)}{y_2(c,u)}=\frac{u^{(4/3)}cB_0\cos(w(-c))+O(u)}{\cos(w(-c))+O(1/u)}, \qquad (13)$$

The eigenvalues of the problem (2) with period boundary condition are the root of (3) or taking into account of (4-13) (after the simple calculation), we get

$$2W(0)+e^{-\frac{2}{3}ud^{3/2}}O(u)+e^{-\frac{2}{3}ud^{3/2}}O(u)$$

$$+y_1(d,u)y_2(c,u)[m(d,u)-p(c,u)]=0$$

or

$$2e^{-\frac{2}{3}ud^{3/2}}W(0)+e^{-\frac{4}{3}ud^{3/2}}O(u)+e^{-\frac{4}{3}ud^{3/2}}O(u)$$

$$+O(\frac{1}{u^{1/6}})y_2(c,u)[m(d,u)-p(c,u)]=0. \qquad (14)$$

Since $m(d,u)$ and $p(c,u)$ have different asymptotic forms, and the first three terms in (14) do not influence the leading terms of zeros of $\Delta_n(u)=0$ (see [4]), so asymptotic distribution of the zeros of $\Delta_n(u)=0$ is therefore asymptotically determined by that zeros of $y_2(c,u)$, where $y_2(c,u)$ is the solution of (2) defined by (5), *i.e.*, the zeros of $\Delta_n(u)$ satisfy

$$W^-_{2n+1,2}(c,u) \to 0; \qquad (15)$$

as $u \to \infty$. From this expression, we see that the asymptotic behavior of the eigenvalues depends essentially on the positive part of the weight function $-\xi$ on $[c,d]$ *i.e.*, on $-\xi$ on $[c,0]$. This is to be expected by the theorem of Atkinson-Mingarelli [7] or the result of Eberhard and Freilling [6].

Analogously to Theorem 3 in [1] we may have:

Theorem 1. Consider the differential equation

$$w'' = (\lambda^2 f(x)+R(x))w, a \le x \le b$$

with boundary conditions $w(a)=0=w(b)$, where for some point $x_0 \in [a,b]$:

1) $f(x_0)=0$

2) $\frac{f(x_0)}{x-x_0}=h(x)$ is a positive and twice continuously differentiable function within (a, b).

3) $R(x)$ is continuously differentiable,

4) The integral

$$H(a)=-\int_a^{x_0}\{\frac{1}{(-f)^{1/4}}\frac{d^2}{dx^2}\left(\frac{1}{(-f)^{1/4}}\right)$$

$$-\frac{R}{(-f)^{1/2}}-\frac{5(-f)^{1/2}}{16(-\xi)^3}\}dx$$

converges where $\xi(x)$ as:

$$\xi(x)=-\{\int_x^{x_0}(-f(t))^{1/2}dt\}^{\frac{2}{3}} \qquad x \le x_0, \qquad (16)$$

$$\xi(x)=\{\int_{x_0}^x f^{1/2}(t)dt\}^{\frac{2}{3}} \qquad x_0 \le x. \qquad (17)$$

Then the asymptotic distribution of the positive eigenvalues of this problem is given by

$$\lambda_m = \frac{m\pi-\pi/4}{\int_a^{x_0}(-f(t))^{1/2}dt}$$

$$+\frac{1}{m\pi}\{\frac{5}{72\int_a^{x_0}(-f(t))^{1/2}dt}+\frac{1}{6}H(a)\}+$$

$$+\frac{1}{4m^2\pi}\{\frac{5}{72\int_a^{x_0}(-f(t))^{1/2}dt}+\frac{1}{6}H(a)\}+ \qquad (18)$$

$$+O(\frac{1}{m^3}).$$

3. Semi-periodic Boundary Condition

The eigenvalues of Equation (2) with Semi-periodic boundary conditions

$$y(a) = -y(b), y'(a) = -y'(b)$$

are the zeros of

$$S_n(u) = W(y_1, y_2)(c) + y_1(c)y'_2(d) + y_1(d)y'_2(c)$$

$$- y_2(c)y'_1(d) - y_2(d)y'_1(c) + W(y_1, y_2)(d)$$

$$= 0$$

Similarly taking into account of (4-13) we obtain:

$$S_n(u) = 2e^{-(2/3)ud^{(3/2)}}W(0)$$

$$-e^{-(4/3)ud^{(3/2)}}O(u) - e^{-(4/3)ud^{(3/2)}}O(u)$$

$$-O(\frac{1}{u^{(1/6)}}y_2(c,u)[m(d,u) - p(c,u)] = 0.$$

Comparing $\Delta_n(u)$ with $S_n(u)$, we see that the only difference is the sign of some expressions which do not influence of the distribution of the eigenvalues therefore the zeros of $S_n(u)$ satisfy

$$W^-_{2n+1,2}(c,u) \to 0.$$

We have thus proved the following as classical case (see [9], chapter 1.4.3).

Theorem 2. If $R(x)$ is a smooth function, then the eigenvalues of the semi-periodic problem and the periodic problem have the same asymptotic expansion as the Dirichlet boundary problem for equation (1), (*i.e.* of the form (18)).

4. Other Cases

We now consider equation (2) with the boundary conditions

$$y'(c) = 0$$
$$y'(d) = 0. \tag{19}$$

In this case we assume that $\int_c^0 \frac{q(t)}{(-t)^{\frac{1}{2}}} \neq 0$, the

eigenvalues of the problem (2), (19) are the roots of

$$y'_1(c,u)y'_2(d,u) - y'_2(c,u)y'_1(d,u) = 0.$$

or, taking into account (5), (7), (9), (11) (after the simple calculation)we obtain

$$e^{-(2/3)ud^{(3/2)}}O(u) + e^{(2/3)ud^{(3/2)}}O(u^{(-1/6)})y'_2(c,u) = 0.$$

Dividing throughout by $u^{(-1/6)}e^{(2/3)ud^{(3/2)}}$ and letting $u \to \infty$ we see that the eigenvalues of this case satisfy

$$\frac{d}{d\xi}W_{2n+1,2}(c,u) \to 0 \quad \text{as} \quad u \to \infty \tag{20}$$

From (7) and (20), we have found that for large u

$$e^{i\{2w(-c)+\pi\}} = \frac{F + iH}{F - iH} \tag{21}$$

where

$$F = (uA'(-c) + u^{\frac{7}{3}}dB(-c))R_2(-c)$$

$$-(-c)^{\frac{1}{2}}(A(-c)u^2 + B'(-c))L_2(-c),$$

$$H = (uA'(-c) + u^{\frac{7}{3}}dB(-c))T_2(-c)$$

$$+(-c)^{\frac{1}{2}}(A(-c)u^2 + B'(-c))K_2(-c).$$

By taking the logarithm in (21) we obtain

$$u_m = \frac{m\pi - \frac{\pi}{4}}{\frac{2}{3}(-c)^{\frac{3}{2}}} + \frac{1}{\frac{4}{3}(-c)^{\frac{3}{2}} \times i}\log\left[\frac{F + iH}{F - iH}\right], \tag{22}$$

where

$$F + iH = u^2(-c)^{\frac{1}{2}}\sum_{k=0}^n\sum_{s=0}^k\frac{(i)^{k-s+1}A_s(c)v_{k-s}}{u^{k+s}(\frac{2}{3}(-c)^{\frac{3}{2}})^{k-s}}$$

$$+u\sum_{k=0}^n\sum_{s=0}^k\frac{A'_{s+1}(c)(i)^{k-s}u_{k-s}}{u^{k+s+2}(\frac{2}{3}(-c)^{\frac{3}{2}})^{k-s}}$$

$$+u^{\frac{7}{3}}\sum_{k=0}^n\sum_{s=0}^k\frac{B_s(c)(i)^{k-s}u_{k-s}}{u^{k+s}(\frac{2}{3}(-c)^{\frac{3}{2}})^{k-s}}$$

$$+(-c)^{\frac{1}{2}}\sum_{k=0}^n\sum_{s=0}^k\frac{B'_s(c) \times (i)^{k-s+1}v_{k-s}}{u^{k+s}(\frac{2}{3}(-c)^{\frac{3}{2}})^{k-s}},$$

$$F - iH = -i(-c)^{\frac{1}{2}}u^2\sum_{k=0}^n\sum_{s=0}^k\frac{(i)^{k-s}A_s(c)u_{k-s}}{v^{k+s}(\frac{2}{3}(-c)^{\frac{3}{2}})^{k-s}}$$

$$+u\sum_{k=0}^n\sum_{s=0}^k\frac{(-i)^{k-s}A'_{s+1}(c)u_{k-s}}{u^{k+s+2}(\frac{2}{3}(-c)^{\frac{3}{2}})^{k-s}}$$

$$+u^{\frac{7}{3}}\sum_{k=0}^n\sum_{s=0}^k\frac{(-i)^{k-s}B_s(c)u_{k-s}}{u^{k+s}(\frac{2}{3}(-c)^{\frac{3}{2}})^{k-s}}$$

$$-i(-c)^{\frac{1}{2}}\sum_{k=0}^n\sum_{s=0}^k\frac{(i)^{k-s}B'_s(c)v_{k-s}}{u^{k+s}(\frac{2}{3}(-c)^{\frac{3}{2}})^{k-s}}.$$

Now we have

$$u^{-\frac{7}{3}}(F + iH) = B_0u_0 + (-c)^{\frac{1}{2}}\frac{iA_0v_0}{u^{\frac{1}{3}}} + \frac{iB_0u_1}{u^{\frac{2}{3}}(-c)^{\frac{3}{2}}},$$

$$-\frac{(-c)^{\frac{1}{2}}A_0 v_1}{u^{\frac{4}{3}}\frac{2}{3}(-c)^{\frac{3}{2}}}+\frac{B_1(c)u_0}{u^2}+\frac{iv_0(A_1+B'_0(c))}{u^{\frac{7}{3}}}+O(u^{\frac{-10}{3}}),$$

$$u^{-\frac{7}{3}}(F-iH)=B_0 u_0-\frac{i(-c)^{\frac{1}{2}}A_0 v_0}{u^{\frac{1}{3}}}-\frac{iB_0 u_1}{u^{\frac{2}{3}}(-c)^{\frac{3}{2}}}$$

$$-\frac{A_0 v_1(-c)^{\frac{1}{2}}}{u^{\frac{4}{3}}\frac{2}{3}(-c)^{\frac{3}{2}}}+\frac{B_1(c)u_0}{u^2}$$

$$-\frac{i(-c)^{\frac{1}{2}}v_0(A_1+B'_0(c))}{u^{\frac{7}{3}}}+O(u^{\frac{-10}{3}}).$$

Consequently

$$\frac{F+iH}{F-iH}=1+2Tiu^{-\frac{1}{3}}-2T^2 u^{-\frac{2}{3}}$$

$$+2i(L-T^3)u^{-1}+O(u^{-\frac{4}{3}}),$$

(23)

where

$$T=\frac{(-c)^{\frac{1}{2}}}{B_0},\qquad L=\frac{5B_0}{48(-c)^{\frac{3}{2}}}.$$

Substituting expression (23) into formula (22) and using the expansion of the logarithm, we find:

$$u_m=\frac{m\pi-\frac{\pi}{4}}{\frac{2}{3}(-c)^{\frac{3}{2}}}$$

$$+\frac{1}{\frac{4}{3}(-c)^{3/2}i}[2iTu^{-\frac{1}{3}}+(2i(L-\frac{1}{3}T^3)u^{-1}+O(u^{-\frac{4}{3}})],$$

or

$$u_m=\frac{m\pi-\frac{\pi}{4}}{\frac{2}{3}(-c)^{\frac{3}{2}}}+\frac{1}{\frac{4}{3}(-c)^{3/2}}2Tu^{-\frac{1}{3}}$$

$$+\frac{1}{\frac{4}{3}(-c)^{3/2}}2(L-\frac{1}{3}T^3)u^{-1}+O(u^{-\frac{4}{3}}).$$

(24)

First and second approximation is

$$u_m=\frac{m\pi-\frac{\pi}{4}}{\frac{2}{3}(-c)^{\frac{3}{2}}}+O(m^{\frac{-1}{3}}),$$

(25)

substituting this approximation (25) into (24), we find the following approximation:

$$u_m=\frac{m\pi-\frac{\pi}{4}}{\frac{2}{3}(-c)^{\frac{3}{2}}}+\frac{2(\frac{3}{2})^{(\frac{2}{3})}}{(m\pi)^{(\frac{1}{3})}\int_c^0\frac{q(t)}{(-t)^{1/2}}dt}+O(\frac{1}{m}).$$

(26)

Let us consider the differential Equation (2) whit

boundary conditions

$$y(c)=0=y'(d).$$

By applying the above similar method, we see that the eigenvalues of this problem are zeros of

$$y_2(c,u)=:W_{2n+1,2}^-(c,u)=0,$$

therefore the higher-order asymptotic distribution of eigenvalues is of the form (18). The eigenvalues of Sturm-Liouville Equation (2) with boundary condition

$$y'(c)=0=y(d)$$

coincide with zeroes of

$$y'_2(c,u)=:\frac{d}{d\xi}W_{2n+1,2}^-(c,u)=0,$$

therefore the distribution of eigenvalues of this case is of the form (26).

Acknowledgements

References

1. Jodayree Akbarfam A., Higher-order asymptotic approximations to the eigenvalues of the Sturm-Liouville problem in one turning point case. *Bull. Iranian Math. Soc.*, No. 2, 37-54 (1997).
2. Olver F.W., *Asymptotic and Special Functions*. Academic Press, New York, (1974).
3. Boumenir A. Eigenvalues of periodic Sturm-Liouville problems by the Shannon-Whittaker sampling theorem. *Uspekhi Mat. Nauk*, **7,6**: 3-96 (1952).
4. Langer R.E. On the zeros of exponential sums and integrals. *Bull. Am. Math. Soc.*, **37**: 213-239 (1931).
5. Dorodnitsyn A.A. Asymptotic laws of distribution of the characteristic values for certain special forms of differential equation of the second order. *Uspekhi Mat. Nauk*, AMS translations 16 (Ser.2), **7,6**: 3-96 (1952).
6. Eberhard W. and Freiling G. The Distribution of the Eigenvalues for Second Order Eigenvalue Problems in the Presence of an Arbitrary Number of Turning Points. Results Math., **21**: 24-41 (1992).
7. Atkinson F.V. and Mingralli A.B. Asymptotic of the number of zeros and of the eigenvalues of general weighted Sturm-Liouville problems. *J. Reine Ang. Math.*, **375**: 380-393 (1987).
8. Abramowitz M. and Stegun A. *Handbook of Mathematical Functions*. Dover Publications, Inc., New York (1972).
9. Levitan B.M. and Sargsjan I.S. *Sturm-Liouville and Dirac Operators*. Kluwer Academic Publisher (1991).