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Arens Regularity and Weak Amenability of Certain Matrix Algebras

R. Alizadeh and G.H. Esslamzadeh*

Faculty of Mathematics and Computer Science, Amirkabir University of Technology (Tehran Polytechnic), 424 Hafez Avenue, 15914 Tehran, Islamic Republic of Iran

Abstract

Motivated by an Arens regularity problem, we introduce the concepts of matrix Banach space and matrix Banach algebra. The notion of matrix normed space in the sense of Ruan is a special case of our matrix normed system. A matrix Banach algebra is a matrix Banach space with a completely contractive multiplication. We study the structure of matrix Banach spaces and matrix Banach algebras. Then we investigate Arens regularity and weak amenability of certain matrix algebras which are built on matrix Banach algebras. In particular we show that for such algebras both of Arens regularity and weak amenability problems can be reduced to the same problem for a considerably smaller algebra.

Keywords: Banach algebras; Weak amenability; Arens regularity

1. Introduction

Since Arens' original paper [1], Arens products have been used as an important tool in the study of Banach algebras and their duals. This subject is now well developed and there is a vast literature on it. A result of Civin and Yood [6, Theorem 6.1] implies that every operator algebras are Arens regular. Unital operator algebras are indeed complete operator spaces with a completely contractive multiplication [5]. On the other hand there are Arens regular Banach algebras which are not operator algebras; for example see [8,9]. These facts suggest the following question.

Which properties of operator algebras imply their Arens regularity?

The notion of weak amenability was introduced by Bade, Curtis and Dales [2]. Since then weak amenability of various known Banach algebras has been examined. In particular every C^*-algebra is weakly amenable [11]. Again one might ask, which properties of a C^*-algebra force it to be weakly amenable?

The above questions motivated us to consider matrix algebras over Banach algebras, which roughly speaking, look like operator algebras. Our approach is based on appropriate weakening of matrix norm conditions of operator spaces. Our conditions are strong enough so that the class of matrix Banach algebras behave nicely and weak enough to include algebras with non-operator norm such as l^1, in this class. Indeed we drop the assumption of contractivity of M_n module actions from the definition of matrix normed space in the sense of Ruan.

Although the abstract operator space theory starts with the notion of matrix normed space, but due to the quantization goal, it moves quickly to the special case of operator spaces; consequently less attention has been paid to general matrix normed spaces. Here we consider a class of matrix normed systems which contain the

* *E-mail: hesslam@aut.ac.ir*

class of matrix normed spaces as a proper subclass.

This paper is organized as follows. In Section 2 we introduce our terminology. In Section 3, we introduce matrix Banach spaces and investigate their structure. Various examples are also provided in this section. In the Section 4 we investigate weak amenability of the algebra of approximable matrices over commutative matrix Banach algebras. In Section 5, we identify the topological centers of the algebra of approximable matrices over a matrix Banach algebra.

2. Notations

Throughout all vector spaces and algebras are over $\mathbb{C}$, A is a Banach algebra, A-module means Banach A-bimodule and dual A-module means the dual X^* of an A-module X with its natural A-module structure.

Let X be a vector space. We denote the vector space of all $m\times n$ matrices on X by $M_{m\times n}(X)$ and $M_{n\times n}(X)$ by $M_n(X)$. In particular we denote $M_n(\mathbb{C})$ by M_n. Let $1\le i$, $j\le n$ and $x\in X$. We denote the $n\times n$ [resp. $\mathbb{N}\times\mathbb{N}$] matrix on X whose ij-th entry is x and all other entries are zero, by $x\otimes E_{ij}^n$ [resp. $x\otimes E_{ij}^\infty$]. If X is a unital algebra, then we denote $1\otimes E_{ij}^n$ [resp. $1\otimes E_{ij}^\infty$] by E_{ij}^n [resp. E_{ij}^∞]. Given $B\in M_{m\times n}(X)$ and $C\in M_{k\times l}(X)$, the direct sum $B\oplus C\in M_{(m+k)\times(n+l)}(X)$ is defined by $B\oplus C=\begin{bmatrix} B & 0\\ 0 & C\end{bmatrix}$. Let $E\in M_{m\times n}(X)$. By a *submatrix* of E we mean a $k\times l$ block F in E where $1\le k\le m$ and $1\le l\le n$. We denote the set of all matrices $E\in M_n(X)$ which contain F as a submatrix and zero elsewhere by E_F.

Let X be a normed space, $n\in\mathbb{N}$, $f\in M_n(X)^*$ and $1\le i,j\le n$. Define $<f_{ij}, x>=<f, x\otimes E_{ij}^n>$ $x\in X$. Then $f_{ij}\in X^*$. Conversely every $(f_{ij})\in M_n(X^*)$ can be considered as an element of $M_n(X)^*$ whose actions defined by

$$<(f_{ij}),(x_{ij})>=\sum_{i,j=1}^{n}<f_{ij},x_{ij}>,(x_{ij})\in M_n(X).$$

Therefore we can identify $M_n(X^*)$ with $M_n(X)^*$. Henceforth we consider $M_n(X^*)$ with the norm which inherits from $M_n(X)^*$.

The *first* and *second Arens multiplications on* A^{**} that we denote by "." and "Δ", respectively, are defined in three steps. For $a,b\in A$, $f\in A^*$ and $m,n\in A^{**}$, the elements $f.a$, $a\Delta f$, $m.f$, $f\Delta m$ of A^* and $m.n$, $m\Delta n$ of A^{**} are defined in the following way:

$$<f.a,b>=<f,ab>\quad <a\Delta f,b>=<f,ba>$$
$$<m.f,b>=<m,f.b>\quad <f\Delta m,a>=<m,a\Delta f>$$
$$<m.n,f>=<m,n.f>\quad <m\Delta n,f>=<n,f\Delta m>$$

When we refer to A^{**} without explicit reference to any of Arens products, we mean A^{**} with the first Arens product.

For fixed $n\in A^{**}$ the map $m\mapsto m.n$ [resp. $m\mapsto n\Delta m$] is weak* – weak* continuous, but the map $m\mapsto n.m$ [$m\mapsto m\Delta n$] is not necessarily weak* – weak* continuous, unless n is in A. The *first topological center* $Z_1(A^{**})$ is defined by

$$\begin{aligned}Z_1(A^{**})&=\{n\in A^{**}:\text{The map } m\mapsto n.m \text{ is weak}^*\text{-weak}^*\text{continuous}\}\\&=\{n\in A^{**}: n.m=n\Delta m \text{ for all } m\in A^{**}\}.\end{aligned}$$

The second topological center $Z_2(A^{**})$ is defined similarly. If $Z_1(A^{**})=A^{**}$, then A is called *Arens regular*. In this case $Z_2(A^{**})=A^{**}$ as well.

3. Structure of Matrix Banach Spaces and Matrix Banach Algebras

In this section we introduce matrix Banach spaces and matrix Banach algebras. Then we study their basic properties.

Definition. Let X be a complex vector space, and $n\in\mathbb{N}$. We say that a norm $\|.\|$ on $M_n(X)$ is

(i) *free-position*, if for every $x\in X$ and positive integers $\le$ $1\le i,j,k,l\le n$ the equality $\|x\otimes E_{ij}^n\|=\|x\otimes E_{kl}^n\|$ holds.

(ii) *monotone*, if for every $B\in M_n(X)$ and every submatrix F of B, we have $\|B'\|\le\|B\|$ for all $B'\in E_F$.

(iii) *permutation invariant*, if for every $B,B'\in M_n(X)$, where B' is obtained by interchanging two rows or two columns of B, the equality $\|B\|=\|B'\|$ holds.

(iv) *unitary invariant*, if for every unitary matrix $U\in M_n$ and $B\in M_n(X)$ the equalities $\|UB\|=\|BU\|=\|B\|$ hold.

Clearly every unitary invariant norm is permutation invariant and every permutation invariant norm is a free-position norm.

Definition. Let X be a vector space and for every $n \in \mathbb{N}$, let $\|.\|_n$ be a norm on $M_n(X)$. We say that $(X, \{\|.\|_n\})$ is a *matrix normed system* if for every positive integer n, $\|.\|_n$ is a free-position and monotone norm on $M_n(X)$ and for every $m, n \in \mathbb{N}$ where $m > n$, the inclusion $\phi_{n,m} : M_n(X) \longrightarrow M_m(X)$, $\phi(A) = A \oplus 0$, is an isometry. As in [14] we say that a matrix normed system $(X, \{\|.\|_n\})$ satisfies the L^∞ condition if for all positive integers m, n and for all $B \in M_n(X)$, $C \in M_m(X)$ the equality $\|B \oplus C\|_{m+n} = max\{\|B\|_n, \|C\|_m\}$ holds. We say that a matrix normed system $(X, \{\|.\|_n\})$ is a *permutation invariant* matrix normed system, if for every positive integer n, $\|.\|_n$ is a permutation invariant norm on $M_n(X)$. Unitary invariant matrix normed system is defined similarly.

Remark 3.1. Suppose $(X, \{\|.\|_n\})$ is a matrix normed system. Then

(i) For every $(x_{ij}) \in M_n(X)$ we have,

$$\|x_{kl}\|_1 \le \|(x_{ij})\| \le \sum_{ij} \|x_{ij}\|_1 .$$

(ii) $(X, \{\|.\|_n\})$ is complete for some $n \in \mathbb{N}$, if and only if it is complete for every $n \in \mathbb{N}$.

(iii) It is easy to check that the concepts of matrix normed space [14] and unitary invariant matrix normed system coincide. To see this let $(X, \{\|.\|_n\})$ be a matrix normed space, n be a positive integer, u be a unitary matrix and x be an element of $M_n(X)$. We have

$$\|x\|_n = \|u^* ux\|_n \le \|u^*\| \|ux\|_n = \|ux\|_n .$$

Also $\|ux\|_n \le \|u\| \|x\|_n = \|x\|_n$. Conversely let $(X, \{\|.\|_n\})$ be a unitary invariant matrix normed system. Let n be a positive integer, α be an element of M_n with the operator norm 1 and x be an element of $M_n(X)$. Using singular value decomposition, we conclude that there exist two unitary matrices u and v such that $\alpha = (u+v)/2$. Now we have $\|\alpha x\| = \|ux/2 + vx/2\| \le \|ux/2 + ux/2\| + \|vx/2\| = \|x\|$. Similarly $\|x\alpha\| \le \|x\|$.

Definition. Let $(X, \{\|.\|_n\})$ be a matrix normed system. If $\|.\|_n$ is complete for some $n \in \mathbb{N}$ (and hence all $n \in \mathbb{N}$), then $(X, \{\|.\|_n\})$ is called a *matrix Banach space* (MBS).

Definition. Let A be an algebra and $(A, \{\|.\|_n\})$ be a MBS. For every $n \in \mathbb{N}$, consider $M_n(A)$ with the usual matrix product. If $\|.\|_n$ is an algebra norm on $M_n(A)$ for every n, that is

$$\|BC\|_n \le \|B\|_n \|C\|_n , \quad B, C \in M_n(A), \quad n \in \mathbb{N}$$

then $(A, \{\|.\|_n\})$ is called a *matrix Banach algebra* (MBA). We say that a MBA $(A, \{\|.\|_n\})$ is *unital* if A is unital and for every positive integer n the equality $\|I_n\|_n = 1$ holds.

Remark 3.2. In our definition of "matrix normed system" we do not assume contractivity of M_n-module actions on $M_n(X)$. Indeed by Remark 3.1(iii) and Ruan's Theorem [7] [respectively Blecher's Theorem] a MBS [respectively MBA] is an operator space [respectively operator algebra] *only if* it is unitary invariant and satisfies L^∞-condition.

Examples 3.3. (i) Let X be a Banach space. If we equip every $M_n(X)$ with the l^1 norm, then X turns into a permutation invariant MBS. But X is not a unitary invariant MBS. This is the greatest MBS structure on X. We denote this MBS with X^{max}. Also if X is a Banach space, then we can equip every $M_n(X)$ with the l^∞ norm. With this structure X turns into a permutation invariant MBS which is not unitary invariant. This is the least MBS structure on X. We denote this MBS with X^{min}. Clearly X^{min} satisfies A condition.

(ii) Let $p \in [1, \infty)$ and L^∞. We can equip A with the operator norm which inherits from A. With this structure $a \in A$ turns into a permutation invariant MBA.

(iii) Let X be a reflexive Banach space with a Schauder basis $\{x_n\}$ whose unconditional constant equals to one. For every positive integer n, we can endow $\mathbb{C}^n$ with the norm it inherits from X, by taking

$$\|(c_1, c_2, \ldots, c_n)\|_{[n]} = \|c_1 x_1 + c_2 x_2 + \ldots + c_n x_n\|$$

We can equip M_n with the operator norm $\|.\|_n$ of

$B(\mathbb{C}^n)$, where $\mathbb{C}^n$ is equipped with the above norm. With the order induced by the $\{x_n\}$, X is a Banach lattice. So with this structure, $\mathbb{C}$ is a MBA.

Definition. Let X be a MBS and $M_\infty(X)$ be the linear space $\{(x_{ij}) \mid i, j \in \mathbb{N}, x_{ij} \in X\}$. For $B \in M_\infty(X)$ let ${}_nB$ be the truncation of B to $M_n(X)$ (*i.e.* ${}_nB$ is $n \times n$ matrix in the top left corner of B) and ${}^nB$ be the element ${}_nB \oplus 0$ of $M_\infty(X)$. Define

$$\|B\| = Sup\{\|{}_nB\|_n \mid n \in \mathbb{N}\} = lim\,\|{}_nB\|_n\,,\ B \in M_\infty(X)\,.$$

We call every $B \in M_\infty(X)$ for which $\|B\| < \infty$, a *bounded matrix*. We denote the space of all bounded matrices on X with above norm by $M_\infty^b(X)$. An element in the closure of the inductive limit $\xrightarrow{\lim} M_n(X)$ in $M_\infty^b(X)$ is called an *approximable matrix*. We denote the set of all approximable matrices in $M_\infty^b(X)$ by $K_\infty(X)$.

Let A be a MBA and $A, B \in K_\infty(A)$. It is easy to see that sequence $\{{}^nA\,{}^nB\}$ is a norm convergent sequence in $K_\infty(A)$. We define

$$AB = \lim {}^nA\,{}^nB \qquad A, B \in A.$$

It is easy to see that for $A = (a_{ij})$ and $B = (b_{ij})$ in $K_\infty(A)$ the equality $AB = (\sum_{k=1}^{\infty} a_{ik} b_{kj})$ holds.

Remark. Let X be a MBS and Y be a Banach space. By identifying $M_n(B(X, Y))$ with $B(M_n(X), Y)$ in the usual way, we can equip $B(X, Y)$ with a MBS structure. In particular we can equip X^* with a MBS structure.

Proofs of the following lemmas are not complicated, but for convenience of readers we present their proofs in Section 6.

Lemma 3.4. If X is a MBS, then $M_\infty^b(X)$ is a Banach space.

Lemma 3.5. Let X be a MBS and $(x_{ij}) \in M_\infty(X)$. Then $(x_{ij}) \in K_\infty(X)$ if and only if $\lim_{n\to\infty}^n (x_{ij}) = (x_{ij})$.

Lemma 3.6. If A is a MBA, then $K_\infty(A)$ is a Banach algebra.

Lemma 3.7. Let X be a MBS and Y is a Banach space. Then $M_\infty^b(B(X, Y))$ is isometrically isomorphic with $B(K_\infty(X), Y)$. In particular $M_\infty^b(X^*) = K_\infty(X)^*$.

4. Weak Amenability of Approximable Matrices

In this section we study the weak amenability of $K_\infty(A)$. We show that for every permutation invariant MBA A which satisfies L^∞ condition and constructed on a unital commutative weak amenable Banach algebra A, $K_\infty(A)$ is weakly amenable.

Theorem 4.1. Let A be a permutation invariant MBA which satisfies the L^∞ condition. If A is unital commutative and weakly amenable, then $K_\infty(A)$ is weakly amenable.

Proof. Let A be weakly amenable, $a \in A$, $(f_{ij}) \in K_\infty(A)^*$ and $(a_{ij}) \in K_\infty(A)$. Then we have

$$< (f_{ij})(a_{ij}), a \otimes E_{kl}^\infty > = < (f_{ij}), (a_{ij})[a \otimes E_{kl}^\infty] > = < (f_{ij}),$$

$$\sum_{s=1}^\infty [a_{sk} a \otimes E_{sl}^\infty] > = \sum_{s=1}^\infty < f_{sl}, a_{sk} a > = < \sum_{s=1}^\infty f_{sl} a_{sk}, a > .$$

Thus the following equality holds

$$[(f_{ij})(a_{ij})]_{kl} = weak * - lim_n \sum_{s=1}^n f_{sl} a_{sk}\,. \tag{1}$$

Similarly we have

$$[(a_{ij})(f_{ij})]_{kl} = weak * - lim_n \sum_{s=1}^n a_{ls} f_{ks}\,. \tag{2}$$

Suppose $D : K_\infty(A) \longrightarrow K_\infty(A)^*$ is a bounded derivation. Now for positive integers i, j, k, l we can define D_{ij}^{kl} by

$$D_{ij}^{kl} : A \longrightarrow A^* \quad D_{ij}^{kl}(a) = [D(a \otimes E_{ij}^\infty)]_{kl}\,.$$

Using (1) and (2) we can conclude that for every $a, b \in A$ and for every positive integers i, j, k, l, m the following identity holds

$$\begin{aligned} D_{ij}^{kl}(ab) &= D([a \otimes E_{im}^\infty][b \otimes E_{mj}^\infty])_{kl} \\ &= [D(a \otimes E_{im}^\infty)[b \otimes E_{mj}^\infty]]_{kl} + [[a \otimes E_{im}^\infty] \\ &\quad + D(b \otimes E_{mj}^\infty)]_{kl} = D_{im}^{ml}(a) b \delta_{jk} + a \delta_{il} D_{mj}^{km}(b) \end{aligned} \tag{3}$$

Where δ is the Kronecker's delta. Thus for every positive integer m, D_{mm}^{mm} is a bounded derivation which is zero by assumption. On the other hand we have the following identity

$$[D((a_{rs}))]_{ij} = [D(lim_n \sum_{r,s=1}^{n} (a_{rs} \otimes E_{rs}^{\infty}))]_{ij} = lim_n \sum_{r,s=1}^{n} D_{rs}^{ij}(a_{rs}). \quad (4)$$

From (3) we conclude that for all positive integers i, j and m,

$$D_{ij}^{ji}(a) = D_{im}^{mi}(1)a - aD_{jm}^{mj}(1) + D_{mm}^{mm}(a).$$

Since every bounded derivation from A into A^* is zero, then

$$D_{ij}^{ji}(a) = D_{im}^{mi}(1)a - aD_{jm}^{mj}(1). \quad (5)$$

Also we have

$$0 = [D(E_{kk}^{\infty} E_{ii}^{\infty})]_{ik} = \sum_{s=1}^{\infty} D_{kk}^{sk}(1)\delta_{si} + \sum_{s=1}^{\infty} \delta_{ks} D_{ii}^{is}(1) = D_{kk}^{ik}(1) + D_{ii}^{ik}(1)$$

and hence

$$D_{kk}^{ik}(1) = -D_{ii}^{ik}(1). \quad (6)$$

From (4) and (6) we conclude that

$$\begin{aligned}[D((a_{rs}))]_{ij} &= \lim_n \sum_{k,l=1}^{n} D_{kl}^{ij}(a_{kl}) \\ &= \lim_n [\sum_{k=1}^{n} D_{ki}^{ij}(1)a_{ki} + \sum_{l=1}^{n} a_{jl} D_{jl}^{ij}(1) \\ &\quad - D_{ji}^{ij}(1)a_{ji} - a_{ji} D_{ji}^{ij}(1) + D_{ji}^{ij}(a_{ji})] \quad (7) \\ &= \lim_n [\sum_{k=1}^{n} D_{kk}^{kj}(1)a_{ki} \\ &\quad + \sum_{k=1}^{n} a_{jk} D_{kk}^{ik}(1)] + D_{ij}^{ji}(a_{ji}).\end{aligned}$$

If for every positive integer i, j we define $D_{kj} = D_{kk}^{kj}$, then from (7) we conclude that

$$[(D(a_{rs}))]_{ij} = [(D_{rs}(1))(a_{rs}) - (a_{rs})(D_{rs}(1))]_{ij} + D_{ij}^{ji}(a_{ji}). \quad (8)$$

On the other hand for every positive integer n we can consider M_n as a subalgebra of $M_n(A)$ and equip it with the norm which inherits from $M_n(A)$. Let $S = \{u_{n1}, ..., u_{nm}\}$ be the set of elements of M_n with exactly one nonzero entry which is 1 or -1, in every row and column. Since A is permutation invariant and satisfies L^{∞} condition, the following identity holds.

$$\|u_{nk}^t\|_n = \|u_{nk}\|_n = 1 \quad 1 \le k \le m. \quad (9)$$

Also we have $u_{nk}u_{nk}^t = I_n$ for every $1 \le k \le m$. Clearly S spans M_n. Now set

$$g_n = -1/m \sum_{k=1}^{m} u_{nk}^t D(u_{nk}).$$

By (9) we have $\|g_n\| \le \|D\|$ and for $1 \le i, j \le m$,

$$D(^n[u_{ni}u_{nj}]) = ^n u_{ni} D(^n u_{nj}) + D(^n u_{ni})^n u_{nj}.$$

Thus

$$D(^n u_{nj}) + ^n u_{ni}^t D(^n u_{ni})^n u_{nj} = ^n u_{nj}(^n[u_{ni}u_{nj}])^t D(^n[u_{ni}u_{nj}]).$$

Divide both sides of the above identity by m and sum over i while keeping j fixed to obtain $D(^n u_{nj}) -^n [g_n u_{nj}] = -^n [u_{nj} g_n]$. Since $\{u_{ni}u_{nj}\}_{i=1}^{m}$ is another listing of the elements of S and S spans M_n, then

$$D(^n u) =^n [g_n u - u g_n] \quad u \in M_n.$$

Let $g_n = (g_{nkl})$, then for $1 \le i, j \le n$ we have

$$D(E_{ij}^{\infty}) =^n (g_{nkl})E_{ij}^{\infty} - E_{ij}^{\infty n}(g_{nkl}) \quad (10)$$

Also from (8) we have

$$[D((E_{ij}^{\infty}))]_{rs} = lim_n \sum_{k=1}^{n} [D_{ks}(1)\delta_{kr} - \delta_{sk} D_{rk}(1)] + D_{rs}^{sr}(\delta_{sr}). \quad (11)$$

From (10) and (11) the following equalities hold for every $1 \le i, j \le n$

$$\begin{cases} D_{ij}(1) = g_{nij} & 1 \le i \ne j \le n \\ D_{ij}^{ji}(1) = g_{nii} - g_{njj} & 1 \le i, j \le n. \end{cases} \quad (12)$$

From (12) we can conclude that for all positive integers m, n where $m > n$ the following equality holds

$$g_{nij} = g_{mij}, 1 \le i \ne j \le n.$$

We set $g_{ij} = g_{nij}$ for $i, j, n \in \mathbb{N}$ and $i \ne j$. Since the sequence $\{diag(g_{n11}, ..., g_{nnn}, 0, 0, ...)\}$ is a bounded sequence with the bound $\|D\|$, then it has a subsequence that converges to an element $\{diag(g_{11}, ..., g_{nn}, ...)\}$ in $K_{\infty}(A)^*$ in the weak* topology. Thus $g=(g_{ij})$ is an element of $K_\infty(A)^*$. From (8) and (12) we conclude that the equality $D(a)=ga-ag, a\in K_\infty(A)$ holds. Therefore $K_\infty(A)$ is weakly amenable. □

Using the argument of the above Theorem we can prove the following Lemma.

Lemma 4.2. Let A be a unital, commutative and permutation invariant MBA which satisfies the equality $\|diag(\lambda_1,...\lambda_n)\|_n=1$ for every positive integer n and every $\lambda_1,...,\lambda_n\in\{-1,1\}$. If A is weakly amenable, then $K_\infty(A)$ is weakly amenable.

Let X be a reflexive Banach space with a Schauder basis $\{x_n\}$ whose unconditional constant is equal to one. If we equip □ with the MBS structure of Example 3.2(III), then $M_\infty^b(□)$ and $B(X)$ are isometrically isomorphic as Banach spaces and $K_\infty(□)$ and $K(X)$ are isometrically isomorphic as Banach algebras.

The following result follows immediately from [10, Theorem 4.5] but still it is worth to be mentioned here, because our approach is totally different.

Theorem 4.3. Let X be a reflexive Banach space with a Schauder basis whose unconditional constant is equal to one. Then $K(X)$ is weakly amenable.

Proof. We equip □ with the MBS structure of the Example 3.3(III). Clearly $(□,\|.\|_n)$ satisfies the conditions of the Lemma 4.2 and hence the weak amenability of $K_\infty(□)$ can be earned from the weak amenability of □. But by the above statement, $K(X)=K_\infty(□)$ and hence $K(X)$ is weakly amenable. □

5. Topological Center of Approximable Matrices

In this section we study the topological centers of the algebra of approximable matrices. Recall that a MBS A is approximable if $K_\infty(A)=M_\infty^b(A)$. The dual of an operator space is an example of an approximable MBS.

Theorem 5.1. Let A be a MBA with approximable dual. Then $Z_i(K_\infty(A)^{**})=M_\infty^b(Z_i(A^{**}))$, $i=1,2$.

proof. Since A^* is approximable then we have

$$K_\infty(A)^*=M_\infty^b(A^*)=K_\infty(A^*).$$

Hence

$$K_\infty(A)^{**}=M_\infty^b(A^{**}).$$

Now Let $a\in A$, $f\in A^*$, $(a_{ij})\in K_\infty(A)$, $(b_{ij}^*)\in K_\infty(A)^*$ and $(m_{ij}^{**}),(n_{ij}^{**})\in K_\infty(A)^{**}$. Then we have

$$\begin{aligned}&<(b_{ij}^*)(a_{ij}),a\otimes E_{kl}^\infty>\\&=<(b_{ij}^*),(a_{ij})[a\otimes E_{kl}^\infty]>=<(b_{ij}^*),\\&\textstyle\sum_{s=1}^\infty[a_{sk}a\otimes E_{sl}^\infty]>=\sum_{s=1}^\infty<b_{sl}^*,a_{sk}a>\\&=<\textstyle\sum_{s=1}^\infty b_{sl}^*a_{sk},a>.\end{aligned}$$

Thus

$$[(b_{ij}^*)(a_{ij})]_{kl}=\text{weak}*-\lim_n\textstyle\sum_{s=1}^n b_{sl}^*a_{sk}. \qquad (1)$$

Also we have

$$\begin{aligned}&<(m_{ij}^{**})(b_{ij}^*),a\otimes E_{kl}^\infty>\\&=<(m_{ij}^{**}),(b_{ij}^*)[a\otimes E_{kl}^\infty]>=<(m_{ij}^{**}),\\&\textstyle\sum_{s=1}^\infty[b_{ks}^*a\otimes E_{ls}^\infty]>=\sum_{s=1}^\infty<m_{ls}^{**},b_{sk}^*a>\\&=<\textstyle\sum_{s=1}^\infty m_{ls}^{**}b_{ks}^*,a>.\end{aligned}$$

Thus

$$[(m_{ij}^{**})(b_{ij}^*)]_{kl}=\text{weak}*-\lim_n\textstyle\sum_{s=1}^n m_{ls}^{**}b_{ks}^*. \qquad (2)$$

Therefore

$$\begin{aligned}&<(m_{ij}^{**})(n_{ij}^{**}),f\otimes E_{kl}^\infty>\\&=<(m_{ij}^{**}),(n_{ij}^{**})[a\otimes E_{ks}^\infty]>=<(m_{ij}^{**}),\\&\textstyle\sum_{s=1}^\infty[n_{sl}^{**}f\otimes E_{ks}^\infty]>=\sum_{s=1}^\infty<m_{ks}^{**},n_{sl}^{**}f>\\&=<\textstyle\sum_{s=1}^\infty m_{ks}^{**}n_{sl}^{**},f>.\end{aligned}$$

So the first Arens product on $M_n(A)^{**}$ can be expressed by the following identity.

$$[(m_{ij}^{**})(n_{ij}^{**})]_{kl}=\text{weak}*-\lim_n\sum_{s=1}^n m_{ks}^{**}n_{sl}^{**}. \qquad (3)$$

Similarly we have

$$\begin{cases} (i)[(a_{ij})\Delta(b^*_{ij})]_{kl} = \text{weak}*-\lim_n \sum_{s=1}^n a_{ls}\Delta b^*_{ks} \\ (ii)[(b^*_{ij})\Delta(m^{**}_{ij})]_{kl} = \text{weak}*-\lim_n \sum_{s=1}^n b^*_{sl}\Delta m^{**}_{sk} \\ (iii)[(m^{**}_{ij})\Delta(n^{**}_{ij})]_{kl} = \text{weak}*-\lim_n \sum_{s=1}^n m^{**}_{ks}\Delta n^{**}_{sl}. \end{cases} \quad (4)$$

From (3) and (iii) of (4) we conclude that $\mathrm{Z}_i(K_\infty(A)^{**}) = M^b_\infty(Z_i(A^{**}))$. □

Corollary 5.2. Let A be a unital MBA with approximable dual. Then A is Arens regular if and only if $K_\infty(A)$ is Arens regular.

Theorem 5.3. Let A be a unital MBA with L^∞ condition. If $K_\infty(A)$ is Arens regular then A^* is approximable.

Proof. Suppose A^* is not approximable. Then there exist an element of (f_{ij}) in $M^b_\infty(A^*)$ such that the sequence $\{^n(f_{ij})\}$ is not Cauchy. Therefore there exist $\varepsilon > 0$ and positive integers $k_1 < k_2 < ... < k_n < k_{n+1} < ...$ such that

$$\left\| {}^{k_{n+1}}(f_{ij}) - {}^{k_n}(f_{ij}) \right\| \geq \varepsilon$$

for every n. Now for every positive integer n there exist $(a_{ijn}) \in \mathrm{Ball}(K_\infty(A))$ with

$$\begin{cases} {}_{k_{n+1}}(a_{ijn}) = (a_{ijn}) \\ {}_{k_n}(a_{ijn}) = 0 \\ \left| < {}^{k_{n+1}}(f_{ij}) - {}^{k_n}(f_{ij}), (a_{ijn}) > \right| + \varepsilon/2 > \left\| {}^{k_{n+1}}(f_{ij}) - {}^{k_n}(f_{ij}) \right\|. \end{cases}$$

we have

$$\lim_m \lim_n < (f_{ij}), (a_{ijn})[\textstyle\sum_{i=1}^m E^\infty_{ii}] >= 0$$

and

$$\lim_n \lim_m < (f_{ij}), (a_{ijn})[\textstyle\sum_{i=1}^m E^\infty_{ii}] >$$
$$= \lim_n < (f_{ij}), (a_{ijn}) > .$$

Since for every n we have

$$| < (f_{ij}), (a_{ijn}) > | = | < {}^{k_{n+1}}(f_{ij}) - {}^{k_n}(f_{ij}), (a_{ijn}) > | > \varepsilon/2$$

then $\lim_n < (f_{ij}), (a_{ijn}) > \neq 0$. Hence by 1.5.11(f) of [13] $K_\infty(A)$ is not Arens regular. □

Corollary 5.4. Let A be a unital Arens regular MBA with L^∞ condition. Then $K_\infty(A)$ is Arens regular if and only if A^* is approximable.

Corollary 5.5. For every unital operator algebra A, the Banach algebra $K_\infty(A)$ is Arens regular.

6. Proofs of Lemmas 3.4 to 3.7

Proof of Lemma 3.4. Let $\{B_n\}_{n=1}^\infty$ be a Cauchy sequence in $M^b_\infty(X)$ and set $x_{ijn} = (B_n)_{ij}$. Then by Remark 3.1(i) for arbitrary positive integers i, j, the sequence $\{x_{ijn}\}$ is Cauchy and hence there is a $B = (x_{ij}) \in M_\infty(X)$ such that $\lim_n x_{ijn} = x_{ij}$. Now fix $n > 0$ and choose $m > 0$ such that $\left\| x_{ij} - x_{ijm} \right\|_1 < 1/n^2$ for $1 \leq i, j \leq n$. Then

$$\left\| {}_nB \right\|_n \leq \left\| {}_n(B - B_m) \right\|_n + \left\| {}_n(B_m) \right\|_n$$
$$\leq \sum_{i,j=1}^n \left\| x_{ij} - x_{ijm} \right\|_1 + \left\| B_m \right\| \leq 1 + M$$

where M is a upper bound for $\{\|B_n\|\}$. Therefore $(x_{ij}) \in M^b_\infty(X)$. Let $\varepsilon > 0$ and consider $N_1 > 0$ such that for every $m \geq N_1$ the inequality $\left\| B_m - B_{N_1} \right\| < \varepsilon/4$ holds and choose $N_2 > N_1$ such that $\left\| B - B_{N_1} \right\| < \left\| {}_{N_2}(B - B_{N_1}) \right\|_{N_2} + \varepsilon/4$. Also choose $N > N_2$ such that $\sum_{i,j=1}^{N_2} \left\| x_{ij} - x_{ijN} \right\|_1 < \varepsilon/4$. Now for $m \geq N$ we have

$$\left\| B - B_m \right\| \leq \left\| B - B_{N_1} \right\| + \left\| B_{N_1} - B_m \right\|$$
$$< \left\| {}_{N_2}(B - B_{N_1}) \right\|_{N_2} + \varepsilon/4 + \varepsilon/4$$
$$< \left\| {}_{N_2}(B - B_N) \right\|_{N_2} + \left\| {}_{N_2}(B_N - B_{N_1}) \right\|_{N_2} + \varepsilon/2$$
$$< \textstyle\sum_{i,j=1}^{N_2} \left\| x_{ij} - x_{ijN} \right\|_1 + \left\| B_N - B_{N_1} \right\| + \varepsilon/2$$
$$< \varepsilon/4 + \varepsilon/4 + \varepsilon/2 = \varepsilon.$$

Therefore $\mathrm{M}^b_\infty(A)$ is a Banach space. □

Proof of Lemma 3.5. Let $(x_{ij}) \in K_\infty(X)$ and T_n be a sequence in $\xrightarrow{\lim} M_n(X)$ such that $\lim T_n = (x_{ij})$.

For every positive integer n, let k_n be the least positive integer that $[T_n]_{ij} = 0$ for all $i, j > k_n$. Now let

$\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that $\|(x_{ij}) - T_N\| < \varepsilon/2$. For all $m \geq n \geq k_N$ we have

$$ {}^m(x_{ij}) - {}^n(x_{ij}) = \sum_{i=n+1}^{m} \sum_{j=1}^{m} [x_{ij} \otimes E_{ij}^{\infty}] $$
$$ + \sum_{i=1}^{n} \sum_{j=n+1}^{m} [x_{ij} \otimes E_{ij}^{\infty}]. $$

But $\sum_{i=n+1}^{m}\sum_{j=1}^{m}[x_{ij} \otimes E_{ij}^{\infty}]$ and $\sum_{i=1}^{n}\sum_{j=n+1}^{m}[x_{ij} \otimes E_{ij}^{\infty}]$ are submatrices of ${}^m[(x_{ij}) - T_N]$. So

$$ \|{}^m(x_{ij}) - {}^n(x_{ij})\| \leq 2\|(x_{ij}) - T_N\| < \varepsilon $$

and hence $\lim_{n\to\infty}^{n}(x_{ij}) = (x_{ij})$. The converse statement is clear. □

Proof of Lemma 3.6. Let $E = (a_{rs}), F = (b_{rs}) \in K_\infty(A)$, $\varepsilon > 0$ be given and $n \in \mathbb{N}$. For $1 \leq i, j \leq n$, choose m_{ij} such that $\left\|\sum_{k=s}^{\infty} a_{ik}b_{kj}\right\|_1 < \varepsilon/n^2$, for every $s \geq m_{ij}$. Setting $m = max\{m_{ij} \mid 1 \leq i, j \leq n\}$, we have

$$ \|{}_n[(a_{ij})(b_{ij})] - {}_n[{}^m(a_{ij})^m(b_{ij})]\|_n $$
$$ = \left\|{}_n(\sum_{k=1}^{\infty} a_{ik}b_{kj}) - {}_n(\sum_{k=1}^{m} a_{ik}b_{kj})\right\|_n $$
$$ = \left\|{}_n(\sum_{k=m+1}^{\infty} a_{ik}b_{kj})\right\|_n $$
$$ \leq \sum_{i,j=1}^{n} \left\|\sum_{k=m+1}^{\infty} a_{ik}b_{kj}\right\|_1 $$
$$ < \sum_{i,j=1}^{n} \varepsilon/n^2 = \varepsilon. $$

Hence EF is an element of $K_\infty(A)$. Therefore with this multiplication $K_\infty(A)$ turns into a Banach algebra. □

Proof of Lemma 3.7. Let $(f_{ij}) \in M_\infty^b(B(X, Y))$. Define,

$$ f((x_{ij})) = \lim_n \sum_{i,j=1}^{n} f_{ij}(x_{ij}), \quad (x_{ij}) \in K_\infty(X). $$

It is easy to see that $M_\infty^b(B(X,Y))$ is isomertically isomorphic with the $B(K_\infty(X),Y)$ via the following map

$$ \phi : M_\infty^b(B(X,Y)) \to B(K_\infty(X),Y), (f_{ij}) \mapsto f \ . \ \square $$

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