Journal of Sciences, Islamic Republic of Iran 13(4): 357-362 (2002)
National Center For Scientific Research, ISSN 1016-1104

MATRIX VALUATION PSEUDO RING (MVPR) AND AN EXTENSION THEOREM OF MATRIX VALUATION

M. Arian-Nejad*

Department of Mathematics, University of Zanjan, Zanjan, Islamic Republic of Iran

Abstract

Let R be a ring and V be a matrix valuation on R. It is shown that, there exists a correspondence between matrix valuations on R and some special subsets $\sum$(MVPR) of the set of all square matrices over R, analogous to the correspondence between invariant valuation rings and abelian valuation functions on a division ring. Furthermore, based on Malcolmson's localization, an alternative proof for the following result is presented. "There exists a natural bijection between the matrix valuations on R and valuated epic R-fields."

Keywords: Matrix valuation; Epic field; Localization; Valuation ring

Introduction

In the classical commutative and non-commutative field theory, the equivalence of valuations and valuation rings are considered as a natural base for the study of these notions. In this paper we present a similar equivalence for matrix valuations by defining MVPRs.

In a commutative ring R, the existence of a valuation function on R is a sufficient condition for the existence of a valuated epic R-field. Conversely, the restriction to R of any valuation on an epic R-field is a valuation on R. A generalization of this result for a non-commutative ring R, based on Cohn's localization [1] and using the notion of matrix valuation, is given in [7] as: "There exists a natural bijection between the matrix valuations on R and the valuated epic R-fields."

In this note we give an alternative simpler proof for this result, which is based on Malcolmson's localization [5,6]. This proof gives the valuation on epic fields directly and simplifies the computation. For an extensive study of the notion of matrix valuation one is referred to [2,3,8,9,10].

Definitions and Preliminaries

In what follows, R, $M(R)$ and $GL(R)$ will represent, an associative ring with unit, the set of all square matrices, and the set of invertible matrices over R. respectively. A Krull valuation on R is a function $\upsilon : R \to \Gamma \bigcup \{\infty\}$, where Γ is a totally ordered abelian additive group such that for all $a,b \in R$,

υ.1) $\upsilon(ab) = \upsilon(a) + \upsilon(b)$,

υ.2) $\upsilon(a+b) \geq min\{\upsilon(a), \upsilon(b)\}$,

υ.3) $\upsilon(a) = \infty$ if and only if $a = 0$.

Let D be a division ring, a subring R of D is called an

* *E-mail: arian@mail.znu.ac.ir*

invariant valuation ring of D if:

i) $d \in D \Rightarrow d \in R$ or $d^{-1} \in R$,

ii) $dRd^{-1} = R$ for all $d \in D$.

The existence of this subring is equivalent to the existence of a Krull valuation on D.

For any two matrices A and B over R we define the *diagonal sum* of A and B as:

$$A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

If A and B both are of size $n \times n$, and agree except possibly in the first column, say $A=(A_1, A_2, A_3, \ldots)$, $B=(B_1, A_2, A_3, \ldots)$, then their *determinantal sum* with respect to the first column is defined as: $A \nabla B = (A_1 + B_1, A_2, A_3, \ldots)$. Determinantal sum with respect to other columns or rows are defined similarly. A square matrix A is said to be *non-full* if it can be written as $A=PQ$, where P is an $n \times r$, Q an $r \times n$, and $r < n$; otherwise, A is called a *full matrix*. In a division ring, full matrices are invertible matrices and vice versa. A collection $\wp$ of square matrices over R is said to be a *matrix ideal* of R if it satisfies the following conditions:

MI.1 $\wp$ includes all non-full matrices.

MI.2 If $A,B \in \wp$ and if $A \nabla B$ is defined, then $A \nabla B \in \wp$.

MI.3 If $A \in \wp$, then $A \oplus B \in \wp$ for all $B \in M(R)$.

MI.4 $A \oplus 1 \in \wp$ implies $A \in \wp$.

Clearly a matrix ideal is proper if it does not contain the element 1. A matrix ideal $\wp$ is said to be *prime* if it is proper and

MI.5 $A \oplus B \in \wp \Rightarrow A \in \wp$ or $B \in \wp$.

A *matrix valuation* on R is a function V on $M(R)$ with values in $\Gamma \cup \{\infty\}$ such that:

MV.1 $V(A \oplus B) = V(A) + V(B)$, for all $A,B \in M(R)$.

MV.2 $V(A \nabla B) \geq \min\{V(A), V(B)\}$, for all $A,B \in M(R)$ such that $A \nabla B$ is defined.

MV.3 $V(A)$ is unchanged if any row or column is multiplied by -1.

MV.4 $V(1) = 0$.

MV.5 $V(A) = \infty$ for any non-full matrix A over R.

It is clearly seen that the axioms of prime matrix ideal may be obtained by writing down the conditions on the set $\wp = V^{-1}(\infty)$. We may note some consequences of MV.1-MV.5, which we use later.

MV.6 If $A \nabla B$ is defined and $V(A) \neq V(B)$, Then $V(A \nabla B) = \min\{V(A), V(B)\}$.

MV.7 $V(A)$ remains unchanged under any permutation of rows (or columns).

MV.8 $V\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = V\begin{pmatrix} A & 0 \\ D & B \end{pmatrix} = V(A) + V(B)$ for any C,D of appropriate size.

MV.9 $V(AB) = V(A) + V(B)$ for square matrices A,B of the same order.

Malcolmson's Method

A brief review of Malcolmson's method is useful. A pair $Q = (\Sigma, S)$ is called a *prime pair* if Σ is a multiplicative set of matrices and S is a proper matrix ideal on R, such that if $A,B \in M(R)$ and $A \oplus B \in S$, $A \in \Sigma$, then $B \in S$. In this case Σ and S do not contain any matrix in common. Given a prime pair $Q = (\Sigma, S)$, denote by T_Q the set of all triples (f,a,x) of matrices over R, where $a \in \Sigma$ (say of size $n \times n$), f is $1 \times n$ and x is $n \times 1$. Define a relation ~ on T_Q by requiring $(f,a,x) \sim (g,b,y)$, if and only if

$$\begin{pmatrix} a & 0 & x \\ 0 & b & -y \\ f & g & 0 \end{pmatrix} \in S.$$

The relation ~ is an equivalence relation. Denote the set of equivalence classes, $T_Q/\sim$ by R_Q. Denote the equivalence class of (f,a,x) by $(f/a\backslash x)$, and define:

$$(f/a\backslash x) + (g/b\backslash y) = ((f\ \ g)/\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\backslash\begin{pmatrix} x \\ y \end{pmatrix}),$$

$$(f/a\backslash x).(g/b\backslash y) = ((f\ \ 0)/\begin{pmatrix} a & -xg \\ 0 & b \end{pmatrix}\backslash\begin{pmatrix} 0 \\ y \end{pmatrix});$$

$$-(f/a\backslash x) = (f/a\backslash -x).$$

Define a map $E: R \to R_Q$ such that $E(r) = (1/1\backslash r)$ for $r \in R$. The above definition of operations gives rise to a well-defined ring structure on R_Q, with additive and multiplicative identities $E(0)$ and $E(1)$, in which E is a ring homomorphism. If $\wp$ is a prime matrix ideal of R, denote by $-\wp$ the set $M(R)\backslash\wp$, which is multiplicative and the pair $Q=(-\wp, \wp)$ is a prime pair. In this case R_Q

(or $R_{\wp}$ for simplicity) will be a division ring and

$$(f/a\backslash x)^{-1} = ((0\ \ 1)/\begin{pmatrix} a & x \\ f & 0 \end{pmatrix}\backslash\begin{pmatrix} 0 \\ -1 \end{pmatrix}).$$

Hence R has an epic R-field as a subset of R_Q generated by the image of E.

Results

1. Identifying Matrix Valuation Pseudo Rings (MVPRs)

Let V be a matrix valuation on a division ring D and let $\Sigma_V = \{A \in M(D) : V(A) \geq 0\}$.

This subset of $M(D)$ has the following properties:

(1) it contains all non-full matrices.
(2) $1 \in \Sigma_V$.
(3) $A,B \in \Sigma_V \Rightarrow A \oplus B \in \Sigma_V$.
(4) $A,B \in \Sigma_V$ and $A\nabla B$ is defined $\Rightarrow A\nabla B \in \Sigma_V$. (*)
(5) $A \oplus I \in \Sigma_V \Rightarrow A \in \Sigma_V$.
(6) $A \in GL(D) \Rightarrow A \in \Sigma_V$ or $A^{-1} \in \Sigma_V$.

It is possible to obtain some other properties of Σ_V, but they are not essential in our present study. Now, suppose that there exists a subset Σ of $M(D)$, which has the above six properties, then the following also hold:

(7) $A \oplus B \in \Sigma \Rightarrow B \oplus A \in \Sigma$.
(8) A,B are of the same size and $A \oplus B \in \Sigma \Rightarrow AB \in \Sigma$.
(9) $A \in \Sigma \Leftrightarrow B \oplus\ \ A \oplus\ B^{-1} \in \Sigma$ for all $B \in GL(D)$.

One can obtain these last three conclusions by invoking the kind of arguments used in [1.p.396].

In the following we introduce a method which shows that, given such a subset Σ on *M(D),* there exists a matrix valuation V on D such that $\Sigma_V = \Sigma$. We do this in seven steps:

(I). Define the following relation on $GL(D)$.

$A \sim B \Leftrightarrow \overline{\alpha} = \overline{\beta}$ or $\overline{\alpha} = \overline{-\beta}$, where $\overline{\alpha} = DetA$, $\overline{\beta} = DetB$,

Where "Det" denotes the Dieudonne' determinant [4, p.133-140]. This relation is an equivalence relation. Let the set of all individual classes be $S = GL(D)/\sim$.

(II). The following operation, similar to diagonal sum, on the elements of S gives a well defined group structure on S. This group is commutative.

$$\oplus : \begin{array}{ccc} S \times S & \to & S \\ ([A],[B]) & \mapsto & [A \oplus B] \end{array}$$

(III). Let $H = \{ A \in GL(D); A, A^{-1} \in \Sigma \}$ and let $L=H/\sim$. Then the group $(L,\oplus)$ is a subgroup of $(S,\oplus)$.

(IV). Now consider the factor group $\Gamma = S/L$ with the induced operation, which is an abelian group.

$$\Gamma = S/L = \{[A] \oplus L;\ [A] \in S\},$$

$$\oplus : \begin{array}{ccc} \Gamma \times \Gamma & \to & \Gamma \\ ([A] \oplus L, [B] \oplus L) & \mapsto & [A \oplus B] \oplus L. \end{array}$$

(V). The abelian group Γ with the following relation, is a totally ordered group.

For all $\overline{[A]}, \overline{[B]} \in \Gamma, \overline{[A]} \geq \overline{[B]} \Leftrightarrow A \oplus B^{-1} \in \Sigma$

The proof of (V). It must be shown that this is, a well-defined, total relation that admits the following conditions: transitivity, equality and consistency with the group operation.

First we check that "≥" is well-defined. Let $\overline{[A]} = \overline{[A']}, \overline{[B]} = \overline{[B']}$ and $\overline{[A]} \geq \overline{[B]}$. By definition $[A] \oplus L = [A'] \oplus L$, which implies $[A] \oplus [A'^{-1}] \oplus L = L$ or $[A \oplus A'^{-1}] \oplus L = L$. Therefore $[A \oplus A'^{-1}] \in L$ or equivalently $A \oplus A'^{-1} \in H$ and this implies that $A \oplus A'^{-1}$, $A^{-1} \oplus A' \in \Sigma$. Similarly, $B \oplus B'^{-1}$, $B^{-1} \oplus B' \in \Sigma$. Now, if $\overline{[A]} \geq \overline{[B]}$, then $A \oplus B^{-1} \in \Sigma$, therefore $A \oplus B^{-1} \oplus B \oplus B'^{-1} \in \Sigma$ and then $B'^{-1} \oplus A \oplus B^{-1} \oplus B \in \Sigma$, which implies $B'^{-1} \oplus A \in \Sigma$ or $B'^{-1} \oplus A \oplus A^{-1} \oplus A' \in \Sigma$, so, $A' \oplus B'^{-1} \in \Sigma$ and consequently $\overline{[A']} \geq \overline{[B']}$.

For the consistency with the group operation, note that by "consistency" we mean: if $\overline{[A]} \geq \overline{[B]}$ and $\overline{[C]} \in \Gamma$, then $\overline{[A]} \oplus \overline{[C]} \geq \overline{[B]} \oplus \overline{[C]}$. So, let $\overline{[A]} \geq \overline{[B]}$ then $B^{-1} \oplus A \in \Sigma$. Since $C \in GL(D)$ and $B^{-1} \oplus A \oplus C \oplus C^{-1} \in \Sigma$, then $A \oplus C \oplus C^{-1} \oplus B^{-1} \in \Sigma$, this implies that $\overline{[A \oplus C]} \geq \overline{[B \oplus C]}$, as desired.

The rest properties can be proved in a similar way. □

(VI). The function V on $M(D)$ defined as follows, is a matrix valuation on D.

$$V: \begin{array}{ccc} M(D) & \to & \Gamma\cup\{\infty\} \\ A & \mapsto & \overline{[A]} \\ non-full & \mapsto & \infty \end{array}$$

The proof of (VI). We will check all of the MV.1-MV.5 conditions of the definition of matrix valuation, as follows:

MV.1 $V(A\oplus B)=\overline{[A\oplus B]}=\overline{[A]}\oplus\overline{[B]}=V(A)\oplus V(B)$.

MV.2 $V(A\nabla B)=\overline{[A\nabla B]}\geq \min\{V(A),V(B)\}$.

For, we consider the following cases:

1. A and B are full matrices which are invertible over a division ring. If $A\nabla B$ is nonfull, then the claim is clear, so let $A\nabla B$ is full. If $\overline{[A]}\leq\overline{[B]}$, then $\overline{[A]}=\min\{V(A),V(B)\}$ and we must show that $\overline{[A\nabla B]}\geq\overline{[A]}$ or equivalently $(A\nabla B)\oplus A^{-1}\in\Sigma$ or equivalently.

$$(A\oplus A^{-1})\nabla(B\oplus A^{-1})\in\Sigma. \qquad (**)$$

(Note that $\overline{[A]}\leq\overline{[B]}$, implies $B\oplus A^{-1}\in\Sigma$, so by $(*)$ Properties of Σ the relation $(**)$ is true.)

2. One of the two matrices, say B, is non-full but A is full, so $B\oplus A^{-1}$ is non-full and hence is in Σ, so $(**)$ is true.

3. Both matrices are non-full. Since D is a division ring, $A\nabla B$ is also non-full, hence the claim is true obviously.

MV.3 This condition is true by definition of S given in (I).

MV.4 $V(1)=[1]\oplus L=L=\overline{O}$.

MV.5 If A is non-full, then $V(A)=\infty$ by definition. □

(VII). Clearly we have $\Sigma=\Sigma_V$.

Now, following the terminology of [1], we define:

Definition. For any ring R if a set $\Sigma\subset M(R)$ has the first five conditions in $(*)$, we call it a *Matrix Pseudo Ring* (briefly MPR). Reasonably any matrix pseudo ring with the sixth condition in $(*)$ could be called a *Matrix Valuation Pseudo Ring* (MVPR).

To sum up, we have proved that:

Proposition 1. *There is a 1-1 correspondence between matrix valuations and MVPRs on a division ring D.*

Note: In the above proposition, to obtain a matrix valuation, it would be possible first to restrict Σ to D which is a non-invariant valuation ring on D, and then extend its corresponding valuation to a matrix valuation on D by Theorem 1 of [7]. But by this method we do not get necessarily the same matrix valuation corresponding to Σ . Furthermore, the following proposition can not be drawn by this method.

Now let R be a ring embeddable in a division ring D. If Σ is an MVPR on R, we can consider $M(R)$ as a subset of $M(D)$. Then by the same process as steps (I)-(VI) and by the restriction of Dieudonne' determinant to $GL(R)$, we obtain its corresponding matrix valuation V on R. Consequently we have:

Proposition 2. *There is a 1-1 correspondence between matrix valuations and pseudo matrix valuation rings on any ring embeddable in a division ring.*

The following theorem states a generalization of the above propositions.

Theorem 1. *There is a 1-1 correspondence between matrix valuations and MVPRs on any ring.*

Proof. Let R be any ring and Σ be its MVPR, which is a proper subset of $M(R)$. Σ contains all of the non-full matrices, hence it contains the least matrix ideal containing non-full matrices. In other words, this matrix ideal is a proper matrix ideal. Hence, the result follows by Proposition 2 and Theorem 7.4.8 of [1]. Noting (VII) above, the other side is clear. □

2. An Alternative Proof for the Extension Theorem of Matrix Valuation.

In this section we give an alternative proof for the extension theorem of matrix valuations, based on Malcolmson's method of localization.

Theorem 2. *Let R be a ring, then each matrix valuation V on R determines an associated epic R-field K with a valuation v; and conversely, every epic R-field*

K with a valuation on it, arises from a matrix valuation on R. This correspondence between matrix valuations on R and valuated epic R-fields is bijective.

Proof. Let K be an epic R-field and v a valuation on it. We can form the associated matrix valuation V on R, simply by defining $V(A) = v(DetA^f)$, for all $A \in GL(R)$ and $V(A)=\infty$, for all $A \in M(R)\setminus GL(R)$; where $f: R \to K$ is the canonical epimorphism, A^f is the image of A by f and "Det" denotes the Dieudonne' determinant, as in [7]. Conversely, let V be a matrix valuation on R, then the set $\wp = V^{-1}(\infty)$ is a prime matrix ideal on R, so there is an associated epic R-field $K_\wp$, [see 1, p. 404]. We directly extend V to a valuation v on $K_\wp$, based on Malcolmson construction for $K_\wp$. Define v: $K_\wp \to \Gamma \cup \{\infty\}$, by

$$v(f/a\setminus x) = V\begin{pmatrix} f & 0 \\ a & x \end{pmatrix} - V(a).$$

Then v is a Krull valuation on $K_\wp$. In what follows, we check the three necessary properties. First the multiplicative property.

(i) Multiplicative property of "v":

$$v\,((f/a\setminus x).(g/b\setminus y))$$

$$= v\,((f\quad 0)/\begin{pmatrix} a & -xg \\ 0 & b \end{pmatrix}\setminus\begin{pmatrix} 0 \\ y \end{pmatrix})$$

$$= V\begin{pmatrix} f & 0 & 0 \\ a & -xg & 0 \\ 0 & b & y \end{pmatrix} - V\begin{pmatrix} a & -xg \\ 0 & b \end{pmatrix}$$

$$= V\begin{pmatrix} f & 0 & 0 \\ a & x & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -g & 0 \\ 0 & b & y \end{pmatrix} - V(a) - V(b)$$

$$= V\begin{pmatrix} f & 0 \\ a & x \end{pmatrix} + V\begin{pmatrix} g & 0 \\ b & y \end{pmatrix} - V(a) - V(b)$$

$$= v(f/a\setminus x) + v(g/b\setminus y).$$

Also

$$v(f/a\setminus x)^{-1} = v((0\quad 1)/\begin{pmatrix} a & x \\ f & 0 \end{pmatrix}\setminus\begin{pmatrix} 0 \\ -1 \end{pmatrix}) = -v(f/a\setminus x).$$

(ii) The function "v" is well defined:

First, note that $(f/a\setminus x) = (g/b\setminus y)$ is equivalent to $(f/a\setminus x)(g/b\setminus y)^{-1} = (1/1\setminus 1)$. Therefore, it is sufficient to prove that $v(f/a\setminus x) = 0$, for $(f/a\setminus x) = (1/1\setminus 1)$. By definition it means that $(f,a,x) \sim (1,1,1)$, thus $\begin{pmatrix} a & 0 & x \\ 0 & 1 & -1 \\ f & 1 & 0 \end{pmatrix} \in \wp$. This by MV.3 and MV.7 implies that the matrix $A = \begin{pmatrix} -1 & f & 0 \\ 0 & a & x \\ 1 & 0 & 1 \end{pmatrix}$ belongs to $\wp$, hence $V(A) = \infty$. But we have $A = \begin{pmatrix} 0 & f & 0 \\ 0 & a & x \\ 1 & 0 & 1 \end{pmatrix} \nabla \begin{pmatrix} -1 & f & 0 \\ 0 & a & x \\ 0 & 0 & 1 \end{pmatrix}$, therefore by MV.6

$$V\begin{pmatrix} 0 & f & 0 \\ 0 & a & x \\ 1 & 0 & 1 \end{pmatrix} = V\begin{pmatrix} -1 & f & 0 \\ 0 & a & x \\ 0 & 0 & 1 \end{pmatrix} \quad or \quad V\begin{pmatrix} f & 0 \\ a & x \end{pmatrix} = V(a).$$

Otherwise a $\in \wp$, which is a contradiction.

(iii) Additive property of "v":

First, note that if $v(f/a\setminus x) \geq 0$, then $v((f/a\setminus x)+(1/1\setminus 1)) \geq 0$. For, we have

$$v((f\quad 1)/\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\setminus\begin{pmatrix} x \\ 1 \end{pmatrix})$$

$$= V\begin{pmatrix} f & 1 & 0 \\ a & 0 & x \\ 0 & 1 & 1 \end{pmatrix} - V\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}.$$

And since

$$\begin{pmatrix} f & 0 & 1 \\ a & x & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} f & 0 & 0 \\ a & x & 0 \\ 0 & 1 & 1 \end{pmatrix} \nabla \begin{pmatrix} f & 0 & 1 \\ a & x & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

This by MV.2 implies that

$$V\begin{pmatrix} f & 1 & 0 \\ a & 0 & x \\ 0 & 1 & 1 \end{pmatrix} = V\begin{pmatrix} f & 0 & 1 \\ a & x & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\geq \min\{V\begin{pmatrix} f & 0 \\ a & x \end{pmatrix}, V(a)\} = V(a),$$

which admits the claim. Now, let $v(f/a\backslash x) \leq v(g/b\backslash y)$, then

$$\begin{aligned} & v((f/a\backslash x)+(g/b\backslash y)) \\ & = v((f/a\backslash x)((1/1\backslash 1)+(f/a\backslash x)^{-1}(g/b\backslash y))) \\ & = v(f/a\backslash x)+v((1/1\backslash 1)+(f/a\backslash x)^{-1}(g/b\backslash y)) \\ & \geq v(f/a\backslash x) \\ & = \min\{v(f/a\backslash x), v(g/b\backslash y)\}, \end{aligned}$$

and this completes the proof. □

Acknowledgements

The author is indebted to the Research Council of the University of Zanjan for Support.

References

1. Cohn P.M. *Free Rings and Their Relations.* Second edition, LMS Monographs No. 19, Academic Press, London, Orlando (1985).
2. Cohn P.M. The construction of valuations on skew fields. *Journal of the Indian Math. Soc.*, **54**: 1-45 (1989).
3. Cohn P.M. Universal field of fractions of a semifir. Proceeding of L.M.S. (1984).
4. Draxl P.K. Skew fields. Cambridge University Press, Lecture note series, 81 (1983).
5. Malcolmson P. A prime matrix ideal yields a skew fields. *J. London Math. Soc.*, **18**(2): 221-233 (1978).
6. Malcolmson P. Construction of universal matrix localizations. Advances in Non-Comm. Ring Theory, Lecture Notes in Math., No. 951, Springer, Berlin, 117-131 pp.(1982).
7. Mahdavi-Hezavehi M. Matrix valuation and their associated skew fields. *Resultate der Mathematik*, **5**: 149-156 (1982).
8. Mahdavi-Hezavehi M. Valuation on simple artinian rings. *Scientia Iraniaca*, **1**(4): 347-351 (1995).
9. Mahdavi-Hezavehi M. Matrix pseudo-valuations on rings and their associated skew fields. ICTP Technical Report, No. 140 (1990).
10. Mojdeh D. Noninvariant matrix valuations and their associated valued division rings. *Indian Journal of Mathematics*, **38**(1): 57-66 (1996).