MODULE HOMOMORPHISMS ASSOCIATED WITH
HYPERGROUP ALGEBRAS

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Abstract
Let X be a hypergroup. In this paper, we study the homomorphisms on certain subspaces of L(X)* which are weak*-weak* continuous.

Keywords: Homomorphisms; Hypergroup algebras; Weak*-weak* continuous

1. Introduction and Notations
The theory of hypergroups was initiated by Dunkl [3], Jewett [7] and has received a good deal of attention from harmonic analysts. It is still unknown whether an arbitrary hypergroup admits a left Haar measure (for more information see [2]). The lack of the Haar measure and involution presents many difficulties, however, we succeed to get some interesting results. Let X be a hypergroup (for more information see [3] or [10]) with convolution measure algebra M(X) and probability measures Mp(X). Recall that L(X) denotes the set of all measures μ∈M(X) for which the mapping x→lμlδx is norm-continuous [6,10]. We assume that X is foundation, i.e. U{supp(μ); μ∈L(X)} is dense in X. It is well known that L(X) is an ideal in M(X) and L(X) has a positive bounded approximate identity bounded by 1 ([6], Lemma 1).

The first Arens product on L(X)** is defined in three steps as follows. For μ,ν∈L(X)*, f in L(X)* and F,G in L(X)**, the elements fμ, Ff of L(X)* and GF of L(X)** are defined by <fμ,ν>=<f,μ*ν>, <Gf,μ*>=<G,μf*> and <FG, f>=<F, Gf>.

2. Main Results
Let B=L(X)**L(X), we know that B is a Banach subspace of L(X)*. The formulas which define the first Arens product in L(X)** can also be used to define a Banach algebra structure on B* [10]. Finally, for every μ∈L(X), ν∈M(X) and f∈L(X)*, we define <vf,μ> =<f,μ*ν> and <fv,μ>=<f,v*μ>, so that M(X)⊆B*. Also, we define <mf,ν>=<m,fv> for any m∈B*, f∈L(X)* and ν∈L(X). Most of our notation in this paper coming from [6,10]. In this paper, we will characterize some homomorphisms which are weak*-weak* continuous (see below).

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2. Main Results

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Let $A$ be a Banach algebra with a bounded approximate identity. It is well known that $A^*$ and $(A^*)^*$ with the first Arens product are Banach algebras [1]. In addition, we define $\langle n, x \rangle = n\langle x \rangle$ for $n \in (A^*)^*$, $\epsilon \in A^*$ and $\epsilon \in A$.

We recall that multiplication in a locally convex algebra $A$ is said to be hypcontinuous, if for every neighbourhood $U$ of zero in $A$ and a bounded subset $C$ of $A$, there exists a neighbourhood $V$ of zero such that $CV \subset UC \subset U$. The following Lemma shows that if multiplication in a Banach algebra $A$ with a bounded approximate identity, is hypcontinuous in the weak-topology, then $A^\ast$ factors on the left, i.e. $A^\ast A = A^\ast$ [9].

**Lemma 2.5.** Let $A$ be a Banach algebra with a bounded approximate identity, and let the multiplication with weak-topology on $A$ be hypcontinuous. Then $A^\ast$ factors on the left.

**Proof.** Let $h \in A^\ast$ and $B_i$ be unit ball in $A$. By assumption, weak-topology on $A$ is hypcontinuous. Therefore there exists a finite subset $\{f_1, f_2, ..., f_n\}$ in $A^\ast$ and $\epsilon_0 > 0$ such that $B_i$ is $\epsilon$-open in $A^\ast$ and $\epsilon_0 < \epsilon$ for any $\epsilon \in \{1, 2, ..., n\}$, $a \in A$, $\langle x, a \rangle > 0$ and $a \in A$. Now, let $a \in A$ and $\langle x, a \rangle > 0$ for all $i \in \{1, 2, ..., n\}$. For $b \in B_i$, we have $\langle b, a \rangle = 0$, and so $hA \cap \{f_1, f_2, ..., f_n\} = \emptyset$. By ([11], Theorem 1.2), $hA$ is a closed subspace of $A^\ast$. On the other hand, if $e_\ast$ is a bounded approximate identity in $A$, then $e_\ast h = h$ is well defined and $hA = hA$.

**Theorem 2.6.** Assume $X$ is such that weak-topology on $L(X)$ is hypcontinuous. Let $T : L(X) \to L(X)$ be a bounded linear map such that $T(f \delta) = T(f) \delta$ for all $f \in L(X)$ and $\|x\| = \|x\|_1$. Then $T \in \text{Hom}_{\text{hyp}}(L(X), L(X))$.

**Proof.** Let $h \in A^\ast$ and $B_i$ be unit ball in $A$. By assumption, weak-topology on $A$ is hypcontinuous. Therefore there exists a finite subset $\{f_1, f_2, ..., f_n\}$ in $A^\ast$ and $\epsilon_0 > 0$ such that $B_i$ is $\epsilon$-open in $A^\ast$ and $\epsilon_0 < \epsilon$ for any $\epsilon \in \{1, 2, ..., n\}$, $a \in A$, $\langle x, a \rangle > 0$ and $a \in A$. Now, let $a \in A$ and $\langle x, a \rangle = 0$ for all $i \in \{1, 2, ..., n\}$. For $b \in B_i$, we have $\langle b, a \rangle = 0$, and so $hA \cap \{f_1, f_2, ..., f_n\} = \emptyset$. By ([11], Theorem 1.2), $hA$ is a closed subspace of $A^\ast$. On the other hand, if $e_\ast$ is a bounded approximate identity in $A$, then $e_\ast h = h$ is well defined and $hA = hA$.
Theorem 2.7. Let A be a Banach algebra with a bounded approximate identity bounded by 1, and T ∈ Homₐₐ(ₐₐ,ₐₐ). The following statements are equivalent:

1) There exists a n ∈ (ₐₐ) such that an ∈ A for all a ∈ A, and T = Tₙ.
2) T is weak⁎-weak* continuous.

Proof. Let T = Tₙ and an ∈ A for any a ∈ A. Let (fₙ) be a net in A such that fₙ → f (f ∈ A') in the weak*-topology. For a ∈ A, we have <an,fₙ> → <an,f> and so <Tₙ(fₙ),a> → <Tₙ(f),a> which shows that T is weak⁎-weak* continuous.

To prove the converse, let T ∈ Homₐₐ(ₐₐ,ₐₐ). By ([1], Theorem 1.1), there exists a n ∈ (ₐₐ) such that T = Tₙ. Now, let a ∈ A. By assumption, T is weak⁎-weak* continuous and so Tₙ(a) ∈ A is weak⁎-continuous. It follows that Tₙ(ₐ) ∈ A ([11], Chapter 3). On the other hand, <Tₙ(a),f> = <ca,Tₙ(f)> = <an,f> where f ∈ A', i.e. Tₙ(a) = an. Consequently an ∈ A for any a ∈ A.

This completes our proof.

Corollary 2.8. Let A be a Banach algebra with a bounded approximate identity bounded by 1. If all operators T in Homₐₐ(ₐₐ,ₐₐ) are weak⁎-weak* continuous, then (ₐₐ) = Z where Zₙ = {n ∈ (ₐₐ)}; the mapping m → nm is weak⁎-weak* continuous.

Proof. Suppose all operators T in Homₐₐ(ₐₐ,ₐₐ) are weak⁎-weak* continuous, and let n ∈ (ₐₐ). Then Tₙ ∈ Homₐₐ(ₐₐ,ₐₐ) is weak⁎-weak* continuous. By Theorem 2.7, an ∈ A for any a ∈ A. A standard argument using the Cohen-Hewitt factorization Theorem shows that AA = A. Now, let m → m in the weak⁎-topology, and let f → B. There exist g ∈ B and a ∈ A such that f = ga. Therefore <nmₙ,f> = <nmₙ,g> = <mₙ,g> and <nmₙ,f> = <m,gan>. This shows that <nmₙ,f> → <m,gan>, i.e. n ∈ Zₙ. Consequently Zₙ = (ₐₐ).

Corollary 2.9. Let G be a locally compact group. Then all operators T in Homₐₐ(L¹(G), L¹(G)) are weak⁎-weak* continuous if and only if G is compact.

Proof. By ([8], Theorem 1), we have Zₙ(LUC(G)) = M(G). On the other hand, LUC(G) = M(G) ⊕ C(G) (5), Lemma 1.1. The results follows from Corollary 2.8.

For some Banach algebras A, the subspace {n ∈ (ₐₐ)} of B has been studied by Lau and Ulger in [9]. In the following Theorem we will study {n ∈ B, L(X)n⊂L(X)} = M(X).

Theorem 2.10. Let X be a hypergroup. Then {n ∈ B, L(X)n⊂L(X)} = M(X).

Proof. Since L(X) is an ideal in M(X), we have M(X)n⊂M(X) = M(X). For the reverse inclusion, let n ∈ B; L(X)n⊂L(X). So the mapping v → vn from L(X) into L(X) is a right multiplier. By ([6], Proposition 1), there exists a measure μ in M(X) such that vμ = vn for any v ∈ L(X). Now, let (eₙ) be a bounded approximate identity in L(X) and eB. Then <eₙμ,f> = n,f> (for all α) implies <μ,f> = <eₙ,f>, i.e. μ = n. This completes our proof.

Corollary 2.11. Assume X is such that Zₙ(B) = M(X). Then L(X) is an ideal in B' if and only if X is compact.

Proof. Let L(X) be an ideal in B', and let n ∈ B'. It is easy to see that the operator Tₙ is weak⁎-weak* continuous. Consequently by Corollary 2.8, B' = Zₙ(B) = M(X). But B' = M(X) ⊕ C(X) (11), Theorem 4), and so C(X) = {0}, i.e. X is compact.

To prove the converse, let X be compact. Then B' = M(X), and so the operator Tₙ(n ∊ B') is weak⁎-weak* continuous. Theorem 2.7 shows that L(X) is a right ideal in B'. On the other hand, by definition X is commutative [3,6,10], so that L(X) is an ideal in B'.

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References