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A SHORT PROOF FOR THE EXISTENCE OF HAAR MEASURE ON COMMUTATIVE HYPERGROUPS

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Abstract

In this short note, we have given a short proof for the existence of the Haar measure on commutative locally compact hypergroups based on functional analysis methods by using Markov-Kakutani fixed point theorem.

Keywords: Hypergroups; Haar measure; Markov-Kakutani

Introduction

A fundamental open question about hypergroups is the existence of a Haar measure for any hypergroup (for a definition the reader can consult with Jewett [6]). If a hypergroup K is compact or discrete, then K possesses a Haar measure. All known examples have a Haar measure [6, s5]. Spector in [11] claims that any commutative hypergroup possesses a Haar measure but as Ross in [9] mentioned there are several technical problems in his proof. Ross in [9] has given a lengthy proof for the existence of the Haar measure on commutative hypergroups. Recently Izzo in [5] has given a short proof of the existence of Haar measure on a commutative locally compact group by using the Markov-Kakutani fixed-point theorem [1, pp. 151-152]. Based on his idea, we give a short proof of the existence of the Haar measure on commutative hypergroups. For definitions and notations we follow Jewett [6].

For the reader's convenience, we include the Markov-Kakutani fixed point theorem. Let $\cal S$ be a compact convex subset of a Hausdorff topological vector space and $\cal F$ be a commutative family of continuous affine mappings of $\cal S$ into $\cal S$ that is abelian. Then there exists $p\in \cal S$ such that $\Lambda(p) = p$ for all $\Lambda\in \cal F$ (for a proof see [1]).

Note 1.1. For a vector space X, let $X^{\#}$ be the space of all linear functionals on X with the weak topology induced by X. Then, if C is a closed subset of $X^{\#}$ such that the set $\{\Lambda x:\, \Lambda\in C\}$ is bounded, for any $x\in X,$ then C is compact [3, pp. 423-424].

Lemma 1.2. Let K be a hypergroup and U a symmetric neighborhood of the identity $e\in K$. Then there exists a subset M of K such that for any finite subset $\{a_1, a_2,\cdots a_n\}$ of K, the set $a_1*a_2*\cdots*a_n*U*U$ contains at least one element of M and the set $a_1*a_2*\cdots*a_n*U$ contains at most one element of M.

Proof. Let ${\cal A}=\{T\subseteq K:\, \mbox{for any} p\neq q\in T, \mbox{there is a finite subset}\{a_1,a_2,\cdots,a_n\} \mbox{of} K \mbox{such that} p\notin q*A*\br{A},\mbox{where}\br{A}=U*\br{a_n}*\cdots *\br{a_1}\}$.

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Then $\cal A$ is non-empty (all single subsets of K are in $\cal A$) and any chain $\{T_\al\}_{\al\in I}$ in $\cal A$ has an upper bound $\cup_{\al\in I} T_\al$. So by Zorn's Lemma $\cal A$ has a maximal element M. By using [6, 4.1A, 4.1B], we have $M\cap a*U*U\neq \emptyset$. Now for $\{a_1, a_2,\cdots,a_n\}$ an arbitrary finite subset of K, we have $ M\cap a_1*a_2*\cdots*a_n*U*U=M\cap(\cup_{x\ina_1*a_2*\cdots*a_n}\,x*U*U)=\cup_{x\ina_1*a_2*\cdots*a_n}\,(M\cap x*U*U)\neq\emptyset$.

To show that M intersects $a_1, a_2,\cdots, a_n*U*U$ at most at one point, let there are s_1 and s_2 in M that $s_1\neq s_2$ and $s_i\in a_1*a_2*\cdots*a_n*U$ for $i=1,2.$ Then by using [6, 4.1A, 4.1B] we have $s_1\in s_2*A*\br{A}$, where A is $U*\br{a_n}*\cdots*\br{a_2}$ and this contradicts $M\in {\cal A}$. So the proof of the Lemma is complete.

Theorem 1.3. Every commutative hypergroup K has a left Haar measure.

Proof. Let $C_{00}(K)^{\#}$ be the space of all linear functionals on $C_{00}(K)$. We consider on $C_{00}(K)^{\#}$ the weak topology generated by $C_{00}(K)$. It is clear that if there exists a $\Lambda\in C_{00}(K)^{\#}$ such that $f(\Lambda) =0$ for all $f\in C_{00}(K),$ then $\Lambda = 0$. So $C_{00}(K)^{\#}$ with this topology is a locally convex space [4, p. 50]. Let U be a fixed symmetric neighborhood of the identity $e\in K$ with compact closure. Let $\cal S$ be the set of all positive linear functionals Λ on $C_{00}(K)$ that satisfy the following two conditions:

(i) $\Lambda(f)\leq 1$ whenever $f\leq 1$ in $C_{00}^+(K)$ and $spt f\subseteq a_1*a_2*\cdots*a_r*U$ for some finite subset $\{a_1, a_2, \cdots,a_r\}$ in K,

(ii) $\Lambda(f)\geq 1$ whenever $f\leq 1$ in $C_{00}^+(K)$ and $f=1$ on $a_1*a_2*\cdots*a_r*U*U$ for some finite subset $\{a_1, a_2,\cdots,a_r\}$ in K.

Then one can easily check that $\cal S$ is closed and convex. Moreover, any $f\in C_{00}^+(K)$ can be written as a finite sum of non-negative continuous functions, each of which has support in $a*U$ for some $a\in K.$ To see this, let $spt f=C,$ (compact set). Then $C\subseteq \cup_{1\leq i\leq n}\, a_i*U$ for some $a_i\in K$, $1\leq i\leq n$. By the partition of unity on compact sets, there are $h_i\in C_{00}^+(K)$ such that $0<\frac{h_i}{f}\leq 1$ on C. That is for any $x\in C$, $0<h_i(x)\leq f(x)$ and $h_1(x)+h_2(x)+\cdots +h_n(x)= f(x)$. Now it follows from (i) that the set $\{ \Lambda(f):\, \Lambda\in \cal S \}$ is bounded. So by Note 1.1., $\cal S$ is compact.

To see $ \cal S$ is non-empty, let M be as in Lemma 1.2. Put $\Lambda(f) = \sum_{s\in M}\, f(s)$, then $\Lambda \in {\cal S}$. Indeed, if $f\in C_{00}^+(K)$ and $f\leq 1$ with $spt f\subseteq a_1*a_2*\cdots*a_n*U$ for some $a_i\in K, 1\leq i\leq n$, then by Lemma 1.2., M intersects $a_1*a_2*\cdots*a_n*U$ at most at one point. Hence $\Lambda(f) \leq 1$. If $f\in C_{00}^+(K)$ and $f = 1$ on $a_1*a_2*\cdots*a_n*U*U$ for some $a_i\in K, 1\leq i\leq n$, then again by Lemma 1.2., M intersects $a_1*a_2*\cdots*a_n*U*U$ at least at one point. So $\Lambda(f) \geq 1$.

For each $x\in K$ and $\Lambda\in \cal S$, let $T_x:\, C_{00}(K)^{\#}\to C_{00}(K)^{\#}$ is defined by $T_x\Lambda(f)= \Lambda(\,_xf)$ for $f\in C_{00}(K)$ where$\,_xf(y)=f(xy)$. Then it is easy to see that T_x is affine and $T_x(\cal S)\subseteq \cal S$. Indeed, let $\Lambda\in \cal S$. If $f\in C_{00}^+(K)$ and $f\leq 1$ with $spt f\subseteq a_1*a_2*\cdots*a_n*U$ for some $a_i\in K,\quad 1\leq i\leq n$, then $\,_xf\in C_{00}^+(K)$ [6, 4.2E] and $\,_xf \leq 1$ with $spt(_xf)\subseteq \br{x}*a_1*a_2*\cdots*a_n*U$. So by (i) $\Lambda (_xf)\leq 1$. If $f\in C_{00}^+(K)$ and $f= 1$ on $a_1*a_2*\cdots*a_n*U*U$ for some $a_i\in K, 1\leq i\leq n$, then $\,_xf\in C_{00}^+(K)$ and $\,_xf=1$ on $\br{x}*a_1*a_2*\cdots*a_n*U*U$. So by (ii), $\ Lambda(_xf)\geq 1$.

Also T_x is continuous, since if $\lim_\al\, \Lambda_\al=\Lambda$ in ${\cal S}$, then for any $f\in C_{00}(K)$, $\lim_\al|T_x\Lambda_\al(f)-T_x\Lambda (f)| = \ lim_\ al|\ Lambda_\ al(_xf)-\ Lambda (_xf)| = 0$. Moreover for $x,y\in K$, $T_x(T_y\Lambda) = T_{x*y}\, \Lambda = T_{y*x}\, \Lambda = T_y (T_x\ Lambda)$ for any $\Lambda\in C_{00}(K)^{\#}$. This shows that the family ${\cal F}=\{T_x:\,x\in K\}$ and $\cal S$ (as above) have all properties in Markov-Kakutani fixed-point theorem. So there exists $\Lambda_0\in \cal S$ such that $T_x\,\Lambda_0= \Lambda_0$ for all $x\in K$. In other words $T_x(\Lambda_0f) =\Lambda_0(_xf) = \Lambda_0(f) \mbox{for all} a\in K \mbox{and} f\in C_{00}(K)$.

Now since all elements of $\cal S$ are non-zero positive linear functionals on $C_{00}(K)$, by [6, s5.2] the proof is complete.

Remark 1.4. Can the above proof be modified to show that every amenable hypergroup has a left Haar measure, using Day's generalization of Markov-Kakutani fixed-point theorem [2, Theorem 1] (see also [7, Theorem 4.2]). (For an extension to hypergroups see

[10, Theorem 3.3.1].)

It is attempted such modification, but there is a problem in the continuity of action of hypergroup K on $\cal S$ (metioned earlier) defined by $(x, \Lambda) \longmapsto T_x\Lambda \mbox{where} T_x\Lambda (f)=\Lambda(_xf) \mbox{for} f\in C_{00}(K)$.

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