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TOPOLOGICALLY STATIONARY LOCALLY COMPACT SEMIGROUP AND AMENABILITY

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Abstract

In this paper, we investigate the concept of topological stationary for locally compact semigroups. In [4], T. Mitchell proved that a semigroup S is right stationary if and only if $m(S)$ has a left Invariant mean. In this case, the set of values $\mu(f)$ where μ runs over all left invariant means on $m(S)$ coincides with the set of constants in the weak* closed convex hull of right translates of f. The main purpose of this paper is to prove a topological analogue (which is also a generalization) of this theorem for locally compact semigroups.

Keywords: Topological stationary; Topological left invariant mean

1. Introduction

Let S be a locally compact Hausdorff semigroup. Let $CB(S)$ be the algebra of all continuous functions on S and $C_0(S)$ be the subalgebra of $CB(S)$ consisting of functions which vanish at infinity. Let $M(S)$ be the Banach space subalgebra of all bounded regular Borel (Signed) measures on S with total variation norm. Let

$$M_0(S) = \{\mu \in M(S) : \mu \geq 0, \|\mu\| = 1\}$$

be the set of all probability measures in $M(S)$.

It is known that $M(S) = C_0(S)^*$ via the correspondence $\mu \to \overline{\mu}$ where $\overline{\mu}(f) = \int f d\mu$ for any f in $C_0(S)$, [3, Sec. 14]. Consider the continuous dual $M(S)^*$ of $M(S)$. Denote by 1, the element 1 in $M(S)^*$ such that $1(\mu) = \mu(S)$ for any μ in $M(S)$. Also if T is a Borel subset of S, we define the *characteristic functional* χ_T of T in $M(S)^*$ by $\chi_T(\mu) = \mu(T)$ for any μ in $M(S)$.

Let X be a linear subspace of $M(S)^*$ containing 1. An element M in X^* is called a *mean* on X if $M(1) = 1$ and $M(F) \geq 0$, whenever $F \geq 0$ as a functional in $M(S)^*$, i.e., $F(\mu) \geq 0$ for all $\mu \geq 0$. An equivalent definition for a mean is that

$$\inf\{F(\mu) : \mu \in M_0(S)\} \leq M(F) \leq \sup\{F(\mu) : \mu \in M_0(S)\}$$

for any F in X. Also $M \in X^*$ is a mean if and only if $\|M\| = M(1) = 1$. The set of all means on X is a weak* compact convex subset of X^*. Each probability measure μ in $M_0(S)$ is a mean on X if we put $\mu(F) = F(\mu)$, for any F in X. An application of

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Hahn-Banach separation Theorem shows that $M_0(S)$ is weak* dense in the set of all means on X.

For $F \in M(S)^*$ and $\mu \in M(S)$, define $l_\mu F \in M(S)^*$ by

$$(l_\mu F)\nu = (\mu \odot F)\nu = F(\mu * \nu), \quad \nu \in M(S)$$

and define $r_\mu F \in M(S)^*$ by

$$(r_\mu F)\nu = (F \odot \mu)\nu = F(\nu * \mu), \quad \nu \in M(S)$$

For $M \in M(S)^{**}$ and $F \in M(S)^*$ define $M \odot F \in M(S)^*$ by

$$(M \odot F)\ \mu = M(F \odot \mu)\ , \quad \mu \in M(S)$$

and for M, $N \in M(S)^{**}$ define $M \odot N \in M(S)^{**}$ by

$$(M \odot N)F = M(N \odot F), \quad F \in M(S)^*$$

see [1] for details.

For $s \in S$, ε_s denotes the Dirac measure at s. The convolutions $\varepsilon_s * \mu$ and $\mu * \varepsilon_s$ are defined for all f in $C_0(S)$ as following

$$\int f d\varepsilon_S * \mu = \int\int f(xy) d\varepsilon_S(x) d\mu(y)$$
$$= \int f(sy) d\mu(y) = \int (l_S f)(y) d\mu(y)$$

and

$$\int f d\mu * \varepsilon_S = \int\int f(xy) d\mu(x)\, d\varepsilon_S(y)$$
$$= \int f(xs) d\mu(x) = \int (r_S f)(x) d\mu(x)$$

We denote the natural isometric embedding of $M(S)$ into $M(S)^{**}$ by Q.

$\mathcal{L}[\mathcal{R}]$ is the set of all left [right] translations of $M(S)^*$ by elements of S (i.e., $l_{\varepsilon_s} F = \varepsilon_s \odot F\left[r_{\varepsilon_s} F = F \odot \varepsilon_s\right]$ for each $s \in S$ and $F \in M(S)^*$).

We denote $\Lambda = \mathrm{Co}(\mathcal{L}) =$ convex hull of $\mathcal{L}$, and $\mathcal{B} = \mathrm{Co}(\mathcal{R})$. For $F \in M(S)^*$, $\mathfrak{Z}_{\mathcal{R}}(F) \subseteq M(S)^*$ $[\mathfrak{Z}_{\mathcal{L}}(F) \subseteq M(S)^*]$ is given by

$$\mathfrak{Z}_{\mathcal{R}}(F) = w^* - cl(Co(\mathcal{R} F)) = w^* - cl(\mathcal{B} F)$$
$$= w^* - cl\{r_\mu F : \mu \in M_0(S)\}$$

$$\mathfrak{Z}_{\mathcal{L}}(F) = w^* - cl(Co(\mathcal{L} F)) = w^* - cl(\Lambda f)$$
$$= w^* - cl\{l_\mu F : \mu \in M_0(S)\}$$

$\mathfrak{R}_{\mathcal{R}}(F) = \{a: a \text{ is real}, a.1 \in \mathfrak{Z}_{\mathcal{R}}(F)\}$

$\mathfrak{R}_{\mathcal{L}}(F) = \{a: a \text{ is real}, a.1 \in \mathfrak{Z}_{\mathcal{L}}(F)\}$

REMARK. If $a \in \mathfrak{R}_{\mathcal{R}}(F)$ then there is a net $\{\mu_\alpha\}$ in $M_0(S)$ such that $\{r_{\mu_\alpha} F\}$ converges weak* to $a.1$. Similarly if $a \in \mathfrak{R}_{\mathcal{L}}(F)$ then is a net $\{\mu_\alpha\}$ in $M_0(S)$ such that $\{l_{\mu_\alpha} F\}$ converges weak* to $a.1$.

LEMMA 1.1. a) If M, N are means on $M(S)^*$, so is $M \odot N$;

b) If $\mu \in M(S), M, N \in M(S)^{**}$, Then $Q\mu \odot M = r_\mu^* M$ and $M \odot Q\mu = l_\mu^* M$;

c) For fixed μ in $M(S)$, $Q\mu \odot M$ is w^*-w^* continuous in the second variable and $M \odot Q\mu$ is w^*-w^* continuous in the first variable, for each $M \in M(S)^{**}$;

d) For fixed $M \in M(S)^{**}$, The map $N \to N \odot M$ is w^*-w^* continuous;

e) $Q: M(S) \to M(S)^{**}$ is an isomorphism of the algebra $M(S)$ into $M(S)^{**}$, i.e., $Q\mu \odot Q\nu = Q(\mu * \nu)$, for any μ, ν in $M(S)$;

f) If M is topological left invariant (i.e., for each $\mu \in M_0(S)$ and $F \in M(S)^*$, $M(\mu \odot F) = M(F)$) and N is a mean on $M(S)^*$, then $N \odot M = M$.

Proof. (a), (b), (c) and (d) are obvious [2, Sec. 2, (B)]. We know that Q is isometry of $M(S)$ into $M(S)^{**}$ and also is linear. For F in $M(S)^*$, we have

$$(Q\mu \odot Q\nu)(F) = (l_\mu^* Q\nu)(F)$$
$$= Q\nu(l_\mu F)$$
$$= Q\nu(\mu \odot F)$$
$$= (\mu \odot F)(\nu)$$
$$= F(\mu * \nu)$$
$$= Q(\mu * \nu)(F)$$

Thus $Q\mu \odot Q\nu = Q(\mu * \nu)$, which (e) is proved. Now, by weak* density of the set $M_0(S)$ in the set of means on $M(S)^*$, there is a net $\{\mu_\alpha\}$ in $M_0(S)$ such that $\mu_\alpha \to N$ in weak* topology of $M(S)^{**}$ (we consider μ_α as a mean, $\mu_\alpha(F) = F(\mu_\alpha)$ for each

$F \in M(S)^*$). Then by (d), $\mu_\alpha \odot M \to N \odot M$ weak*. Now

$$\mu_\alpha \odot M = l^*_{\mu_\alpha} M = M$$

Since M is topological left invariant, hence $N \odot M = M$ which (f) is proved.

2. Topological Stationary Semigroups

T. Mitchell [4] proved that a semigroup S is right stationary if and only if $m(S)$ has a left invariant mean. In this section we investigate the concept of topological stationary for locally compact semigroups and we present topological analogue of results of T. Mitchell.

DEFINITION 2.1. Let S be a locally compact semigroup. S is called *topological right stationary* [*topological left stationary*] whenever $\mathfrak{R}_{\mathcal{R}}(F)[\mathfrak{R}_{\mathcal{L}}(F)]$ is nonempty, for all F in $M(S)^*$.

REMARKS. a) If $\mathfrak{R}_{\mathcal{R}}(F)$ is nonempty, then there exists a net $\{\mu_\alpha\}$ in $M_0(S)$ such that $\{r_{\mu_\alpha} F\}$ converges weak* to a constant functional in $M(S)^*$ for each $F \in M(S)^*$. Similarly if $\mathfrak{R}_{\mathcal{L}}(F)$ is nonempty, then there is a net $\{\mu_\alpha\}$ in $M_0(S)$ such that $\{l_{\mu_\alpha} F\}$ converges weak* to a constant functional in $M(S)^*$ for each $F \in M(S)^*$.

b) Definition 2.1 is a topological analogue as well as an extension of the definition of T. Mitchell [4] for discrete semigroups.

DEFINITION 2.2. For each M in $M(S)^{**}$, define a mapping $M_R : M(S)^* \to M(S)^*$ by $(M_R(F))(\mu) = M(F \odot \mu)$ for any $F \in M(S)^*$ and $\mu \in M(S)$. The operator M_R is called the *topological right introversion* of M. Similarly the *topological left introversion* $M_L : M(S)^* \to M(S)^*$ is defined by $(M_L(F))(\mu) = M(\mu \odot F)$ for any $F \in M(S)^*$ and $\mu \in M(S)$.

LEMMA 2.3. a) $M_R : M(S)^* \longrightarrow M(S)^*$ is bounded and linear. Moreover $\| M_R(F) \| \le \| M \| \| F \|$ and $M_R(F) = M \odot F$ for any $F \in M(S)^*$.

b) If $M \in M(S)^{**}$, $F \in M(S)^*$, μ, $\nu \in M(S)$, then $M_L(\mu \odot F) = \mu \odot M_L(F)$.

c) For M, $N \in M(S)^{**}$, $M \odot N$ is topological left invariant if M is topological left invariant.

d) If $M_\alpha \to M$ in norm topology of $M(S)^{**}$, then $(M_\alpha)_R \to M_R$ in uniform operator topology.

Proof. a) Clearly M_R is linear. For any $F \in M(S)^*$, μ, $\nu \in M(S)$, we have

$$\begin{aligned}|(M_R(F))(\mu)| &= |M(F) \odot \mu| \\ &\le \| M \| \| F \odot \mu \| \\ |(F \odot \mu)(\nu)| &= |F(\mu * \nu)| \\ &\le \| F \| \| \mu * \nu \| \\ &\le \| F \| \| \mu \| \| \nu \|\end{aligned}$$

Thus $\| F \odot \mu \| \le \| F \| \| \mu \|$, and so

$$|(M_R(F))(\mu)| \le \| M \| \| F \| \| \mu \|$$

Hence $\| M_R(F) \| \le \| M \| \| F \|$. Also

$$\begin{aligned}(M_R(F))(\mu) &= M(F \odot \mu) \\ &= (M \odot F)(\mu)\end{aligned}$$

thus $M_R(F) = M \odot F$.

b) If $\mu \in M(S)$, and $F \in M(S)^*$, we have

$$\begin{aligned}(M_L(\mu \odot F))(\nu) &= M(\nu \odot (\mu \odot F)) \\ &= M((\mu * \nu) \odot F) \\ &= (M_L(F))(\mu * \nu) \\ &= (\mu \odot M_L(F))(\nu)\end{aligned}$$

So $M_L(\mu \odot F) = \mu \odot M_L(F)$.

c) Let M be topological left invariant, then for each $\mu \in M_0(S)$ and $F \in M(S)^*$ we have

$$\begin{aligned}l^*_\mu(M \odot N)(F) &= ((M \odot N) \odot \mu)(F) \\ &= ((M \odot N)(\mu \odot F) \\ &= M(N_L(\mu \odot F)) \\ &= M(\mu \odot N_L(F)) \\ &= M(N_L(F)) \\ &= (M \odot N)(F)\end{aligned}$$

d) First note that

$$[((M_\alpha)_R - M_R)(F)](\mu)$$

$$= ((M_\alpha)_R(F))(\mu) - (M_R(F))(\mu)$$
$$= M_\alpha(F \odot \mu) - M(F \odot \mu)$$
$$= (M_\alpha - M)(F \odot \mu)$$
$$= ((M_\alpha - M)_R(F))(\mu)$$

So $(M_\alpha)_R - M_R = (M_\alpha - M)_R$ and by (a) we have,

$$\| ((M_\alpha)_R - M_R)(F) \| = \| (M_\alpha - M)_R(F) \|$$
$$\leq \| M_\alpha - M \| \| F \|$$

now if $M_\alpha \to M$ in norm topology of $M(S)^{**}$ then $(M_\alpha)_R \to M_R$ in uniform operator topology.

LEMMA 2.4. a) $M_L : M(S)^* \longrightarrow M(S)^*$ is bounded and linear. Moreover $\| M_L(F) \| \leq \| M \| \| F \|$ and $M_L(F) = F \odot M$ for any $F \in M(S)^*$;

b) If $M \in M(S)^{**}$, $F \in M(S)^*$, μ, $\nu \in M(S)$, then $M_R(F \odot \mu) = M_R(F) \odot \mu$;

c) For $M, N \in M(S)^{**}$, $M \odot N$ is topological right invariant if N is topological right invariant;

d) If $M_\alpha \to M$ in norm topology of $M(S)^{**}$, then $(M_\alpha)_L \to M_L$ in uniform operator topology.

Proof. Similar to the proof of Lemma 2.3.

DEFINITION 2.5. A linear subspace X of $M(S)^*$ is said to be *topological left* [*right*] *introverted*, if for any mean M on $M(S)^*$, $M_L(X) \subseteq X [M_R(X) \subseteq X]$.

THEOREM 2.6. Let X be a topological left intorverted and topological left invariant linear subspace of $M(S)^*$ containing the constants. Then the following statements are equivalent:

a) X has a topological left invariant mean.

b) For any $F \in X$, there is a mean M on X such that

$$M(\mu \odot F) = M(F) \quad \text{for every} \quad \mu \in M_0(S)$$

Proof. (a) $\Rightarrow$ (b), is clear.

(b) $\Rightarrow$ (a), For each $F \in X$, define $\Re_F = \{ M : M$ is a mean on $M(S)^*$, $M(\mu \odot F) = M(F)$ for any $\mu \in M_0(S) \}$

By assumption $\Re_F$ is nonempty. (BY Hahn-Banach Theorem any mean on X can be extended to a mean on $M(S)^*$. We show that the family $\{\Re_F : F \in X\}$ has the finite intersection property. When n=1, by assumption $\Re_{F_1}$ is nonempty. Assume $\cap_{i=1}^{n-1} \Re_{F_i}$ is nonempty. And let $M \in \cap_{i=1}^{n-1} \Re_{F_i}$. Let $F_1, ..., F_n \in X$, since X is topological left introverted, $M_L(F_n) \in X$. Put $F = M_L(F_n)$. For this $F \in X$ there is a mean $N \in \Re_F$ on X such that $N(\mu \odot F) = N(F)$ for each $\mu \in M_0(S)$. By Lemma 1.1 (a), $N \odot M$ is a mean on X. We show that $N \odot M \in \cap_{i=1}^{n} \Re_F$.

For $1 \leq i \leq n-1$ and for each $\mu \in M_0(S)$, we have

$$(M_L(F_i))(\mu) = M(\mu \odot F_i) = M(F_i), \quad (M \in \Re_{F_i})$$

therefore for each $\mu \in M_0(S)$

$$(M_L(F_i))(\mu) = (M(F_i).1)(\mu)$$

Hence

$$M_L(F_i) = M(F_i).1$$

and it follows that for $\mu \in M_0(S)$,

$$(N \odot M)(\mu \odot F_i) = N(M_L(\mu \odot F_i))$$
$$= N(\mu \odot M_L(F_i)) \quad \text{(Lemma 2.3 (b))}$$
$$= N(\mu \odot (M(F_i).1))$$
$$= N(M(F_i).1)$$
$$= N(M_L(F_i))$$
$$= (N \odot M)(F_i)$$

Now, if $\mu \in M_0(S)$, then

$$(N \odot M)(\mu \odot F_n) = N(M_L(\mu \odot F_n))$$
$$= N(\mu \odot M_L(F_n))$$
$$= N(\mu \odot F)$$
$$= N(F)$$
$$= N(M_L(F_n))$$
$$= (N \odot M)(F_n)$$

consequently $N \odot M \in \cap_{i=1}^{n} \Re_{F_i}$. By weak* compactness of the unit ball in $M(S)^{**}$ and the fact that $\Re_F$ is weak* closed subset of the unit ball of $M(S)^{**}$, it follows that $\cap\{\Re_F : F \in X\}$ is nonempty. Since X is topological left invariant linear subspace of $M(S)^*$, any mean in this intersection is a topological left invariant mean on X.

Remark. There is a different proof for the above theorem, when $X = M(S)^*$ as following [6].

***Necessity*:** It is enough to show that the existence of such M_F, for each $F \in M(S)^*$ implies that for any μ, $\mu_2 \in M_0(S)$, $d((\mu_1 - \mu_2 * M_0(S)), 0) = 0$, where $d(\mu_1 - \mu_2 * M_0(S),$ $0) = \inf\{\|(\mu_1-\mu_2)*\mu\| : \mu \in M_0(S)\}$ [6, Proposition 2.11, p. 488]. Fix μ_1, μ_2 in $M_0(S)$ and let $X = (\mu_1 - \mu_2)\odot M(S)^* = \{(\mu_1-\mu_2)\odot F : F \in M(S)^*\}$, then X is a subspace of $M(S)^*$. For any $F \in M(S)^*$, we must have $\inf\{((\mu_1-\mu_2)\odot F)(\nu) : \nu \in M_0(S)\} \le 0$. Hence there is a mean M in $M(S)^{**}$ depending on X such that $M(F) = 0$ for any $F \in X$ (apply Theorem 2.12 in [6] to X). Since $M_0(S)$ is weak*-dense in the set of means of $M(S)^*$, there is a net $\{\mu_\alpha\}$ in $M_0(S)$ such that $\mu_\alpha \to M$ weak* in $M(S)^{**}$. In particular, for any $F \in M(S)^*$, if $G = (\mu_1 - \mu_2)\odot F$ (Note that G is in X), then $F((\mu_1-\mu_2)*\mu_\alpha) = G(\mu_\alpha) \to M(G) = 0$. That is, 0 is in the weak closure of $(\mu_1-\mu_2)*M_0(S)$ which is equivalent to the part (c)$\Rightarrow$(a) of Theorem 2.12 in [6], hence $M(S)^*$ has a TLIM.

***Sufficiency*:** Obvious.

REMARK. For discrete groups, this Theorem is due to E. Granirer and A.T.M. Lau [5] and the first proof follows the idea in [5].

THEOREM 2.7. Let X be a topological left introverted topological left invariant linear subspace of $M(S)^*$ containing the constants. The following statements are equivalent:

a) X has a topological left invariant mean,

b) X is topologically right stationary,

c) For any $F \in X$ and $a \in \Re_{\mathcal{R}}(F)$, there is a topological left invariant mean M on X such that $M(F) = a$.

Proof. By Definition 2.1, (c) is equivalent to (b).

(a)$\Rightarrow$(b). Assume that X has a topological left invariant mean M. Then there is a net $\{\mu_\alpha\}$ in $M_0(S)$ such that $\mu_\alpha \to M$ in weak* topology of X^*. Let $F \in X$, then

$$(F\odot\mu_\alpha)(\mu) = (\mu\odot F)(\mu_\alpha) = \mu_\alpha(\mu\odot F)$$

$$\to M(\mu\odot F) = (M_L(F))(\mu) = (M(F).1)(\mu)$$

for any $\mu \in M_0(S)$, Hence $\{F\odot\mu_\alpha\}$ converges to the constant functional $M(F).\ 1$ in $M(S)^*$. That is $M(F).\ 1 \in \mathfrak{Z}_{\Re}(F)$ and so $M(F) \in \Re_{\mathcal{R}}(F)$. Hence $\Re_{\mathcal{R}}(F)$ is nonempty, so X is topological right stationary, (b) $\Rightarrow$ (a), For $F \in X$, by definition of $\Re_{\mathcal{R}}(F)$, there is a net $\{\mu_\alpha\}$ in $M_0(S)$ such that $\{F\odot\mu_\alpha\}$ converges weak* to $a.1$ in $M(S)^*$. Without loss of generality, we can assume that $\{\mu_\alpha\}$ converges weak* to some M in $M(S)^{**}$ by weak* compactness of the set of means in $M(S)^{**}$. Consider the mean $M\odot M$ on $M(S)^*$. We show that $M\odot M \in \Re_F$.

For any $\mu \in M(S)$

$$\begin{aligned}(M_L(F))(\mu) &= M(\mu\odot F)\\ &= \lim_\alpha \mu_\alpha(\mu\odot F)\\ &= \lim_\alpha (F\odot\mu_\alpha)(\mu)\\ &= (a.1)(\mu).\end{aligned}$$

Hence $M_L(F) = a.1$. For each $\mu \in M_0(S)$

$$\begin{aligned}(M\odot M)(\mu\odot F) &= M(M_L(\mu\odot F))\\ &= M(\mu\odot M_L(F))\\ &= M(\mu\odot(a.1))\\ &= M(a.1))\\ &= M(M_L(F))\\ &= (M\odot M)(F)\end{aligned}$$

thus $M\odot M \in \Re_F$. So we have proved that for each $F \in X$, $\Re_F$ is nonempty. By Theorem 2.6, X has a topological left invariant mean N in $\cap\{\Re_F : F \in X\}$. Consider $N\odot M$, then

$$\begin{aligned}(N\odot M)(F) &= N(M_L(F))\\ &= N(a.1)\end{aligned}$$

$$= a$$

Since N is a topological left invariant mean, so by Lemma 2.3 (c), $N \odot M$ is topological left invariant mean on X.

COROLLARY 2.8. If S is topological right stationary then S is topological left amenable (i.e., $M(S)^*$ has a TLIM).

Proof. Assume S is topological right stationary, so for each $F \in M(S)^*$, $\mathfrak{K}_{\mathcal{R}}(F)$ is nonempty, say $a \in \mathfrak{K}_{\mathcal{R}}.(F)$ Hence there is a topological left invariant mean M on $M(S)^*$ such that $M(F) = a$. By Theorem 2.7, (c)$\Rightarrow$(a), $M(S)^*$ has a topological left invariant mean.

THEOREM 2.9. Let S be a locally compact semigroup, F_0 an arbitrary element of $M(S)^*$, $a \in \mathbb{R}$ and M a topological right invariant mean on $M(S)^*$. If $M(F_0) = a$ then there exists a net $\{\mu_\alpha\}$ of elements of $M_0(S)$ such that:

(a) For any $F \in M(S)^*$, the net $\{r_{\mu_\alpha} F\}$ converges pointwise to a constant functional,

(b) The net $\{r_{\mu_\alpha} F_0\}$ converges pointwise to $a.1$.

Proof. $M_0(S)$ is weak* dense in the set of means on $M(S)^*$. So there exists a net $\{\mu_\alpha\}$ in $M_0(S)$ such that $\{\mu_\alpha\}$ is weak* convergent to M. For any $F \in M(S)^*$

$$\lim_\alpha((\mu_\alpha)_r(F))(\mu) = \lim_\alpha \mu_\alpha(F \odot \mu)$$
$$= M(F \odot \mu)$$
$$= M(F)$$

Hence $\{(\mu_\alpha)_r(F)\}$ converges pointwise to the constant functional $N.1$ where $N = M(F)$. On the other hand,

$$((\mu_\alpha)_r(F))(\mu) = \mu_\alpha(F \odot \mu)$$
$$= (F \odot \mu_\alpha)(\mu)$$
$$= (r_{\mu_\alpha} F)(\mu)$$

hence $\{r_{\mu_\alpha} F\}$ converges pointwise to constant functional $N.1$, this proves (a). Now since $M(F_0) = a$, then $\{r_{\mu_\alpha} F_0\}$ is the required net in (b).

THEOREM 2.10. Let S be a locally compact semigroup, then the following are equivalent:

a) For every $F \in M(S)^*$, there exists a net $\{\mu_\alpha\}$ in $M_0(S)$ such that $\{r_{\mu_\alpha} F_0\}$ converges pointwise to a constant functional.

b) S is topological left amenable.

c) there exists a net $\{v_\alpha\}$ in $M_0(S)$, such that $\{r_{v_\alpha} F\}$ converges pointwise to a constant functional for each $F \in M(S)^*$.

Proof. (a)$\Rightarrow$(b). By [4, Lemma 3], $\{r_{\mu_\alpha} F\}$ converges weak* to a constant functional. Thus $\mathfrak{K}_{\mathcal{R}}(F)$ is nonempty. Hence S is topological right stationary. Since $M(S)^*$ is left introverted (topological) linear subspace of itself, so by Theorem 2.7, $M(S)^*$ has a topological left invariant mean.

(b)$\Rightarrow$(c). Follows from Theorem 2.9 (a)

(c)$\Rightarrow$(a). Condition (c) is formally stronger than (a).

THEOREM 2.11. Let S be a locally compact semigroup, which is topological left amenable. Let F_0 be an arbitrary element of $M(S)^*$ and a be an arbitrary real number then the following conditions are equivalent:

(a) there exists a net $\{\mu_\alpha\}$ in $M_0(S)$ such that $\{r_{\mu_\alpha} F_0\}$ converges pointwise to $a.1$.

(b) there exists a topological left invariant mean M on $M(S)^*$ such that $M(F_0) = a$.

Proof. (a)$\Rightarrow$(b). By Theorem 2.10, (a) implies (b), $M(S)^*$ has a topological left invariant mean. Also by Theorem 2.7, there exists a topological left invariant mean M on $M(S)^*$ such that $M(F_0) = a$.

(b)$\Rightarrow$(a). By Theorem 2.7, (c) implies (a), S is topological left amenable, and by Theorem 2.9 (b), there exists a net $\{\mu_\alpha\}$ in $M_0(S)$ such that $\{r_{\mu_\alpha} F_0\}$ converges pointwise to $a.1$.

References

1. Arens R.F. The adjoint of a bilinear operator. *Proc. Amer. Math. Soc.*, **2**: 839-848 (1951).
2. Day M.M. Amenable semigroups. *Illinions J. Math.*, **1**: 509-544 (1957).
3. Hewitt E. and Ross K.A. *Abstract Harmonic Analysis I.* Springer-Verlag, Berlin (1963).
4. Mitchell T. Constant functions and left invariant means on semigroups. *Trans. Amer. Math. Soc.*, **119**: 244-261 (1965).
5. Wong J.C.S. Topologically stationary locally compact groups and amenability. *Ibid.*, **144**: 351-363 (1969).
6. Wong J.C.S. and Riazi A. Characterisations of amenable locally compact semigroups. *Pacific J. Math.* **108**:(2):

479-496 (1983).