IMPROVED ESTIMATOR OF THE VARIANCE IN THE LINEAR MODEL

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Abstract

The improved estimator of the variance in the general linear model is presented under an asymmetric linex loss function.

Keywords: Equivariant estimator; Normal variance estimator; Improved estimator; Linex loss function

1. Introduction

Consider the canonical form of the general linear model and suppose \( X \sim N(\mu, \tau I) \) and \( U \sim N(0, \tau I) \) are to be independently observed. On the basis of these observations, \( \tau \) is to be estimated, where the loss function is given by

\[
L(\tau, \delta) = b \left[ a \left( \frac{\delta}{\tau} - 1 \right)^{a-1} - a \left( \frac{\delta}{\tau} - 1 \right) - 1 \right],
\]

where \( a \neq 0 \) is a shape parameter and \( b > 0 \) is a scale parameter. This loss function which was introduced by Varian [1] and was extensively discussed by Zellner [2], is useful when overestimation is regarded as more serious than underestimation or vice versa. In this regard see Parsian and Sanjari Farsipour [3].

A sufficient statistic in this problem is \((X, T)\), where if \( ||.|| \) denotes the usual Euclidean norm, \( T = ||U||^2 \).

2. MLE and Bayes Estimators

With \( U \) unobserved, we can write down the likelihood function, given our normality assumptions, and easily obtain the maximum likelihood estimator. The likelihood function is

\[
L(\mu, \tau) = (2\pi)^{-\frac{n}{2}} e^{-\frac{1}{2\tau} U'U} \left\{ -\frac{1}{2\tau} (X - \mu)'(X - \mu) - \frac{1}{2\tau} U'U \right\}.
\]

So we have \( X \) as an MLE of \( \mu \), and \( \frac{1}{2} \sum_{i=1}^{n} U_i^2 \) as an MLE of \( \tau \). Now, we calculate the risk function relative to the loss function in (1.1) of \( T = \sum_{i=1}^{n} U_i^2 \), we have

\[
R(\tau, \hat{\tau}) = e^{-a(1-a)^{-\frac{a}{2}}} - \frac{am}{2} + a - 1
\]

(2.1)

Now, let \( \lambda = \tau^{-1} \), and introducing a diffuse prior, as the one cited in the article by Zellner [1], \( i.e. \), \( \pi(\lambda) = \frac{1}{\lambda} \), we can derive an optimal estimate that minimizes the posterior expected loss of our loss function in (1.1), as a solution of the following equation

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\[ E_\beta [\Lambda e^{\alpha \delta_b} \mid T = t] = e^a E_\beta [\Lambda \mid T = t]. \] (2.2)

Hence, the Bayes estimator is
\[ \delta_B = \frac{1}{e^a}(1 - e^{-\beta t})T. \]

Now we are able to obtain the risk function associated with this estimator as the following equation
\[ R(\lambda, \delta_B) = \frac{1}{2} \left( 1 + e^{-\beta t} \right)^n e^{-\alpha a} + \frac{n}{2} e^{-\frac{\alpha a}{2}} - \frac{n}{2} + a - 1, \] (2.3)

and we can compare it with that we already derived under the assumption that \( U \) is observed. Obviously \( \delta_B \) works better than \( T \), since it is the best invariant estimator, and \( T \) is an invariant estimator.

For the loss function of the form
\[ L(\delta, \lambda) = (\delta - \lambda)^2, \]
the problem was solved by some authors such as Brewster and Zidek [4] as well as Hodges and Lehmann [5].

### 3. Improved Estimators

The problem remains invariant under the transformation group \( A \) under which
\[
\begin{align*}
(X, T) &\rightarrow (\alpha T + \beta, \alpha^2 T) \\
(\mu, \tau) &\rightarrow (\alpha T \mu + \beta, \alpha^2 \tau) \\
\delta &\rightarrow \alpha^2 \delta
\end{align*}
\] (3.1)

where \( \alpha > 0, \beta \in \mathbb{R} \) and \( \Gamma \) is a \( p \times p \) orthogonal matrix. It follows that any nonrandomized \( A \)-invariant estimator of \( \tau \) is of the form \( cT \), for some constant \( c > 0 \). Since \( A \) acts transitively on the parameter space, the risk function of \( cT \),
\[ R(\mu, \tau; \delta) = E_{\mu, \tau} \left[ \frac{\rho(\phi(z)T)}{\tau} \right] = R(\xi_1, \xi_2, \ldots, \xi_p), \]

where \( \xi = (\xi_1, \xi_2, \ldots, \xi_p) \), and \( \xi_i = |x_i|^2 + 1 \), \( i = 1, \ldots, p \). We can see that the risk of such an estimator is

Let \( \mathcal{H} \) denote the subgroup of \( A \) obtained by requiring in (3.1) that \( \beta = 0 \) and that \( \Gamma \) be a diagonal orthogonal matrix. Any \( \mathcal{H} \)-invariant estimator is of the form \( \phi(Z)T \), where \( Z = (Z_1, Z_2, \ldots, Z_p) \) and \( Z_i = X_i | T_i^2, i = 1, \ldots, p \).

Now, for any estimator \( \phi(Z)T \) define \( \phi^*(z) = \min \{ \phi(z), \phi_o(z) \} \), then let
\[ R(\xi; \phi) = E_\xi \left[ E[\rho(\phi(Z)T) \mid Z, K] \right] \]

and for the loss function (1.1), \( c^* \) is a multiplier of \( \sum_{i=1}^{p} X_i^2 \) [3].

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where \( \xi = (\xi_1, \xi_2, \ldots, \xi_p) \) and \( \xi_i = |x_i|^2 + 1 \), \( i = 1, \ldots, p \).

Since we deal only with \( \mathcal{H} \)-invariant estimators, we may assume without loss of generality that \( \tau = 1 \).

On the other hand, \( X_i^2 \) has a chi-squared distribution with \( 1 + 2K_i \) degrees of freedom, where \( K_i \) denotes a Poisson random variable with mean \( \lambda_i = \frac{1}{\sigma^2} \), and the \( K_i, i = 1, \ldots, p \), are independent of each other and of \( T \). Let \( K = (K_1, K_2, \ldots, K_p) \), the joint density of \( T \) and \( Z \) conditional on \( K = k \) is
\[ f_{T,Z}(t, z \mid k) \propto t^{\frac{p}{2} + K} e^{-\frac{t}{2} \lambda_k} \Pi_{i=1}^{p} e^{-\frac{z_i^2}{2}}, \]

Independent of \( \xi \), where \( k = \sum_{i=1}^{p} k_i \).

Now, since the loss (1.1) is strictly convex, it uniquely minimized at \( \phi(z) \) satisfying
\[ E[\rho(\phi(z)T) \mid Z = z, K = k] = 0 \]

which is equivalent to
\[ E[Te^{a\phi k} \mid Z = z, K = k] = e^a E[T \mid Z = z, K = k]. \]

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k, it follows that \( R(\phi^*(z) \mid z, k) < R(\phi(z) \mid z, k) \), see Figure 2.1, which is also cited in Maatta and Casella [6] in the univariate set up. Therefore, for any \( \xi, R(\hat{\xi}, \phi^*) \leq R(\hat{\xi}, \phi) \) with inequality if \( P_\xi(\phi^*(Z) \neq \phi(z)) > 0 \). Now, let \( \phi(z) = c^* = \frac{1}{2\alpha}(1 - e^{-\frac{z^2}{2\alpha^2}}) \), then to find \( \phi_c(z) \) in this case, note that

\[
R(c \mid z, O) \propto \int \rho(c t) t^{(n+1)p-1} e^{-\frac{z^2}{4(1+||z||^2)}} dt.
\]

So, using the transformation \( t \rightarrow t(1+||z||^2) \), we can see that

\[
R(c \mid z, O) \propto \int \rho(\tilde{c} T) T^{\frac{1}{2}} e^{-\frac{\tilde{c}^2}{4t(1+||z||^2)}} dt
\]

\[
\propto E\left[ \rho(\tilde{c} T) T^{\frac{1}{2}} \right]
\]

where \( \tilde{c} = c/(1+||z||^2) \), so the minimum is attained at \( \tilde{c} = \phi_c(z)/(1+||z||^2) \). For finding the value of \( \tilde{c} \), using (2.2), \( \tilde{c} \) must satisfy the following relation

\[
E\left[ T^{\frac{1}{2}} e^{\tilde{c} T} \right] = e^{\alpha} E\left[ T^{\frac{1}{2}} + 1 \right]
\]

which is obtained by

\[
\tilde{c} = \frac{1}{2\alpha} \left( 1 - e^{-\frac{\delta^*}{2\alpha}} \right)
\]

Hence, \( \phi_c(z) = \frac{1}{2\alpha}(1 - e^{-\frac{\delta^*}{2\alpha}})(1+||z||^2) \), and so by the above discussion \( e^* T \) is dominated by

\[
\delta^* = \min \{ e^*, \tilde{c} (1+||z||^2) \} T.
\]

References