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Limiting Properties of Empirical Bayes Estimators in a Two-Factor Experiment under Inverse Gaussian Model

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Abstract

The empirical Bayes estimators of treatment effects in a factorial experiment were derived and their asymptotic properties were explored. It was shown that they were asymptotically optimal and the estimator of the scale parameter had a limiting gamma distribution while the estimators of the factor effects had a limiting multivariate normal distribution. A Bootstrap analysis was performed to illustrate the theoretical results empirically.

Keywords: Bayes and empirical Bayes procedures; Bootstrap; Factorial experiments; Inverse Gaussian

Introduction

In our previous study [3], the empirical Bayes estimators of the scale parameter and treatment effects for a factorial experiment under an Inverse Gaussian model were derived. To provide a concise background for the sake of easier reference, let for each i and j, the observations Y_{ijk} be a random sample from an Inverse Gaussian distribution. That is,

$$f(y_{ijk} \mid \theta_{ij}, \lambda) =$$

$$\{\lambda / 2\pi y_{ijk}^3\}^{1/2} \exp\{-\lambda (y_{ijk} - \theta_{ij})^2 / 2 y_{ijk} \theta_{ij}^2\}$$

$$y_{ijk}, \lambda, \theta_{ij} > 0, \ \mathrm{i} = 1, \ldots, \mathrm{I}, \quad \mathrm{j} = 1, \ldots, \mathrm{J}, \quad k = 1, \ldots, n.$$

Here the mean is modeled in terms of drift, i.e., the sum of factor effects:

$$\theta_{ij}^{-1} = \mu + \alpha_i + \beta_j + \gamma_{ij},$$

$$\sum_{i=1}^{I} \alpha_i = \sum_{j=1}^{J} \beta_j = \sum_{i=1}^{I} \gamma_{ij} = \sum_{j=1}^{J} \gamma_{ij} = 0.$$

Taking account of constraints imposed on factor effects, the parameter vector $\mathbf{\Phi}$, is defined as:

$$\mathbf{\Phi} = [\mu, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}_1, \ldots, \boldsymbol{\gamma}_i, \ldots, \boldsymbol{\gamma}_{I-1}], \ \alpha = [\alpha_1, \ldots, \alpha_{I-1}],$$

$$\boldsymbol{\beta} = [\beta_1, \ldots \beta_{J-1}], \ \boldsymbol{\gamma}_i = [\gamma_{i1}, \ldots, \gamma_{i,J-1}], \ \mathrm{i} = 1, \ldots, \mathrm{I} - 1.$$

Then the likelihood function is:

$$L(\mathbf{\Phi}, \lambda | \mathbf{y}) \propto \lambda^{nIJ/2} \exp\{-n\lambda [IJr_{\ldots} - 2\mathbf{\Phi}' d + \mathbf{\Phi}' \mathbf{M} \mathbf{\Phi}] / 2\}$$

where

$$\mathbf{y} = [y_{111}, y_{112}, \ldots, y_{ijk}, \ldots, y_{IJn}], \ \theta_{ij}^{-1} = \mathbf{X}'_{ij} \mathbf{\Phi},$$

$$\mathbf{X}' = (X_{11}, X_{12}, \ldots, X_{IJ}),$$

$$\mathbf{D} = \mathrm{diag}\{y_{11.}, \ldots, y_{ij.}, \ldots, y_{IJ.}\},$$

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$$y_{ij.} = n^{-1}\sum_{k=1}^{n} y_{ijk} \;,\; \mathbf{M = X'DX} \,,\, \mathbf{d} = \sum_{i=1}^{I}\sum_{j=1}^{J} \mathbf{X}_{ij} \;,$$

$r_{ijk} = y_{ijk}^{-1}$

$$r_{...} = (nIJ)^{-1}\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{n} r_{ijk} = (nIJ)^{-1} r_{+++} \,.$$

The following conjugate priors are used for λ and $\mathbf{\Phi}$:

$\lambda \sim Gamma(a, b/2)$ and $(\mathbf{\Phi} \mid \lambda) \sim N(\eta, \lambda^{-1}\Delta)$,

with

$\boldsymbol{\eta} = [\eta_1, \eta_2, \ldots, \eta_{IJ}]$, $\Delta = diag\{\delta_1^2, \ldots, \delta_{IJ}^2\}$.

The posterior means of λ and $\mathbf{\Phi}$ provide the Bayes estimators under squared error loss. The empirical Bayes estimators are obtained by substituting the estimators of hyperparameters (a_0, b_0) and $(\boldsymbol{\eta}_0, \Delta_0)$ for corresponding values in the Bayes estimators. These estimators are:

$$\lambda_{EB1} = 2k_0 / [Q_1(\boldsymbol{\eta}_0)] \tag{1.1}$$

with estimated variance

$$\text{var}(\lambda \mid \mathbf{y}) = 4k_0 / [Q_1(\boldsymbol{\eta}_0)]^2 = [\lambda_{EB1}]^2 / k_0 \,, \tag{1.2}$$

and

$$\mathbf{\Phi}_{EB1} = \boldsymbol{\eta}_0^* = (n\Delta_0\mathbf{M} + \mathbf{I})^{-1}(n\Delta_0\mathbf{d} + \boldsymbol{\eta}_0) \,, \tag{1.3}$$

with estimated variance

$$\text{var}(\mathbf{\Phi} \mid \mathbf{y}) = [2(k_0 - 1)]^{-1} Q_1(\boldsymbol{\eta}_0)\Psi_0 \,. \tag{1.4}$$

In (1.1) to (1.4),

$k_0 = 0.5(nIJ + 2a_0)$, $\mathbf{\Psi}_0 = (n\Delta_0\mathbf{M} + \Delta_0^{-1})^{-1}$,

$Q_1(\boldsymbol{\eta}_0) = r_{+++} + b_0 + \boldsymbol{\eta}'\Delta_0^{-1}\boldsymbol{\eta} - \boldsymbol{\eta}_0^{*\prime}\mathbf{\Psi}_0^{-1}\boldsymbol{\eta}_0^*$.

Using the results given by Chhikara and Folks [2], the moments of the marginal distributions of $R_{ijk} = Y_{ijk}^{-1}$, and those of various means $R_{ij.}$, $R_{i..}$, $R_{.j.}$ and $R_{...}$ are also obtained. It is known that:

$$V_{ij} = \sum_{k=1}^{n} (Y_{ijk}^{-1} - Y_{ij.}^{-1}) \sim \lambda^{-1}\chi_{n-1}^2 \,,$$

i = 1,…,I, j=1,…,J.

Let $\overline{V}$ and c be the sample mean and coefficient of variation of V_{ij}. Then, for $c^2 > 2/(n-1)$ we obtain:

$a_0 = [2(n-1)c^2 + n - 3]/[(n-1)c^2 - 2]$,

$b_0 = 2(a_0 - 1)\overline{V}/(n-1)$,

and $a_0 = b_0 = 0$. Furthermore, in the two-way table of observations, each cell has n replicates R_{ijk}. Let S_{ij}^2, $S_{i.}^2$, $S_{.j}^2$, and S^2 denote the sample variances of $R_{ij.}$, $R_{i..}$, $R_{.j.}$ and $R_{...}$, respectively. By equating the sample and theoretical moments, we will have:

$\eta_{1,0} = r_{i..} - b_0/2(a_0 - 1)$, $\eta_{i+1,0} = r_{i..} - r_{...}$,

$\eta_{I+j,0} = r_{.j.} - r_{...}$, $\eta_{I+i(J-I)+j,0} = r_{ij.} - r_{i..} - r_{.j.} + r_{...}$

i = 1,…,I, j=1,…,J. (1.5)

Similarly, we obtain:

$$\begin{aligned}
\delta_{1,0}^2 &= 2(a_0 - 1)S^2/b_0 - \eta_{1,0}/nIJ \\
&\quad - b_0[2(a_0 - 1) + nIJ]/2nIJ(a_0 - 1)(a_0 - 2), \\
\delta_{i+1,0}^2 &= 2(a_0 - 1)S_{i.}^2/b_0 - (\eta_{1,0} + \eta_{i+1,0})/nJ - \delta_{1,0}^2 \\
&\quad - b_0[2(a_0 - 1) + nJ]/2nJ(a_0 - 2), \\
\delta_{I+j,0}^2 &= 2(a_0 - 1)S_{.j}^2/b_0 - (\eta_{1,0} + \eta_{1+j,0})/nI - \delta_{1,0}^2 \\
&\quad - b_0[2(a_0 - 1) + nI]/2nI(a_0 - 1)(a_0 - 2), \\
\delta_{I+i(J-1)+j,0} &= 2(a_0 - 1)S_{ij}^2/b_0 \\
&\quad - (\eta_{1,0} + \eta_{i+1,0} + \eta_{I+j,0} + \eta_{I+i(J-1)+j})/n \\
&\quad - (\delta_{1,0}^2 + \delta_{i+1,0}^2 + \delta_{I+j,0}^2) \\
&\quad - b_0[2(a_0 - 1) + n]/2n(a_0 - 1)(a_0 - 2).
\end{aligned} \tag{1.6}$$

Now, we arrange them as:

$\boldsymbol{\eta}_0 = [\eta_{1,0}, \eta_{2,0}, \ldots, \eta_{IJ,0}]$ and

$\Delta_0 = diag\{\delta_{1,0}^2, \delta_{2,0}^2, \ldots, \delta_{IJ,0}^2\}$.

The large sample properties of the empirical Bayes estimators, λ_{EB1} and Φ_{EB1}, are derived in section 2. In section 3, the limiting distribution of an alternative estimator of $\mathbf{\Phi}$ respective to a truncated normal prior is studied. The findings from a limited Bootstrap study are reported in section 4.

2. Large Sample Properties

To establish the large sample properties of our empirical Bayes estimators, we utilize the following well-known facts. In deriving the estimates of the hyperparameters we have used the sample means of some random variables. Thus for large sample size, n, all these means converge in probability to their respective expected values. Their continuous functions behave in the same manner, as well. Thus, we have:

Lemma 1. When $n \to \infty$, we have:

$$a_0 \xrightarrow{p} a\ ,\ b_0 \xrightarrow{p} b\ ,\ k_0 \xrightarrow{p} k\ ,$$

$$\boldsymbol{\eta}_0 \xrightarrow{p} \boldsymbol{\eta}\ ,\ \Delta_0 \xrightarrow{p} \Delta\ ,$$

which imply:

$$Q_1(\boldsymbol{\eta}_0) \xrightarrow{p} Q_1(\boldsymbol{\eta})\ .$$

Proof. These are the straight forward implications of the weak law of large numbers. □

Consequently, we have:

Theorem 1. As $n \to \infty$, the $\lambda_{EB1} \xrightarrow{p} \lambda_{B1}$. Moreover, its asymptotic distribution converges to a gamma distribution, i.e.,

$$\lambda_{EB1} \sim Gamma[k, Q_1(\boldsymbol{\eta})/2]\ .$$

Proof. The first part follows from Lemma 1 and the continuous nature of λ_{EB1} as a function of (a_0, b_0) and $(\boldsymbol{\eta}_0, \Delta_0)$.

To prove the second part, note that:

$$\lambda_{EB1} = \{2[k + (k_0 - k)]/Q_1(\boldsymbol{\eta})\}[Q_1(\boldsymbol{\eta}_0)/Q_1(\boldsymbol{\eta}_1)]^{-1}\ ,$$

where by virtue of Lemma 1 and Slutsky's theorem, the first factor on the right hand side converges in distribution to λ_{B1}, the Bayes estimator of λ, and the second factor approaches in probability to 1. However, in our previous study [3] it was shown that λ_{B1}, had a posterior distribution of gamma as stated above. □

To make the above result more practical, we prove:

Corollary 1. For large n, the unknown parameters in the limiting distribution of λ_{EB1} can be substituted by their estimates, while the statement still remains valid, i.e.,

$$\lambda_{EB1} \xrightarrow{D} Gamma[k_0, Q_1(\boldsymbol{\eta}_0)/2]\ .$$

Proof. By Theorem 1, the asymptotic cumulant generating function of λ_{EB1} is:

$$C(t) = -k\ \log[1 - 2it/Q_1(\boldsymbol{\eta})]$$

$$= -[(k - k_0) + k_0]\log\{1 - \frac{2it}{Q_1(\boldsymbol{\eta}_0)} \cdot \frac{Q_1(\boldsymbol{\eta}_0)}{Q_1(\boldsymbol{\eta})}\}$$

which by Lemma 1 is asymptotically equivalent to:

$$C(t) = -k_0 \log[1 - 2it/Q_1(\boldsymbol{\eta}_0)]\ .$$

Hence the proof is complete. □

To obtain the asymptotic distribution of $\boldsymbol{\Phi}_{EB1}$, we appeal to the multivariate central limit theorem along with Slutsky's theorem. Finally, at the last stage, we apply the δ-method to derive asymptotically equivalent distribution. These techniques are well expounded by Rao [5] and Serfling [6].

Let

$$\boldsymbol{R}_k = [R_{11k}, \ldots, R_{ijk}, \ldots, R_{IJk}]\ , \qquad k = 1, \ldots, n\ .$$

By assumption $\mathbf{R}_k$ are independent and identically distributed random vectors with the marginal moments

$$E(\mathbf{R}_k) = \rho = \mathbf{X}\boldsymbol{\eta} + [b/2(a-1)]\mathbf{1}_{IJ}\ ,$$

with $\boldsymbol{\rho} = [\rho_{11}, \ldots, \rho_{IJ}]$ and $\mathbf{1}_{IJ} = [1,1,\ldots,1]$.

Explicitly

$$\rho_{ij} = X'_{ij}\boldsymbol{\eta} + b/2(a-1)$$

$$= \eta_1 + \eta_{i+1} + \eta_{I+j} + \eta_{I+I(J-1)+j} + b/2(a-1)$$

and

$$Var(\mathbf{R}_k) = \Sigma = [b/2(a-1)]\{\mathbf{X}'\Delta\mathbf{X} + \mathbf{C} + [b/2(a-1)(a-2)]\mathbf{E}\} \qquad (2.1)$$

with

$$C = diag\{[X'_{11}\eta + b/2(a-2)], \ldots, [X'_{ij}\eta + b/2(a-2)], \ldots, [X'_{IJ}\eta + b/2(a-2)]\}$$

and $\mathbf{E} = \mathbf{1}_{IJ}\mathbf{1}'_{IJ}$.

Now, it is obvious that all second moments are finite. Thus, the multivariate central limit theorem applies to:

$$\boldsymbol{R}_{\cdot} = n^{-1}\sum_{k=1}^{n}\boldsymbol{R}_k$$

$$= [R_{11\cdot}, \ldots, R_{1J\cdot}, \ldots, R_{i1\cdot}, \ldots, R_{iJ\cdot}, \ldots, R_{I1\cdot}, \ldots, R_{IJ\cdot}].$$

That is,

$$\sqrt{n}\sum^{-1/2}(\boldsymbol{R}_{\cdot} - \boldsymbol{\rho}) \xrightarrow{D} N(\mathbf{0}, \mathbf{I})\ . \qquad (2.2)$$

This leads us to the limiting distribution of $\boldsymbol{\eta}_0$.

Theorem 2. The asymptotic distribution of $\boldsymbol{\eta}_0$ is a multivariate normal, i.e.,

$$\sqrt{n}\,(IJ)[\mathbf{A}\textstyle\sum\mathbf{A}']^{-1/2}[\boldsymbol{\eta}_0-\boldsymbol{\eta}]\xrightarrow{D}N\,(\mathbf{0},\mathbf{I}),$$

where

$$\eta=(IJ)^{-1}\mathbf{A}\rho-[b/2(a-1)]\mathbf{e}.$$

Proof. From (1.5), it is observed that $\boldsymbol{\eta}_0$ is a linear function of $\boldsymbol{R}.$, which can be written as:

$$\boldsymbol{\eta}_0=(IJ)^{-1}\mathbf{A}\boldsymbol{R}.-[b_0/2(a_0-1)]e \qquad (2.3)$$

where $\mathbf{e}=[\mathbf{1},\mathbf{0}]$ is a column vector of order IJ, with a 1 in the first row, and zero elsewhere.

$\mathbf{A}$ is an IJ×IJ block matrix defined by the following vectors and sub-matrices. Let $\mathbf{0}_m$ and $\mathbf{1}_m$ be m-vectors of zeros and 1's, respectively. Define $\mathbf{E}_i$ as a matrix of order (I-J)×J with $\mathbf{1}'_J$ in the i-the row and $\mathbf{0}'_J$ elsewhere. Let

$$\mathbf{A}_\mathrm{i}=\mathrm{I}\mathbf{E}_\mathrm{i}-\mathbf{1}_{(\mathrm{I}-1)}1'_\mathrm{J},\ \mathrm{i}=1,\ldots,\mathrm{I}-1,\ \text{and}\ \mathbf{A}_\mathrm{I}=-\mathbf{1}_{(\mathrm{I}-1)}1'_\mathrm{J},$$

and

$$\mathbf{B}=\mathrm{J}[\mathbf{I},\mathbf{0}_{(\mathrm{J}-1)}]-\mathbf{1}_{(\mathrm{J}-1)}\mathbf{1}'_\mathrm{J}.$$

Then, $\mathbf{A}$ has the following structure which is conformable to left multiplication with $\mathbf{R}.$,

$$\mathbf{A}=\begin{bmatrix}
\mathbf{1}'_J & \mathbf{1}'_J & \cdots & \mathbf{1}'_J & \cdots & \mathbf{1}'_J & \mathbf{1}'_J\\
\mathbf{A}_1 & \mathbf{A}_2 & \cdots & \mathbf{A}_i & \cdots & \mathbf{A}_{(I-1)} & \mathbf{A}_I\\
\mathbf{B} & \mathbf{B} & \cdots & \mathbf{B} & \vdots & \mathbf{B} & \mathbf{B}\\
(I-1)\mathbf{B} & -\mathbf{B} & \cdots & -\mathbf{B} & \cdots & -\mathbf{B} & -\mathbf{B}\\
-\mathbf{B} & (I-1)\mathbf{B} & \cdots & -\mathbf{B} & \cdots & -\mathbf{B} & -\mathbf{B}\\
\vdots & \vdots & & \vdots & & \vdots & \vdots\\
-\mathbf{B} & -\mathbf{B} & \cdots & (I-1)\mathbf{B} & \cdots & -\mathbf{B} & -\mathbf{B}\\
\vdots & \vdots & & \vdots & & \vdots & \vdots\\
-\mathbf{B} & -\mathbf{B} & \cdots & -\mathbf{B} & \cdots & (I-1)\mathbf{B} & -\mathbf{B}\\
-\mathbf{B} & -\mathbf{B} & \cdots & -\mathbf{B} & \cdots & -\mathbf{B} & (I-1)\mathbf{B}
\end{bmatrix}$$

Since in (2.3), $b_0/2(a_0-1)\xrightarrow{P}b/2(a-1)$, it follows that

$$\boldsymbol{\eta}_0=(IJ)^{-1}\mathbf{AR.}-[b_0/2(a_0-1)]e\xrightarrow{D}(IJ)^{-1}\mathbf{AR.}-[b/2(a-1)]e,$$

Thus, by Slutsky's theorem the result follows. □

Now, we are able to find the asymptotic distribution of $\boldsymbol{\Phi}_{\mathrm{EB}1}$.

Theorem 3. The empirical Bayes estimator

$$\boldsymbol{\Phi}_{\mathrm{EB}1}=(\mathrm{n}\,\Delta_0\mathbf{M}+\mathbf{I})^{-1}(\mathrm{n}\,\Delta_0\mathbf{d}+\boldsymbol{\eta}_0)$$

has a limiting multivariate normal distribution, i.e.,

$$\sqrt{n}\,(IJ)[\textstyle\sum^*]^{-1/2}[\boldsymbol{\Phi}_{EB1}-\boldsymbol{\Phi}_{B1}]\xrightarrow{D}N\,(\mathbf{0},\mathbf{I}),$$

with

$$\boldsymbol{\Phi}_{B1}=(\mathrm{n}\,\Delta\mathbf{M}+\mathbf{I})^{-1}(\mathrm{n}\,\Delta\mathbf{d}+\boldsymbol{\eta})\ \text{and}$$
$$\textstyle\sum^*=(\mathrm{n}\Delta\mathbf{M}+I)^{-1}(\mathbf{A}\sum\mathbf{A}')(\mathrm{n}\Delta\mathbf{M}+I)^{-1}.$$

Proof. First, we employ the Cramer-Wold device, Serfling [6]. Note that by Lemma 1, for large n,

$$(\mathrm{n}\Delta_0\mathbf{M}+\mathbf{I})=n(\Delta_0-\Delta)\mathbf{M}+(\mathrm{n}\Delta\mathbf{M}+\mathbf{I})\xrightarrow{P}(\mathrm{n}\Delta\mathbf{M}+\mathbf{I})$$

and

$$(\mathrm{n}\Delta_0\mathbf{d}+\boldsymbol{\eta}_0)=\mathrm{n}(\Delta_0-\Delta)\mathbf{d}+(\mathrm{n}\Delta\mathbf{d}+\boldsymbol{\eta}_0)\xrightarrow{D}(\mathrm{n}\Delta\mathbf{d}+\boldsymbol{\eta}_0)$$

Applying Slutsky's theorem at this stage, we have

$$\boldsymbol{\Phi}_{EB1}=[\mathrm{n}(\Delta_0-\Delta)\mathbf{M}+(\mathrm{n}\Delta\mathbf{M}+\mathbf{I})]^{-1}[\mathrm{n}(\Delta_0-\Delta)\mathbf{d}+(\mathrm{n}\Delta\mathbf{d}+\boldsymbol{\eta}_0)]\xrightarrow{D}[\mathrm{n}\Delta\mathbf{M}+\mathbf{I}]^{-1}[\mathrm{n}\Delta\mathbf{d}+\boldsymbol{\eta}_0],$$

which is a linear function of $\boldsymbol{\eta}_0$ whose asymptotic distribution is normal. Thus, the result follows from Theorem 2. □

In the above results, $\boldsymbol{\Delta}$ and $\sum$ are unknown, making the theorem of limited use in practice. It will be more useful, however, if the replacement of $\boldsymbol{\Delta}$ and $\sum$ by their estimates $\sum_0$ and $\boldsymbol{\Delta}_0$ can be justified. This is what we have in

Theorem 4. The empirical Bayes estimator

$$\boldsymbol{\Phi}_{\mathrm{EB}1}=(\mathrm{n}\Delta_0\mathbf{M}+\mathbf{I})^{-1}(\mathrm{n}\Delta_0\mathbf{d}+\boldsymbol{\eta}_0)$$

has an asymptotic multivariate normal distribution with mean $\boldsymbol{\Phi}_{B1}$ and variance

$$\textstyle\sum_0^*=(\mathrm{n}\Delta_0\mathbf{M}+\mathbf{I})^{-1}(\mathbf{A}\sum_\mathbf{0}\mathbf{A}')(\mathrm{n}\Delta_0\mathbf{M}+\mathbf{I})^{-1}.$$

That is,

$$\sqrt{\mathrm{n}}(IJ)[\textstyle\sum_0^*]^{-1/2}[\boldsymbol{\Phi}_{\mathrm{EB}1}-\boldsymbol{\Phi}_{\mathrm{B}1}]\xrightarrow{D}N\,(\mathbf{0},\mathbf{I}).$$

Proof. By the consistent nature of $\boldsymbol{\Delta}_0$ and $\sum_0$, it is obvious that $(\Sigma_0^* - \Sigma) \xrightarrow{P} \mathbf{0}$. Thus,

$$\sqrt{\mathrm{n}}(IJ)[(\Sigma_0^* - \Sigma^*) + \Sigma^*]^{-1/2}[\mathbf{\Phi}_{EB1} - \mathbf{\Phi}_{B1}]$$

$$\xrightarrow{D} \sqrt{\mathrm{n}}(IJ)[\Sigma^*]^{-1/2}[\mathbf{\Phi}_{EB1} - \mathbf{\Phi}_{B1}].$$

which completes the proof. □

3. The Case of Restricted Prior

In our previous study [3], a univariate normal distribution truncated from the left at zero was used as an alternative prior for μ which is a positive parameter. Thus,

$$\mathbf{q}_2(\mu|\lambda) \propto \lambda^{1/2}[\delta_1 \mathbf{N}(\tau)]^{-1} \exp\{-\lambda(\mu - \eta_1)^2 / 2\delta_1^2\}$$

$$\propto [\mathbf{N}(\tau)]^{-1} \mathbf{q}_1(\mu|\lambda)$$

$$\lambda, \delta_1, \eta_1, \mu > 0 ,$$

where $\mathbf{N}(\tau)$ is the value of the standard normal distribution function at $\tau = \lambda^{1/2}\eta_1^* \psi_{11}^{-1}$.

This modification of the previous prior renders a posterior for $(\mathbf{\Phi} \mid \lambda, \boldsymbol{y})$ as:

$$\mathbf{q}_2(\mathbf{\Phi} \mid \lambda, y) \propto [\mathbf{N}(\xi)]^{-1} \mathbf{q}_1(\mathbf{\Phi} \mid \lambda, y) , \tag{3.1}$$

where $\xi = \mathbf{E}(\mu \mid \lambda, y)/\mathrm{Var}(\mu \mid \lambda, y)$ and $\mathbf{q}_1(\mathbf{\Phi} \mid \lambda, y)$ is the posterior corresponding to the unrestricted prior studied in section 2.

Using the partitions

$$\mathbf{\Phi} = [\mu, \mathbf{\Phi}_2] , \ \boldsymbol{\eta} = [\eta_1, \boldsymbol{\eta}_2] , \ \boldsymbol{\eta}^* = [\eta_1^*, \boldsymbol{\eta}_2^*],$$

and

$$\mathbf{\Psi} = \begin{bmatrix} \psi_{11} & \mathbf{\Psi}_{12} \\ \mathbf{\Psi}_{21} & \mathbf{\Psi}_{22} \end{bmatrix}$$

in the well-known identities

$$\mathbf{\Psi}_{22.1} = \mathbf{\Psi}_{22} - \mathbf{\Psi}_{21}\psi_{11}^{-1}\mathbf{\Psi}_{12} \ \text{ and } \ |\mathbf{\Psi}| = \psi_{11}|\mathbf{\Psi}_{22.1}| ,$$

We are able to decompose the quadratic form in the exponent of the posterior (3.1). Hence, we obtain the marginal posteriors $\mathbf{q}_2(\mu \mid y)$ and $\mathbf{q}_2(\mathbf{\Phi}_2 \mid y)$ whose means provide the Bayes estimators of μ and $\mathbf{\Phi}_2$, respectively. To obtain the empirical Bayes estimators, we need some sort of estimates to replace for $\boldsymbol{\eta}$ and $\mathbf{\Delta}$ in the Bayes estimators.

To estimate the hyperparameters of the truncated normal prior for μ along with those related to $\mathbf{\Phi}_2$, we again apply the method of moments on $\mathbf{R.}$. This procedure, which has been detailed in our previous study [3], leads to

$$\mathbf{E}(\mathbf{R.}) = \varsigma = \rho + \mathrm{k}\delta_1 \mathbf{1}_{IJ} ,$$

$$\mathrm{Var}\ (\mathbf{R.}) = \mathrm{n}^{-1}\mathbf{\Omega} = \mathrm{n}^{-1}[\Sigma - k_1\mathbf{E} + k_2\mathbf{I}]$$

with

$$k = 0.82[b(a-0.5)/a]^{1/2} ,$$

$$k_1 = \mathrm{k}\delta_1[\eta_1 + \mathrm{k}\delta_1 - b/(a-1)(2a-3)],$$

and

$$k_2 = k\,\delta_1 b/(2a-3) .$$

When the sample mean $\mathbf{R.}$ and the sample variance $\mathbf{T}$ are substituted for the respective theoretical moments, we obtain the following system of equations in $\boldsymbol{\eta}$ and $\mathbf{\Delta}$:

$$\mathbf{R.} = \mathbf{X}\eta + [b_0/2(a_0-1)]k_0\delta_1 \mathbf{1}_{IJ}$$

$$\boldsymbol{T} = \Sigma - k_{1,0}\mathbf{E} + k_{2,0}\mathbf{I} , \tag{3.2}$$

where the subscript 0 in the above quantities indicates the respective estimated values obtained by using a_0 and b_0 in place of a and b, respectively. Upon solving the system of equations (3.2), we arrive at:

$$\overline{\boldsymbol{\eta}} = (\mathbf{X'X})^{-1}[\mathbf{X'R.} - (\mathrm{r}_{\dots} - \eta_1^0)\mathrm{e}_{IJ}] ,$$

and

$$\overline{\mathbf{\Delta}} = (\mathbf{X'X})^{-1}\mathbf{X'T^*X(X'X)}^{-1} , \tag{3.3}$$

where

$$\mathbf{T}^* = [2(a_0-1)/b_0]\{\mathbf{T} + [k_{1,0} - 1/(a_0-2)]\mathbf{E} - k_{2,0}\mathbf{I}\} - \mathbf{D}_1 ,$$

with

$$\mathbf{D}_1 = diag\,\{\mathbf{X}'_{11}\overline{\boldsymbol{\eta}} + [b_0/2(a_0-1)] ,\dots, \mathbf{X}'_{ij}\overline{\boldsymbol{\eta}}$$

$$+ [b_0/2(a_0-1)],\dots, \mathbf{X}'_{IJ}\overline{\boldsymbol{\eta}} + [b_0/2(a_0-1)]\}.$$

Having found these estimates, we define the alternative empirical Bayes estimates relative to the truncated prior as:

$$\lambda_{EB2} = 2k_0/Q_1(\overline{\boldsymbol{\eta}}) , \tag{3.4}$$

with the estimated variance

$$\mathrm{Var}(\lambda_{EB2} \mid y) = 4k_0/[Q_1(\overline{\boldsymbol{\eta}})]^2 .$$

For the factorial effects, we have the alternative estimator

$$\mathbf{\Phi}_{EB2} = \overline{\boldsymbol{\eta}^*} = (n\overline{\mathbf{\Delta}}\,\mathbf{M} + \mathbf{I})^{-1}(n\overline{\mathbf{\Delta}}\mathbf{d} + \overline{\boldsymbol{\eta}}) . \tag{3.5}$$

Now, similar to the case of unrestricted prior, the asymptotic distributions of λ_{EB1} and $\mathbf{\Phi}_{EB2}$ can be derived.

Corollary 2. The asymptotic distribution of the empirical Bayes estimator λ_{EB2} relative to the restricted prior for μ and an unrestricted prior for $\mathbf{\Phi}_2$ is a gamma distribution, i.e.,

$$\lambda_{EB2} \xrightarrow{D} Gamma[k_0, Q_1(\overline{\boldsymbol{\eta}})/2].$$

Due to similarity with Corollary 1, the proof is omitted. □

For the factorial effects, firstly we need to find the limiting distribution of $\overline{\boldsymbol{\eta}}$. To this end, note that by the multivariate central limit theorem, we have:

$$\sqrt{n}\,(\mathbf{\Omega})^{-1/2}[\mathbf{R}. - \varsigma] \xrightarrow{D} N(0, \mathbf{I}). \tag{3.6}$$

Similar to the case of the unrestricted prior

$$\overline{\boldsymbol{\eta}} = (IJ)^{-1}\mathbf{AR}. - [b_0/2(a_0-1) + k_0\overline{\delta_1}]e,$$

where $\mathbf{A}$ and $\mathbf{e}$ have been defined in (2.3). Thus, with regard to (3.6), Lemma 1, and consistency of $\overline{\delta_1}$, it follows that:

$$\sqrt{n}\,(IJ)(\mathbf{A\Omega A'})^{-1/2}[\overline{\boldsymbol{\eta}} - \boldsymbol{\eta}] \xrightarrow{D} N(\mathbf{0}, \boldsymbol{I}).$$

Now, we can obtain the limiting distribution of $\mathbf{\Phi}_{EB2}$, as stated in

Theorem 5. As $n \to \infty$, the empirical Bayes estimator $\mathbf{\Phi}_{EB2}$ converges in distribution to a normal vector, i.e.,

$$\sqrt{n}\,(IJ)[\mathbf{A\Omega A'}]^{-1/2}[\mathbf{\Phi}_{EB2} - \mathbf{\Phi}_{B2}] \xrightarrow{D} N(0, \mathbf{I})$$

with

$$\mathbf{\Phi}_{B2} = \boldsymbol{\eta}^* + 0.21[(k-0.5)\psi_{11}^{-1}Q_1(\boldsymbol{\eta})e/k]^{1/2}\Psi_{.1},$$

where $\mathbf{\Psi}_{.1}$ is the first column of $\mathbf{\Psi}$.

Proof. The proof is similar to the proof of Theorem 3. □

Theorem 6. In Theorem 5, the unknown variance $\mathbf{\Omega}$ can be replaced by its estimator $\overline{\mathbf{\Omega}} = \Sigma_0 - k_{1,0}\mathbf{E} + k_{2,0}\mathbf{I}$, while its statement still remains valid. That is,

$$\sqrt{n}\,(IJ)[\mathbf{A\overline{\Omega}A'}]^{-1/2}[\mathbf{\Phi}_{EB2} - \mathbf{\Phi}_{B2}] \xrightarrow{D} N(\mathbf{0}, \mathbf{I}).$$

Proof. The proof is similar to the proof of Theorem 4. □

4. A Bootstrap Study

In order to verify the performance of our estimators empirically, a small scale Bootstrap analysis is considered. It is performed on a set of data which has been the subject of analyses by Ostle [4], Shuster and Muira [7], and Achcar and Rosales [1]. The estimates reported by Meshkani [3] show that the interaction effects are negligible. Therefore, assuming a nontruncated prior for the main effects, a two-factor model without interactions is considered for the Bootstrap analysis. The Bootstrap samples of size n=10 are chosen from the observations of each treatment combination. They were repeated for B=100 times. The Bootstrap distribution of λ_{EB1} of (1.1) and those of the elements of $\mathbf{\Phi}_{EB1}$ of (1.3) are computed and plotted. Only Three plots of distributions are reported in Figures (4.1) to (4.3). We observe that λ_{EB1} follows a gamma distribution and $\mu = \phi_1$ and $\beta_3 = \phi_5$ have normal distributions. These findings are in good agreement with our theoretical results. Furthermore, utilizing the quantiles of the Bootstrap distribution, the 95% confidence intervals are constructed which are only reported for λ, μ and β_3.

$\lambda \in [0.95,\ 3.23]$, $\mu \in [1.10,\ 1.42]$, and $\beta_3 \in [0.17,\ 0.27]$.

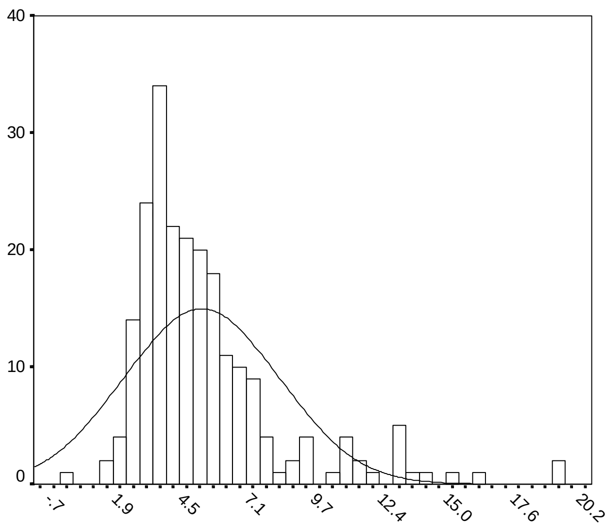

Figure 4.1. The Bootstrap posterior distribution of λ_{EB1}.

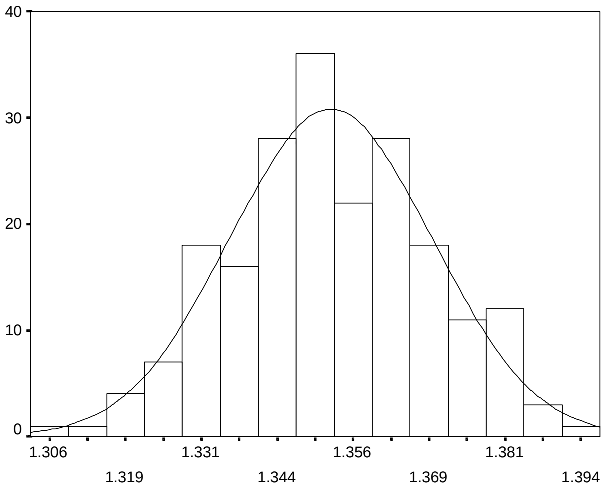

Figure 4.2. The Bootstrap posterior distribution of $\mu = \phi_1$.

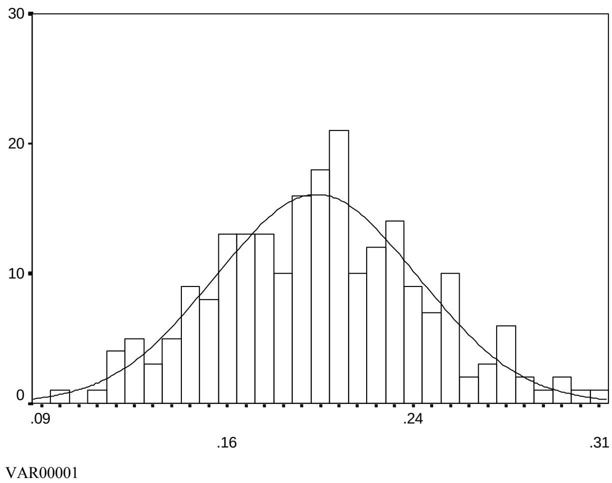

Figure 4.3. The Bootstrap posterior distribution of $\beta_3 = \phi_5$.

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