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EMPIRICAL BAYES ANALYSIS OF TWO-FACTOR EXPERIMENTS UNDER INVERSE GAUSSIAN MODEL

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Abstract

A two-factor experiment with interaction between factors wherein observations follow an Inverse Gaussian model is considered. Analysis of the experiment is approached via an empirical Bayes procedure. The conjugate family of prior distributions is considered. Bayes and empirical Bayes estimators are derived. Application of the procedure is illustrated on a data set, which has previously been analyzed by other authors.

Keywords: Inverse Gaussian; Factorial experiments; Bayesian analysis and empirical Bayes analysis

1. Introduction

There are many types of experiments setups in science and technology where the normal theory is inappropriate for the analysis of factorial experiments. One important class is related to the highly skewed nature of the data, which cannot be removed by the usual transformations. Alternatively, the Inverse Gaussian family of distributions is flexible enough to provide a suitable model for these types of data. Tweedie [16] pioneered work in providing an analogue to analysis of variance for nested classifications concerning observations from an Inverse Gaussian model. Despite the quite striking resemblance between normal analysis of variance and what he called, the Inverse Gaussian analysis of reciprocals, in one-way or nested classifications, the possibilities of developing analogous results for other classifications appeared to be limited, Folks and Chhikara [8], and Chkikara and Folks [7]. However, Shuster and Muira [15] succeeded in providing tests for balanced two-way classifications. Their approach has the disadvantage that it requires many observations in each cell. Such a requirement is hard to fulfil in most experiments. Bhattacharyya and Fries [4] treated the analysis of two-factor experiment with no interaction and obtained explicit solutions to the likelihood equations. They also proved asymptotic consistency and normality of their estimators. Few authors have contributed to the Bayesian analysis of the Inverse Gaussian distribution. Banerjee and Bhattacharyya [5] focussed on distributional results concerning Bayesian inference with an Inverse Gaussian model. Achcar *et al.* [3] also used a Bayesian approach for this family; some similarities to the normal model were found. Banerjee and Bhattacharyya [5] modeled the interpurchase time of a commodity with an Inverse Gaussian model, while employing a natural conjugate prior for population heterogeneity.

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Achcar and Rosales [1,2] are the only records in print for Bayesian analysis of two-factor experiments under an Inverse Gaussian model. They actually follow the approach presented in Bhattacharyya and Fries [4], assuming a non-informative prior density. Consequently, as one expects, because of heavy reliance on the likelihood, their results are not much different from those obtained by the maximum likelihood method of estimation. In this paper, we present an empirical Bayes analysis of two-factor experiments under an Inverse Gaussian model. This model is described in section 2. Section 3 considers the Bayesian analysis relative to two different conjugate priors. The main results are obtained in section 4. A real-life example previously analyzed by Shuster and Muira [15], and later by Achcar and Rosales [2], is reworked in section 5. Since any factorial experiment can be cast into a *two-factor* experiment by using a composite *factor* in place of all factors except the last one, the results of this work can be used in the more complicated designs. Section 6 contains some comments and suggestions for further research.

2. The Model

Consider an experiment with two factors, factor A with I levels indexed by i, and factor B with J levels, numbered by j, with each treatment combination being repeated n times. Observations from this experiment, denoted by y_{ijk} are assumed to follow an Inverse Gaussian model $IG(\theta_{ij}, \lambda)$,

$$Y_{ijk} \sim IG(\theta_{ij}, \lambda),\ i=1,\dots,I,\ j=1,\dots,J,\ k=1,\dots,n.$$

For each i, j, the random variables y_{ijk} are i.i.d. with mean θ_{ij}, and shape parameter λ. The two-parameter Inverse Gaussian density is

$$f(y_{ijk};\theta_{ij},\lambda) =$$

$$\{\lambda/(2\pi\, y_{ijk}^3)\}^{\frac{1}{2}} \exp\{-\lambda\,(y_{ijk}-\theta_{ij})^2 / 2\, y_{ijk}\,\theta_{ij}^2\}$$

$$y_{ijk} > 0,\ \theta_{ij} > 0,\ \lambda > 0,\ i=1,\dots,I,\ j=1,\dots,J,\ k=1,\dots,n. \tag{2.1}$$

In a two-factor experiment with interaction, each cell mean is assumed to be inversely proportional to the drift, while the drift is considered as the sum of factor main effects (α,β) and their interaction (γ). Thus, it is assumed that

$$\theta_{ij}^{-1} = \mu + \alpha_i + \beta_j + \gamma_{ij}\quad i=1,\dots,I,\ j=1,\dots,J, \tag{2.2}$$

$$\sum_{i=1}^{I}\alpha_i = \sum_{j=1}^{J}\beta_j = 0,\ \sum_{i=1}^{I}\gamma_{ij} = \sum_{j=1}^{J}\gamma_{ij} = 0. \tag{2.3}$$

Here, μ denotes the reciprocal of each cell mean when there is no drift. To incorporate the constraint (2.3) in the model, we can define the IJ×1 parameter vector $\mathbf{\Phi}$ as

$$\Phi = \left[\mu \middle| \alpha_1,\dots,\alpha_{I-1} \middle| \beta_1,\dots,\beta_{J-1} \middle| \gamma_{11},\dots,\gamma_{1,J-1} \middle| \cdots \middle| \gamma_{I-1,J},\dots,\gamma_{I-1,J-1} \right] = [\mu,\alpha,\beta,\gamma] \tag{2.4}$$

Then, the likelihood for the whole experiment can be written as

$$L(\Phi,\lambda|y) \propto \lambda^{nIJ/2}\exp\{-\lambda\left[R_{+++} - 2n\,\Phi' d + n\,\Phi' M\Phi\right]/2\} \tag{2.5}$$

In (2.5), the convention used by Bhattacharyya and Fries [4] has been utilized, where we have set

$$y = \left[y_{111},\dots,y_{11k},\dots,y_{11n},\dots,y_{ij1},\dots,\right.$$

$$\left. y_{ijk},\dots,y_{ijn},\dots,y_{IJ1},\dots,y_{IJk},\dots,y_{IJn}\right],$$

$$\theta_{ij}^{-1} = \mu + \alpha_i + \beta_j + \gamma_{ij} = X_{ij}'\mathbf{\Phi},$$

$$y_{ij} = \sum_{k=1}^{n} y_{ijk}/n,$$

$$\mathbf{D} = diag\{y_{11\cdot}, y_{12\cdot}\cdots, y_{IJ\cdot}\},\ \mathbf{M} = \mathbf{X}'\mathbf{D}\mathbf{X},$$

$$d = \sum_{i=1}^{I}\sum_{j=1}^{J} X_{ij},\ R_{ijk} = y_{ijk}^{-1}. \tag{2.6}$$

Summing over an index is shown by a plus sign while averaging is denoted by a dot. Thus, we shall use $R_{i++}, R_{+j+}, R_{ij+}, R_{+++}, R_{i..}, R_{.j.}, R_{ij.}$, and $R_{...}$ as sums and averages, respectively. We intend to use a conjugate prior for λ and $\mathbf{\Phi}$. The following priors have been proposed, see Chhikara and Folks [7], and Banerjee and Bhattacharyya [5]. The prior for λ is chosen from the gamma family and given λ, a normal prior is assumed for $\mathbf{\Phi}$. Thus,

$$\pi(\lambda) \propto \lambda^{a-1}\exp\{-b\lambda/2\},\qquad \lambda,a,b>0,$$

and, given λ, elements of $\mathbf{\Phi}$ are considered independent with either of the following two priors.

Case 1. Unrestricted Parameter Space

In the unrestricted case, the prior distribution for $\mathbf{\Phi}$ is

$$\mathbf{\Phi}|_{\lambda} \sim N(\boldsymbol{\eta}, \lambda^{-1}\mathbf{\Delta}) \tag{2.8}$$

with

$$\boldsymbol{\eta} = [\eta_1,...,\eta_{IJ}], \qquad \mathbf{\Delta} = diag\{\delta_1^2,...,\delta_{IJ}^2\}.$$

Then, the posterior is

$$q_1(\mathbf{\Phi}, \lambda|\mathbf{y}) \propto |\mathbf{\Psi}|^{\frac{-1}{2}} \lambda^{\nu-1} \exp\{-\lambda[Q_1(\boldsymbol{\eta}) + Q_2(\mathbf{\Phi})]/2\} \tag{2.9}$$

where

$$\mathbf{\Psi} = (n\mathbf{\Delta M} + \mathbf{\Delta}^{-1})^{-1},$$

$$\nu = a + (n+1)IJ/2,$$

$$Q_1(\boldsymbol{\eta}) = R_{+++} + b + \boldsymbol{\eta}'\mathbf{\Delta}^{-1}\boldsymbol{\eta} - \boldsymbol{\eta}^{*}{}'\mathbf{\Psi}^{-1}\boldsymbol{\eta}^*,$$

$$\boldsymbol{\eta}^* = (n\mathbf{\Delta M} + I)^{-1}(n\mathbf{\Delta}d + \boldsymbol{\eta}),$$

$$Q_2(\mathbf{\Phi}) = (\mathbf{\Phi} - \boldsymbol{\eta}^*)'\mathbf{\Psi}^{-1}(\mathbf{\Phi} - \boldsymbol{\eta}^*).$$

It is evident from (2.9) that

$$\pi(\lambda|\mathbf{y}) \propto \lambda^{k-1} \exp\{-\lambda Q_1(\boldsymbol{\eta})/2\}, \quad \lambda > 0,$$
$$k = (nIJ + 2a)/2, \qquad Q_1(\boldsymbol{\eta}) > 0, \tag{2.10}$$

and

$$q_1(\mathbf{\Phi}|\lambda, \mathbf{y}) \propto \left|\lambda^{-1}\mathbf{\Psi}\right|^{\frac{-1}{2}} \exp\{-\lambda Q_2(\mathbf{\Phi})/2\}. \tag{2.11}$$

That is, conjugacy holds. Therefore, we can write

$$q_1(\mathbf{\Phi}|\mathbf{y}) \propto \frac{1}{[Q_1(\boldsymbol{\eta}) + Q_2(\mathbf{\Phi})]^{\nu}}, \qquad \mathbf{\Phi} \in R^{IJ},$$

$$\propto \left[1 + \frac{(\mathbf{\Phi} - \boldsymbol{\eta}^*)'\sum^{-1}(\mathbf{\Phi} - \boldsymbol{\eta}^*)}{2a + nIJ}\right]^{-\frac{(2a+nIJ)+IJ}{2}} \tag{2.12}$$

with

$$\sum = \frac{Q_1(\boldsymbol{\eta})}{2a + nIJ}\mathbf{\Psi}$$

which is a multivariate T-type distribution, with 2a+nIJ > 2 degrees of freedom.

Case 2. Restricted Parameter Space

Strictly speaking, one should have $\phi_1 = \mu > 0$. This restriction on ϕ_1 is observed in the prior assigned to ϕ_1. Thus, a normal distribution truncated at zero is considered for ϕ_1, which has density

$$q_2(\phi_1|\lambda) \propto \lambda^{\frac{1}{2}}\left[\delta_1 N(\lambda^{\frac{1}{2}} - \eta_1/\delta_1)\right]^{-1} \exp\{-\lambda(\phi_1 - \eta_1)^2/2\delta_1^2\},$$
$$\phi_1 > 0 \tag{2.13}$$

and the remaining is as in Case 1. In (2.13), ***N***(.) Is the standard normal Cdf. The restriction imposed on ϕ_1 results in a posterior proportional to (2.9).

3. Bayes Estimates

In case 1, from (2.10) we have

$$E(\lambda^m|\mathbf{y}) = [2/Q_1(\boldsymbol{\eta})]^m[\Gamma(k+m)/\Gamma(k)] \tag{3.1}$$

which provides the Bayes estimate of λ relative to the squared error loss:

$$\lambda_{B1} = E(\lambda|\mathbf{y}) = (nIJ + 2a)/Q_1(\boldsymbol{\eta}), \tag{3.2}$$

$$V_{\lambda} = Var(\lambda|\mathbf{y}) = 2(2a + nIJ)/[Q_1(\boldsymbol{\eta})]^2. \tag{3.3}$$

Upon using (2.12), we arrive at

$$\mathbf{\Phi}_{B1} = E(\mathbf{\Phi}|\mathbf{y}) = \boldsymbol{\eta}^* = (n\mathbf{\Delta M} + I)^{-1}(n\mathbf{\Delta}d + \boldsymbol{\eta}), \tag{3.4}$$

$$V_{B1} = Var(\mathbf{\Phi}|\mathbf{y}) = \frac{Q_1(\boldsymbol{\eta})}{2(a-1) + nIJ}\mathbf{\Psi}. \tag{3.5}$$

For case 2, the posterior moments of λ remain unchanged from those for Case 1. However, for $\mathbf{\Phi}$, the restriction on ϕ_1 renders results on $\mathbf{\Phi}$ different from those in (3.4) and (3.5) for Case 1. To this end, let $\mathbf{\Phi}$, $\boldsymbol{\eta}$ and $\mathbf{\Psi}$ be partitioned as

$$\boldsymbol{\eta} = \begin{bmatrix}\eta_1 \\ \eta^{(2)}\end{bmatrix}, \boldsymbol{\eta}^* = \begin{bmatrix}\eta_1{}^* \\ \eta^{*(2)}\end{bmatrix}, \mathbf{\Phi} = \begin{bmatrix}\phi_1 \\ \mathbf{\Phi}^{(2)}\end{bmatrix}, \mathbf{\Psi} = \begin{bmatrix}\psi_{11} & \mathbf{\Psi}_{12} \\ \mathbf{\Psi}_{21} & \mathbf{\Psi}_{22}\end{bmatrix}$$

Then by virtue of the facts $\mathbf{\Psi}_{22.1} = \mathbf{\Psi}_{22} - \mathbf{\Psi}_{21}\psi_{11}^{-1}\mathbf{\Psi}_{12}$ and $|\mathbf{\Psi}| = \psi_{11}|\mathbf{\Psi}_{22.1}|$, $Q_2(\mathbf{\Phi})$ in (2.9) can be written as

$$Q_2(\mathbf{\Phi})=\psi_{11}^{-1}(\phi_1-\eta_1^*)^2 + (\mathbf{\Phi}^{(2)}-\boldsymbol{\eta}^*_{2.1})'\mathbf{\Psi}_{22.1}^{-1}(\mathbf{\Phi}^{(2)}-\boldsymbol{\eta}^*_{2.1}) \quad (3.6)$$

where

$$\eta^*_{2.1} = E(\mathbf{\Phi}^{(2)}|\phi_1,\lambda,\mathbf{y}) = \boldsymbol{\eta}^{*(2)} + \psi_{11}^{-1}(\phi_1-\eta_1)\mathbf{\Psi}_{21} \quad (3.7)$$

and

$$Var(\mathbf{\Phi}^{(2)}|\phi_1,\lambda,\mathbf{y}) = \lambda^{-1}\mathbf{\Psi}_{22.1}. \quad (3.8)$$

From (3.7), upon taking successive expectations with respect to ϕ_1 and λ, we obtain

$$\mathbf{\Phi}_{B2}^{(2)} = E(\mathbf{\Phi}^{(2)}|\mathbf{y}) = \boldsymbol{\eta}^{*(2)} + \left[E(\phi_1|\mathbf{y}) - \eta_1^*\right]\psi_{11}^{-1}\mathbf{\Psi}_{21} \quad (3.9)$$

and

$$\begin{aligned} V_{B2} &= Var(\mathbf{\Phi}^{(2)}|\mathbf{y}) \\ &= [Q_1(\boldsymbol{\eta})/2(k-1)]\mathbf{\Psi}_{22.1} + E[Var(\phi_1|\lambda,\mathbf{y})]\psi_{11}^{-2}\mathbf{\Psi}_{21}\mathbf{\Psi}_{12} \\ &\quad + Var[E(\phi_1|\lambda,\mathbf{y})]\psi_{11}^{-2}\mathbf{\Psi}_{21}\mathbf{\Psi}_{12}. \end{aligned} \quad (3.10)$$

The posterior moments of ϕ_1 are substituted in (3.9) to obtain the Bayes estimators, (See Appendix1):

$$\phi_{1B2} = E(\phi_1|\mathbf{y}) = \eta_1^* + 0.21[\psi_{11}Q_1(\boldsymbol{\eta})]^{1/2}\mathbf{W}, \quad (3.11)$$

and

$$\begin{aligned} V_{1B2} &= Var(\phi_1|\mathbf{y}) \\ &= 0.45\psi_{11}Q_1(\boldsymbol{\eta})/(k-1) \\ &\quad + 0.045\psi_{11}Q_1(\boldsymbol{\eta})[11/(k-1)-\mathbf{W}^2] \\ &\quad - 0.21\eta_1^*[\psi_{11}Q_1(\boldsymbol{\eta})]^{1/2}\mathbf{W}, \end{aligned}$$

with

$$\mathbf{W} \cong [(k-0.5)/k]^{1/2}. \quad (3.12)$$

Now, we substitute the posterior moments of ϕ_1 into (3.9) to obtain a simpler form as:

$$\mathbf{\Phi}_{2B2} = \boldsymbol{\eta}^{*(2)} + \{0.21\mathbf{W}[\psi_{11}^{-1}Q_1(\boldsymbol{\eta})]^{1/2}\}\mathbf{\Psi}_{21}, \quad (3.13)$$

and

$$\begin{aligned} V_{2B2} &= \\ &[Q_1(\boldsymbol{\eta})/2(k-1)][\mathbf{\Psi}_{22} - 0.09(k-1)\mathbf{W}^2\mathbf{\Psi}_{21}\psi_{11}^{-1}\mathbf{\Psi}_{12}] \\ &+ 0.21\mathbf{W}\eta_1^*[\psi_{11}^{-1}Q_1(\boldsymbol{\eta})]^{1/2}\mathbf{\Psi}_{21}\psi_{11}^{-1}\mathbf{\Psi}_{12}. \end{aligned} \quad (3.14)$$

The expressions (3.11) and (3.14) provide the Bayes estimates relative to the restricted prior given in (2.13). In a fully Bayesian analysis, one can utilize MCMCs (Markov Chain Monte Carlo) methods to obtain the posterior moments of interest. We do not pursue such a path, because we have exact formulas for Case 1 and an approximate formula for Case 2, so we can dispense with such methods.

As long as the prior distribution can be assessed, Bayes estimators, either (3.2)-(3.5) or (3.11)-(3.14) could be put into application for two-way classifications. Unfortunately, the situations where these priors can reasonably be assessed are rare. In such cases, we can utilize an empirical Bayes procedure to estimate the prior distributions from the data. By this, we borrow strength from Bayesian logic and objectivity from classical method.

4. Empirical Bayes Estimates

To estimate the prior parameters from the marginal distribution of the observations, one can use any method of estimation. Two more common methods are the method of moments and maximum likelihood. To provide explicit expressions for estimates of a, b, $\boldsymbol{\eta}$, and $\mathbf{\Delta}$ from the data, **y**, we shall first use the method of moments. To this end, we have from Chhikara and Folks [7],

$$V_{ij} = \sum_{k=1}^{n}(Y_{ijk}^{-1} - Y_{ij.}^{-1}) \sim \lambda^{-1}\chi_{n-1}^2.$$

Thus,

$$\begin{aligned} E(V_{ij}) &= E[E(V_{ij}|\lambda)] = (n-1)b/2(a-1), \\ Var(V_{ij}) &= E[Var(V_{ij}|\lambda)] + Var[E(V_{ij}|\lambda)] \\ &= (n-1)(n-3+2a)b^2/4(a-1)^2(a-2). \end{aligned}$$

Let

$$V = \sum_{i=1}^{I}\sum_{j=1}^{J}V_{ij}/IJ, \quad S_V^2 = \sum_{i=1}^{I}\sum_{j=1}^{J}[V_{ij}-V]^2/(IJ-1).$$

And let $C = S_v/V$ be the sample coefficient of variation for V_{ij}. Then,

$$\begin{aligned} a_0 &= [2(n-1)C^2 + n - 3]/[(n-1)C^2 - 2], \\ b_0 &= 2(a_0-1)V/(n-1) \end{aligned} \quad (4.1)$$

which are valid positive estimates of *a* and *b* for n>1, if one has $C^2 > 2/(n-1)$, otherwise, take *a* and *b* equal to zero, *i.e.*, use a noninformative prior. For estimation of $\boldsymbol{\eta}$ and $\boldsymbol{\Delta}$, we need 2IJ equations. These are provided by the following considerations. Let $Z_i, i=1,...,n,$ be i.i.d random variables distributed as $IG(\theta, n\lambda)$. The distributional relations between Z_i, $\bar{Z}$, and Z_i^{-1} have been found in Chhikara and Folks (1989). That is, $\bar{Z} \sim IG(\theta, n\lambda)$ and Z_i^{-1} has mean and variance as stated below:

$$Z_i^{-1} \sim \left[\theta^{-1} + \lambda^{-1} = E(Z_i^{-1}), (\lambda\theta)^{-1} + 2\lambda^{-2} = Var(Z_i^{-1})\right].$$

Let define the sample means $R_{ij.}$, $R_{i..}$, $R_{.j.}$, and $R_{...}$ as in Section 2. Moreover, define the corresponding sample variances as

$$S_{ij}^2 = \sum_{k=1}^{n} (R_{ijk} - R_{ij.})^2 / n(n-1),$$

$$S_j^2 = \sum_{i=1}^{I} (R_{ij.} - R_{.j.})^2 / I(I-1),$$

$$S_i^2 = \sum_{j=1}^{J} (R_{ij.} - R_{i..})^2 / J(J-1),$$

$$S^2 = \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{n} (R_{ijk} - R_{ij.})^2 / nIJ(nIJ-1).$$

Now apply the above general distributional rules on sample means and then obtain their two distributional moments for given values of λ and $\boldsymbol{\Phi}$. Next find the marginal moments by taking expectations with respect to prior distributions of λ and $\boldsymbol{\Phi}$. The moment estimates of $\boldsymbol{\eta}$ and $\boldsymbol{\Delta}$ are the solution of this system of equations (See Appendix 2) along with (4.1). The estimates are:

$$\eta_1^0 = R_{...} - b_0 / 2(a_0 - 1),$$

$$\delta_{1,0}^2 = 2(a_0-1)\, S^2 / b_0 - \eta_1^0 / nIJ$$
$$- b_0\left[2(a_0-1) + nIJ\right](a_0-1)(a_0-2). \qquad (4.2)$$

For $i = 1,..., I-1$,

$$\eta_{i+1}^0 = R_{i..} - \eta_1^0 - b_0 / 2(a_0-1) = R_{i..} - R_{...},$$

$$\delta_{i+1,0}^2 = 2(a_0-1)\, S_i^2 / b_0 - (\eta_1^0 + \eta_{i+1}^0) / nJ - \delta_1^0$$
$$- b_0\left[2(a_0-1) + nJ)\right] / 2nJ(a_0-1)(a_0-2).$$

(4.3)

For $j = 1,..., J-1$,

$$\eta_{I+j}^0 = R_{.j.} - \eta_1^0 - b_0 / 2(a_0-1) = R_{.j.} - R_{...},$$

$$\delta_{I+j,0}^2 = 2(a_0-1)\, S_j^2 / b_0 - (\eta_1^0 + \eta_{I+j}^0) / nI - \delta_1^0$$
$$- b_0\left[2(a_0-1) + nJ\right] / 2nI(a_0-1)(a_0-2). \qquad (4.4)$$

Finally, for $i = 1,..., I-1$, and $j = 1,..., J-1$,

$$\eta_{I+i(J-1)+j}^2 = R_{ij.} - \eta_1^0 - \eta_{i+1}^0 - \eta_{I+j}^0 - b_0 / 2(a_0-1)$$
$$= R_{ij.} - R_{i..} - R_{.J.} + R_{...},$$

$$\delta_{I+i(J-1)+j,0}^2 = 2(a_0-1)\, S_{ij}^2 / b_0$$
$$- \left[\eta_1^0 + \eta_{i+1}^0 + \eta_{I+j}^0 + \eta_{I+i(J-1)+j}^0\right] / n$$
$$- (\delta_{1,0}^2 + \delta_{i+1,0}^2 + \delta_{I+j,0}^2)$$
$$- b_0[2(a_0-1) + n)] / 2n(a_0-1)(a_0-2) \qquad (4.5)$$

which provide

$$\eta^0 = \left[\eta_1^0, \eta_2^0, ..., \eta_{IJ}^0\right], \text{and } \Delta^0 = diag\{\delta_{1,0}^2, \delta_{2,0}^2, ..., \delta_{IJ,0}^2\}. \qquad (4.6)$$

Now, we substitute these estimates into (3.2) and (3.5) to obtain the empirical Bayes estimate relative to unrestricted prior distribution. This gives us

$$\lambda_{EB1} = (nIJ + 2a_0) / Q_1(\eta^0), \qquad (4.7)$$

$$\Phi_{EB1} = (n\Delta^0 M + I)^{-1}(n\Delta^0 + \eta^0) = \eta^{*0}. \qquad (4.8)$$

Posterior variances are estimated by

$$Var(\lambda \big| y) = 2(2a_0 + nIJ) / \left[Q_1(\eta^0)\right]^2 \qquad (4.9)$$

and

$$Var(\Phi \big| y) = \left[Q_1(\eta^0) / 2(k-1)\right]\Psi^0,$$

$$\Psi^0 = \left[n\Delta^0 M + (\Delta^0)^{-1}\right]^{-1}. \qquad (4.10)$$

In Case 2, the only difference in prior is that ϕ_1 has a truncated normal prior. Accordingly, we should alter the posterior and the marginal moments for differences in moments of ϕ_1 (See Appendix 1). In this case,

$$E(\phi|\lambda) = \eta_1 + 0.3\lambda^{-1/2}\delta_1,$$

and

$$Var(\phi_1|\lambda) = 0.91\lambda_1^{-1}\delta_1^2 + 0.3\eta_1\lambda^{-1/2}\delta_1.$$

To account for this difference, the previous moment equations should be modified accordingly. Omitting the details, which can be found in Meshkani [9], we shall give the final results.

Let

$$k(a_0) = 0.21\omega(a_0)b_0^{1/2}, \quad w(a_0) = \Gamma(a_0 - 0.5)/\Gamma(a_0),$$

$$m_0 = \left[b_0^2 \Big/ 4nIJ(a_0-1)^2(a_0-2) \right] [2(a_0-1) + nIJ].$$

Then,

$$\eta_1 = R_{...} - b_0/2(a_0-1) - k(a_0)\delta_1,$$

and

$$S^2 = m_0 + [b_0/2(a_0-1)]\delta_1^2$$
$$+ \left[b_0 \Big/ 2nIJ(a_0-1) \right] [\eta_1 + k(a_0-1)\delta_1]$$
$$- k^2(a_0)\delta_1^2.$$

Absorbing η_1 into S^2 leads to the quadratic equation

$$A\delta_1^2 + B\delta + D = 0$$

with

$$A = 1 - 4(a_0-1)k^2(a_0)/b_0,$$
$$B = k(a_0)\{(1+2nIJ)[\omega(a_0-1) - \omega(a_0)]/nIJ\omega(a_0)$$
$$+ 2(a_0-1)R.../b_0\} - 1,$$

and

$$D = (R.../nIJ) + b_0(a_0 + nIJ)/2nIJ(a_0-1)(a_0-2)$$
$$- 2(a_0-1)S^2/b_0.$$

Thus, we obtain an estimate for δ_1^2 as

$$\delta_1^2 = \begin{cases} (-B/2A)^2 & if \quad B^2 - 4AD \geq 0 \\ D/A & if \quad B^2 - 4AD < 0 \end{cases}.$$

This gives $\tilde{\boldsymbol{\eta}}_1 = R_{...} - b_0/2(a_0-1) - k_0\tilde{\delta}_1$, while other elements of $\tilde{\boldsymbol{\eta}}$ being equal to those given for Case 1. However, for $i = 1,..., I-1$,

$$\tilde{\delta}_{i+1}^2 = 2(a_0-1)S_i^2/b_0$$
$$-[\tilde{\eta}_1 + \tilde{\eta}_{i+1} + \tilde{\delta}_1 k(a_0-1)]/nJ$$
$$-b_0[2(a_0-1) + nJ]/2nJ(a_0-1)(a_0-2)$$
$$-\tilde{\delta}_1^2 + 2k(a_0)[\tilde{\delta}_1 k(a_0) - \tilde{\eta}_1]\tilde{\delta}_1(a_0-1)/b_0$$
$$-2\tilde{\delta}_1[k(a_0-1) - k(a_0)].$$

For $j = 1,..., J-1$,

$$\tilde{\delta}_{I+j} = 2(a_0 \ -1)S_j^2/b_0$$
$$-[\tilde{\eta}_1 + \tilde{\eta}_{I+j} + \tilde{\delta}_1 k(a_0-1)]/nI$$
$$-b_0[2(a_0-1) + nI]/2nI(a_0-1)(a_0-2)$$
$$-\tilde{\delta}_1^2 + 2k(a_0)[\tilde{\delta}_1 k(a_0) - \tilde{\eta}_1]\tilde{\delta}_1(a_0-1)/b_0$$
$$-2\tilde{\delta}_1[k(a_0-1) - k(a_0)],$$

and finally, for $i = 1,..., I-1$, and $j = 1,..., J-1$,

$$\tilde{\delta}_{I+i(J-1)+j} = 2(a_0 \ -1)S_{ij}^2/b_o$$
$$-[\tilde{\eta}_1 + \tilde{\eta}_{i+1} + \tilde{\eta}_{I+j} + \tilde{\eta}_{I+i(J-1)+j} + \tilde{\delta}_1 k(a_0-1)]/nI$$
$$-b_0[2(a_0-1) + n] \Big/ 2n(a_0-1)(a_0-2)$$
$$-[\tilde{\delta}_1^2 + \tilde{\delta}_{i+1}^2 + \tilde{\delta}_{I+j}^2]$$
$$+2k(a_0)[\tilde{\delta}_1 k(a_0) - \tilde{\eta}_1]\tilde{\delta}_1(a_0-1)/b_0$$
$$-2\tilde{\delta}_1[k(a_0-1) - k(a_0)].$$

Thus, we have $\tilde{\boldsymbol{\eta}} = [\tilde{\eta}_1,..., \tilde{\eta}_{IJ}]$, $\tilde{\boldsymbol{\Delta}} = diag\{\delta_1^2,..., \delta_{IJ}^2\}$, which provide the respective empirical Bayes estimates:

$$\lambda_{EB2} = (nIJ + 2a_0) \Big/ Q_1(\tilde{\boldsymbol{\eta}}) \tag{4.11}$$

and

$$Var(\lambda|\mathbf{y}) = 2(nIJ + 2a_0)/[Q_1(\tilde{\boldsymbol{\eta}})]^2, \tag{4.12}$$

$$\tilde{\phi}_{1,EB2} = \tilde{\eta}_1^* + 0.21\omega(k)[\tilde{\psi}_{11}Q_1(\tilde{\boldsymbol{\eta}})]^{1/2} \tag{4.13}$$

and

$$Var(\phi|y) =$$

$$0.045\tilde{\psi}_{11}Q_1(\tilde{\eta})\left[11/(k-1)-\omega^2(k)\right]$$

$$+0.21\tilde{\eta}_1^*\omega(k)\left[\tilde{\psi}_{11}Q_1(\tilde{\boldsymbol{\eta}})\right]^{1/2} \tag{4.14}$$

with

$$\tilde{\boldsymbol{\eta}}^* = \left[\tilde{\eta}_1^*, \tilde{\eta}^*(2)\right] = \left[n\tilde{\Delta}M + I\right]^{-1}\left[n\tilde{\Delta}d + \tilde{\boldsymbol{\eta}}\right],$$

$$\tilde{\boldsymbol{\Psi}} = \left[n\tilde{\Delta}\mathbf{M} + \tilde{\Delta}^{-1}\right] = \begin{bmatrix} \tilde{\psi}_{11} & \tilde{\boldsymbol{\Psi}}_{12} \\ \tilde{\boldsymbol{\Psi}}_{21} & \tilde{\boldsymbol{\Psi}}_{22} \end{bmatrix}.$$

Moreover,

$$\boldsymbol{\Phi}_{EB2}^{(2)} = \tilde{\boldsymbol{\eta}}^{*(2)} + \{0.21\omega(k)\left[\tilde{\psi}_{11}^{-1}Q_1(\tilde{\boldsymbol{\eta}})\right]^{1/2}\}\tilde{\boldsymbol{\Psi}}_{21} \tag{4.15}$$

$$Var(\boldsymbol{\Phi}|y) =$$

$$\left[Q_1(\tilde{\eta}\big/2(k-1))\right]\left[\tilde{\boldsymbol{\Psi}}_{22} - 0.09(k-1)\omega^2(k)\,\tilde{\boldsymbol{\Psi}}_{21}\tilde{\psi}_{11}^{-1}\tilde{\boldsymbol{\Psi}}_{12}\right]$$

$$+021\omega(k)\,\tilde{\boldsymbol{\eta}}_1{}^*\left[\tilde{\psi}_{11}^{-1}Q_1(\tilde{\boldsymbol{\eta}})\right]^{1/2}\tilde{\boldsymbol{\Psi}}_{21}\tilde{\psi}_{11}^{-1}\tilde{\boldsymbol{\Psi}}_{21}. \tag{4.16}$$

Although we have used the method of moments to reach explicit solutions, we could have alternatively used the maximum likelihood procedure to obtain estimates of a, b, $\boldsymbol{\eta}$ and $\boldsymbol{\Delta}$. This method needs numerical maximization which can be done by usual routines. Here, we only outline the procedure and leave the detail for practical data analysis. From (2.5)-(2.8),

$$\pounds = \ell(\mathbf{y}|a,b,\boldsymbol{\eta},\boldsymbol{\Delta})$$

$$= \mathbf{K}\left\{\int_0^\infty \left[\lambda^{v-1}b^a|\boldsymbol{\Delta}|^{-1/2}/\Gamma(a)\right]\exp\{-\lambda/2Q_1(\boldsymbol{\eta})\}\right.$$

$$\left.\times\int \exp-\lambda/2Q_2(\phi)d\phi\right\}$$

$$= \frac{\Gamma(a+nIJ/2)}{\Gamma(a)}\cdot\frac{|\boldsymbol{\Psi}|^{1/2}}{|\boldsymbol{\Delta}|^{1/2}}\frac{(2b)^a}{[Q_1(\boldsymbol{\eta})]^{nIJ/2+a}}$$

Maximizing $\ell(\mathbf{y}|a,b,\boldsymbol{\eta},\boldsymbol{\Delta})$ with respect to $a, b, \boldsymbol{\eta}$ and $\boldsymbol{\Delta}$ would provide the maximum likelihood estimates, denoted by $\hat{a}, \hat{b}, \hat{\boldsymbol{\eta}}$, and $\hat{\boldsymbol{\Delta}}$. Applying them in (3.4) and (3.5) would result in empirical Bayes estimates based on the maximum likelihood procedure. Again, if we observe the restriction on ϕ_1, we shall have the corresponding results. Let these results be expressed as

$$\phi_{EBL} = (n\hat{\boldsymbol{\Delta}}\mathbf{M} + I)^{-1}(n\hat{\boldsymbol{\Delta}}d + \hat{\boldsymbol{\eta}}), \tag{4.17}$$

$$\mathrm{Var}(\boldsymbol{\Phi}|\mathbf{y}) = [Q_1(\hat{\boldsymbol{\eta}})/2(k-1)]\hat{\boldsymbol{\Psi}}, \tag{4.18}$$

with

$$\hat{\boldsymbol{\Psi}} = (n\hat{\boldsymbol{\Delta}}\mathbf{M} + \hat{\boldsymbol{\Delta}}^{-1})^{-1}.$$

The above derivation remains valid for the 2-factor **ANOVA** model without interaction, as well as for one-way **ANOVA**. In these cases, one only need to reduce the order of the vector of parameters and the design matrix, according to the model used and follow the above procedure.

5. An Example

To illustrate our proposed estimators and compare them with other estimators, we analyze an experiment originally reported by Ostle [10] and analyzed by Shuster and Muira [15], and later by Achcar and Rosales [2]. Data in Table 5.1 have resulted from a randomized 2×5 layout with 10 replicates in each cell. The responses consist of the impact resistance of 5 kinds of insulators to shocks when they are cut lengthwise or widthwise. There are 10 replicates for each combination.

Maximum Likelihood Estimates

It can be shown that the maximum likelihood estimates are

$$\hat{\boldsymbol{\Phi}}(ML) = \mathbf{M}^{-1}\mathbf{d}, \qquad \mathbf{d} = [10,0,...,0],$$

$$\hat{\lambda}(ML) = \mathrm{IJ}/[\mathrm{n}\,R_{...} - \mathbf{d}'\mathbf{M}^{-1}\mathbf{d}]$$

whose large-sample variances are

$$Var\left[\hat{\boldsymbol{\Phi}}(ML)\right] = (n\hat{\lambda})^{-1}\mathbf{M}^{-1},$$

$$Var\left[\hat{\lambda}(ML)\right] = 2(nIJ)^{-1}\left[\hat{\lambda}(ML)\right]^2,$$

$$Cov\left[\hat{\boldsymbol{\Phi}}(ML), \hat{\lambda}(ML)\right] = 0.$$

In this example, the diagonal elements of $\mathbf{D}$ are given in the last column of Table 5.1, and $\mathbf{X}'$ is

$$\mathbf{X}' = \begin{bmatrix} 1'_4 & 1 & 1'_4 & 1 \\ 1'_4 & 1 & -1'_4 & -1 \\ I_4 & -1_4 & I_4 & -1_4 \\ I_4 & -1_4 & -I_4 & 1_4 \end{bmatrix}$$

which yield the estimates and their standard errors (S.E.) given in Table 5.2. The asymptotic 95 percent confidence intervals (CI) are also provided in Table

Table 5.1. Observations from a 2*5 factorial experiment with 10 replications

$k \rightarrow$ / $(i,j) \downarrow$	1	2	3	4	5	6	7	8	9	10	Mean ($y_{ij.}$)
(1,1)	1.15	0.84	0.88	0.91	0.86	0.88	0.92	0.87	0.93	0.95	0.919
(1,2)	1.16	0.85	1.00	1.08	0.80	1.01	1.14	0.87	0.97	1.09	0.999
(1,3)	0.79	0.68	0.64	0.72	0.63	0.59	0.81	0.65	0.64	0.75	0.690
(1,4)	0.96	0.82	0.98	0.93	0.81	0.79	0.79	0.86	0.84	0.92	0.870
(1,5)	0.49	0.61	0.59	0.51	0.53	0.72	0.67	0.47	0.44	0.48	0.551
(2,1)	0.89	0.69	0.46	0.85	0.73	0.67	0.78	0.77	0.80	0.79	0.743
(2,2)	0.86	1.17	1.18	1.32	1.03	0.84	0.89	0.84	1.03	106	1.022
(2,3)	0.52	0.52	0.80	0.64	0.63	0.58	0.65	0.60	0.71	0.59	0.623
(2,4)	0.86	1.06	0.81	0.97	0.90	0.93	0.87	0.88	0.89	0.82	0.899
(2,5)	0.52	0.53	0.47	0.47	0.57	0.54	0.56	0.55	0.45	0.60	0.526

Table 5.2. ML Estimates of λ, $\mathbf{\Phi}$ and their asymptotic 95 percent Confidence Intervals

Parameter	MLE	S.E.	95 Percent CI		CI Length
λ	0.98	0.09	0.8	1.16	0.36
μ	1.34	0.82	-0.27	2.95	3.22
α_1	0.04	0.82	-1.57	1.65	3.22
β_1	-0.13	1.58	-3.23	2.96	6.19
β_2	-0.35	1.47	-2.27	2.53	4.80
β_3	0.18	1.72	-3.19	3.55	6.74
β_4	-0.21	1.54	-3.23	2.81	6.04
γ_{11}	-0.09	1.58	-3.19	3.01	6.20
γ_{12}	-0.06	1.47	-2.94	2.82	5.76
γ_{13}	-0.03	1.71	-3.38	3.32	6.70
γ_{14}	0.06	1.54	-2.96	3.08	6.04

Table 5.3. Empirical Bayes estimates

Prior	Unrestricted		Restricted	
Parameter	Estimated	S.E.	Estimated	S.E.
λ	0.7973	0.1057	1.2198	0.1616
μ	1.1502	0.2324	0.9858	0.1462
α_1	-0.0004	0.1302	-0.0006	0.1239
β_1	0.0216	0.2440	0.0177	0.2406
β_2	-0.0555	0.3223	0.0001	0.3131
β_3	0.0212	0.3733	0.0923	0.2107
β_4	-0.0467	0.3193	-0.0001	0.1025
γ_{11}	0.0005	0.1781	-0.0014	0.3202
γ_{12}	0.0012	0.2324	-0.0003	0.1422
γ_{13}	-0.0001	0.2392	0.0005	0.1552
γ_{14}	0.0016	0.2362	-0.0022	0.1471

Table 5.4. The 95 percent credible intervals for Empirical Bayes estimates

Prior	Unrestricted			Restricted		
Parameter	95 Percent CI		CI Length	95 Percent CI		CI Length
λ	0.5901	1.0044	0.4139	0.9031	1.5365	0.6334
μ	0.6947	1.6057	0.9110	0.6992	1.2724	0.5732
α_1	-0.2556	0.2548	0.5104	-0.2434	0.2422	0.4856
β_1	-0.4566	0.4998	0.9564	-0.4539	0.4893	0.9432
β_2	-0.6872	0.5763	0.5950	-0.6136	0.6138	1.2274
β_3	-0.7105	0.7529	1.4664	-0.3207	0.5053	0.8260

β_4	-0.6725	0.5721	1.2516	-0.2010	0.2008	0.4018
γ_{11}	-0.3486	0.3496	0.6982	-0.6290	0.6262	1.2552
γ_{12}	-0.4543	0.4567	0.9110	-0.2790	0.2784	0.5574
γ_{13}	-0.4689	0.4687	0.9376	-0.3037	0.3047	0.6084
γ_{14}	-0.4614	0.4646	0.926	-0.2905	0.2861	0.5766

(5.2). It is clear that except for λ no parameter can be taken different from zero at 5 percent level. However, only the constant μ is different from zero at 10 percent level. But, due to wide confidence intervals one should feel uncertain about these inferences. Better inferences are possible by the empirical Bayes procedure presented below.

Empirical Bayes estimates of a and b are a_0 =6.95 and b_0 =0.26, respectively. The estimates of the model parameters for each of the two cases (unrestricted and restricted) along with the standard errors are shown in Table (5.3). In Table (5.4), the 95 percent credible intervals (CI) based on the marginal posteriors are given. Again, we observe that only λ and μ are inferred to be different from zero at 5 percent level. Although we have reached the same conclusion as the one relative to **MLE**, but here we have much smaller standard errors, which make the inference more precise. In fact, comparing Table (5.2) and (5.3), we note a striking consequence of exploiting the empirical Bayes procedure. The credible intervals in both cases (unrestricted and restricted) are very much shorter than the corresponding confidence intervals given in Table (5.2). The only exception is the intervals for λ which have become somewhat longer for the empirical Bayes procedure.

There are various ways to check the adequacy of proposed model. They include Lack of fit criteria and predictive distribution, to name a few. We shall not discuss such methods for the sake of brevity and refer to standard errors of estimates and their corresponding co-nfidence intervals as determinants of model adequacy.

6. Some Comments

We have proposed an empirical Bayes procedure for analysis of the data from a two-factor experiment when they are assumed to follow an Inverse Gaussian Distribution. Conjugate priors have been used. The reasons are mathematical tractibility and "objectivity" requirements as expounded in Robert [12]. The posterior distributions have been used as the basis of any inference about the factor effects and their interactions. Though not worked there, one could have utilized various alternative priors like Jefferys', reference and diffuse priors. Each alternative merits a separate article. In a data analytic effort, it would be useful to try such alternatives in hope of finding the "best" prior. However, this goal has not been of prime interest in this work. Consequently, we have confined ourself to the conjugate Priors.

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Appendix 1

Observe that from(2.11), we have

$$q_2(\phi_1|\lambda,\mathbf{y}) \propto \left|\lambda^{-1}\psi_{11}\right|^{1/2}\exp\{-\lambda\,\psi_{11}^{-1}(\phi_1-\eta_1^*)^2/2\},$$

$$\phi_1 > 0,$$

which is a normal density truncated at zero. For this distribution,

$$E(\phi_1|\lambda,\mathbf{y}) = \eta_1^* + (\lambda^{-1}\psi_{11})^{1/2}\omega$$

with

$$\omega = \varphi\left[\eta_1^*/(\lambda^{-1}\psi_{11})^{1/2}\right]N\left[\eta_1^*/(\lambda^{-1}\psi_{11})^{1/2}\right]$$

where $\varphi(.)$ is the standard normal pdf and N(.) is its cdf,

$$Var(\phi_1|\lambda,\mathbf{y}) = \lambda^{-1}\psi_{11}\left[1-\omega^2\right] + \eta_1^*\,\omega(\lambda^{-1}\psi_{11})^{1/2}.$$

These moments, however, are too complicated to be useful for estimation purposes. To simplify them, we observe that $\omega(x)=\varphi(x)/N(x)$ is a smooth decreasing function of x. In the literature there are a host of approximations to $N(x)$ (Patel and Read [11]), which can be used to approximate $\omega(x)$ with desired precision. Here, we choose to use the simpler one due to Shah [14],

$$\mathrm{N}(x) = \begin{cases} 0.5 + x(4.4-x)/10 & 0 \le x \le 2.2, \\ 0.99, & 2.2 \le x \le 2.6, \\ 1.00 & x \ge 2.6. \end{cases}$$

Consequently,

$$\omega(x) = \begin{cases} \dfrac{2(2.2-x)}{5+(4.4-x)} & 0 \le x \le 2.2 \\ 0, & x > 2.2 \end{cases}$$

In our problem, $0 \le x \le \infty$ and $0 \le \omega(x) \le 0.8$. Thus, we can approximate $\omega(x)$ by its average value, which is about 0.3. Of course, if one has a better guess of $x = \eta_1^*/(\lambda^{-1}\psi_{11})^{1/2}$, a closer approximation could be obtained. Using this approximation, we have

$$E(\phi_1|\lambda,\mathbf{y}) = \eta_1^* + 0.3(\lambda^{-1}\psi_{11})^{1/2}$$

and

$$Var(\phi_1|\lambda,\mathbf{y}) = 0.91\lambda^{-1}\psi_{11} + 0.3\eta_1^*(\lambda^{-1}\psi_{11})^{1/2}$$

Now, we take expectations with respect to the posterior distribution of λ and obtain the Bayes estimate

$$\phi_{1B2} = E(\phi_1|\ \mathbf{y}) = \eta_1^* + 0.21\mathbf{W}\left[\psi_{11}Q_1(\boldsymbol{\eta})\right]^{1/2}$$

and

$$\begin{aligned} V(\phi_1|\mathbf{y}) &= 0.45\psi_{11}Q_1(\boldsymbol{\eta})/(k-1) \\ &\quad + 0.045\psi_{11}Q_1(\boldsymbol{\eta})\left[11/(k-1)-\mathbf{W}^2\right] \\ &\quad + 0.21\eta_1^*\left[\psi_{11}Q_1(\boldsymbol{\eta})\right]^{1/2}\mathbf{W}. \end{aligned}$$

Appendix 2

Note that

$$(R_{ijk}|\lambda,\mathbf{\Phi}) \sim \left[\theta_{ij}^{-1}+\lambda^{-1}, (\lambda\theta_{ij})^{-1}+2\lambda^{-2}\right],$$

$$(R_{ij.}|\lambda,\mathbf{\Phi}) \sim \left[\theta_{ij}^{-1}+\lambda^{-1}, (n\lambda\theta_{ij})^{-1}+2(n\lambda^2)^{-1}\right],$$

$$(R_{i..}|\lambda,\mathbf{\Phi}) \sim \left[(\mu+\alpha_i)+\lambda^{-1}, (n\lambda J)^{-1}(\mu+\alpha_i)+(2n\lambda^{-2}J)^{-1}\right],$$

$$(R_{.j.}|\lambda,\mathbf{\Phi}) \sim \left[(\mu+\beta_i)+\lambda^{-1}, (n\lambda I)^{-1}(\mu+\beta_i)+(2n\lambda^{-2}I)^{-1}\right],$$

$$(R_{...}|\lambda,\mathbf{\Phi}) \sim \left[\mu++\lambda^{-1}, (n\lambda IJ)^{-1}\mu+(2n\lambda^{-2}IJ)^{-1}\right],$$

To obtain the marginal moments, we take expectations successively with respect to λ and $\mathbf{\Phi}$. In Case 1, we obtain for $i=1,\ldots,I$, $j=1,2,\ldots,J-1$:

$$\begin{aligned} R_{ij.} \sim \{&(\eta_1+\eta_{i+1}+\eta_{I+J}+\eta_{I+i(J-1)+j}+b/2(a-1), \\ &b\Big[\eta_1+\eta_{i+1}+\eta_{I+J}+\eta_{I+i(J-1)+j} \\ &\quad +n(\delta_1^2+\delta_{i+1}^2+\delta_{I+j}^2+\delta_{I+i(J-1)+i}^2\Big]/2n(a-1)\}, \end{aligned}$$

$$\begin{aligned} R_{i..} \sim \{&(\eta_1+\eta_{i+1}+b/2(a-1), \\ &b\left[\eta_1+\eta_{i+1}+nJ(\delta_1^2+\delta_{i+1})\right]/2nJ(a-1), \\ &\quad +b^2\left[2(a-1)+nJ\right]/4nJ(a-1)^2(a-2)]\}, \end{aligned}$$

$$\begin{aligned} R_{.j.} \sim \{&(\eta_1+\eta_{I+1}+b/2(a-1), \\ &b\left[\eta_1+\eta_{I+}\,j+nI(\delta_1^2+\delta_{I+j})\right]/2nI(a-1), \\ &\quad +b^2\left[2(a-1)+nI\right]/4nI(a-1)^2(a-2)]\}, \end{aligned}$$

$$
\begin{aligned}
R_{\ldots} \sim \{(&\eta_1 + b/2(a-1), \\
&b\left[\eta_1 + nIJ\,\delta_1^2\right]/2nIJ(a-1), \\
&\quad + b^2\left[2(a-1) + nIJ\right]/4nIJ(a-1)^2(a-2)\}.
\end{aligned}
$$