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Research Note

NUMERICAL SOLUTION OF LINEAR FREDHOLM AND VOLTERRA INTEGRAL EQUATION OF THE SECOND KIND BY USING LEGENDRE WAVELETS

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Abstract

In this paper, we use the continuous Legendre wavelets on the interval [0,1] constructed by Razzaghi M. and Yousefi S. [6] to solve the linear second kind integral equations. We use quadrature formula for the calculation of the products of any functions, which are required in the approximation for the integral equations. Then we reduced the integral equation to the solution of linear algebraic equation.

Keywords: Fredholm and Volterra integral equations; Legendre wavelets; Operational matrix; Product operation

Introduction

In recent years, wavelets have found their way into many different fields of science and engineering. Wavelets constitute a family of signal functions constructed from dilation and translation of a signal function called the mother wavelet. When the dilation parameter a and the translation parameter b vary continuously, we have the following family of continuous wavelets [1].

$$\psi_{a,b}(t) = |a|^{-\frac{1}{2}}\psi\left(\frac{t-b}{a}\right), \qquad a,b \in R, \quad a \neq 0.$$

Legendre wavelets $\psi_{m,n}(t) = \psi(k,\hat{n},m,t)$ have four arguments: $k = 2,3,\ldots$, $\hat{n} = 2n-1$, $n = 1,2,3,\ldots,2^{k-1}$, m is the order for Legendre polynomials and t is the normalized time. They are defined on the interval $[0,1)$ by:

$$\psi_{m,n}(t) = \begin{cases} (m+\frac{1}{2})^{\frac{1}{2}} 2^{k/2} L_m(2^k t - \hat{n}) & \frac{\hat{n}-1}{2^k} \le t < \frac{\hat{n}+1}{2^k} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

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Hear, $L_m(t)$ are the well-known Legendre polynomials of the order *m*, which are orthogonal to the weight function $w(t)=1$ and satisfy the following recursive formula:

$$L_0(t)=1,$$
$$L_1(t)=t,$$
$$L_{m+1}(t)=\frac{2m+1}{m+1}tL_m(t)-\frac{m}{m+1}L_{m-1}(t),\quad m=1,2,3,\ldots$$

The set of Legendre wavelets are an orthonormal set [6,7].

Function Approximation

A function $f(t)\in L^2[0,1)$ may be expand as:

$$f(t)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}c_{n,m}\psi_{n,m}(t), \tag{2}$$

where

$$c_{n,m}=(f(t),\psi_{n,m}(t)). \tag{3}$$

In (2.2), (. , .) denotes the inner product.

If the infinite series in (2) are truncated, then (2) can be written as:

$$f(t)\approx\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{n,m}\psi_{n,m}(t)=C^T\Psi(t), \tag{4}$$

where C and $\Psi(t)$ are $2^{k-1}M\times 1$ matrixes given by:

$$C=[c_{10},c_{11},\ldots,c_{20},\ldots,c_{2,M-1},\ldots,c_{2^{k-1},0},\ldots,c_{2^{k-1},M-1}]^T$$
$$=[c_1,c_2,\ldots,c_{2^{k-1}M}]^T, \tag{5}$$

and

$$\Psi(t)=[\psi_{1,0}(t),\psi_{1,1}(t),\ldots,\psi_{1,M-1}(t),\ldots,\psi_{2,0}(t),\ldots,$$
$$\psi_{2,M-1}(t),\ldots,\psi_{2^{k-1},0}(t),\ldots,\psi_{2^{k-1},M-1}(t)]^T \tag{6}$$
$$=[\psi_1(t),\psi_2(t),\ldots,\psi_{2^{k-1}M}(t)]^T.$$

Similarly a function $k(t,s)\in L^2[0,1)\times[0,1)$ may be approximated as:

$$k(t,s)\approx\Psi^T(t)K\Psi(s), \tag{7}$$

where K is an $2^{k-1}M\times 2^{k-1}M$ matrix, with:

$$K_{ij}=(\psi_i(t),(k(t,s),\psi_j(s))).$$

The integration of the vector $\Psi(t)$ defined in equation (6) can be obtained as:

$$\int_0^t\Psi(t')dt'=P\Psi(t), \tag{8}$$

where P is the $2^{k-1}M\times 2^{k-1}M$ operational matrix for integration and is given in [6] as:

$$P=\frac{1}{2^k}\begin{bmatrix} L & F & F & \cdots & F & F\\ 0 & L & F & \cdots & F & F\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & L & F\\ 0 & 0 & 0 & \cdots & 0 & L\end{bmatrix} \tag{9}$$

In (9) F and L are $M\times M$ matrixes given by:

$$F=\frac{1}{2^k}\begin{bmatrix} 2 & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0\end{bmatrix}$$

and

$$L = \frac{1}{2^k}\begin{bmatrix} 1 & \frac{1}{\sqrt{3}} & 0 & 0 & \cdots & 0 & 0 \\ \frac{-\sqrt{3}}{3} & 0 & \frac{\sqrt{3}}{3\sqrt{5}} & 0 & \cdots & 0 & 0 \\ 0 & \frac{-\sqrt{5}}{5\sqrt{3}} & 0 & \frac{\sqrt{5}}{5\sqrt{7}} & \cdots & 0 & 0 \\ 0 & 0 & \frac{-\sqrt{7}}{7\sqrt{5}} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{\sqrt{2M-3}}{(2M-3)\sqrt{2M-1}} \\ 0 & 0 & 0 & 0 & \cdots & \frac{-\sqrt{2M-1}}{(2M-1)\sqrt{2M-3}} & 0 \end{bmatrix}$$

The integration of the product of two Legendre wavelet function vectors is obtained as:

$$I = \int_0^1 \Psi(t)\Psi^T(t)dt\ , \tag{10}$$

where I is an identity matrix.

The following property of the product of two Legendre wavelet vector functions will also be used:

$$\Psi(t)\Psi^T(t)C \approx \tilde{C}^T\Psi(t)\,, \tag{11}$$

where C is given in (5) and $\tilde{C}$ is a $2^{k-1}M \times 2^{k-1}M$ matrix, which is called the product operation matrix of Legendre wavelet vector functions [6].

Linear Integral Equations

Fredholm Integral Equation of the Second Kind

Consider the following integral equation:

$$y(t) = \int_0^1 k(t,s)y(s)ds + x(t)\,, \tag{12}$$

where $x \in L^2[0,1), k \in L^2[0,1)\times[0,1)$ and y is an unknown function [2,3]. If we approximate x, y and k by the way mentioned before:

$$\begin{aligned} &x(t) \approx X^T\Psi(t),\ y(t) \approx Y^T\Psi(t), \\ &k(t,s) \approx \Psi^T(t)K\Psi(s)\,. \end{aligned} \tag{13}$$

With substituting in (3.1) we have:

$$\begin{aligned} \Psi^T(t)Y &= \Psi^T(t)X + \int_0^1 \Psi^T(t)K\Psi(s)\Psi^T(s)Yds \\ &= \Psi^T(t)X + \Psi^T(t)K\left(\int_0^1 \Psi(s)\Psi^T(s)ds\right)Y \\ &= \Psi^T(t)(X + KY). \end{aligned}$$

then

$$(I-K)Y = X\ . \tag{14}$$

Volterra Integral Equation of the Second Kind

For the following Volterra integral equation [2,5].

$$y(t) = \int_0^t k(t,s)y(s)ds + x(t)\,, \tag{15}$$

with (8) and (11) we have:

$$\begin{aligned} \int_0^t k(t,s)y(s)ds &\approx \int_0^t \Psi^T(t)K\Psi(s)\Psi^T(s)Yds \\ &= \Psi^T(t)K\int_0^t \Psi(s)\Psi^T(s)Yds \\ &= \Psi^T(t)K\int_0^t \tilde{Y}^T\Psi(s)ds \\ &= \Psi^T(t)K\tilde{Y}^T P\Psi(t) \end{aligned}$$

then

$$y(t) = x(t) + \Psi^T(t)K\tilde{Y}^T P\Psi(t)\ . \tag{16}$$

By evaluating this equation in $2^{k-1}M$ points $\{t_i\}_{i=1}^{2^{k-1}M}$ in the interval [0,1) we have a system of linear equations:

$$\Psi^T(t_i)Y = x(t_i) + \Psi^T(t_i)K\tilde{Y}^T P\Psi(t_i) \qquad i = 1,2,\ldots,2^{k-1}M \tag{17}$$

Numerical Examples

We first let $M = 3$ and $k = 2$. The six basis functions are given by:

$$\left.\begin{aligned} \psi_1(t) &= \sqrt{2} \\ \psi_2(t) &= \sqrt{6}(4t-1) \\ \psi_3(t) &= \sqrt{10}\left(\frac{3}{4}(4t-1)^2 - \frac{1}{2}\right) \end{aligned}\right\} \quad 0 \le t < \frac{1}{2}$$
$$\left.\begin{aligned} \psi_4(t) &= \sqrt{2} \\ \psi_5(t) &= \sqrt{6}(4t-3) \\ \psi_6(t) &= \sqrt{10}\left(\frac{3}{4}(4t-3)^2 - \frac{1}{2}\right) \end{aligned}\right\} \quad \frac{1}{2} \le t < 1 \tag{18}$$

Example 1

$$y(t) = \int_0^1 \ln|t-s|\, y(s)ds + t - \frac{1}{2}\left(t^2 \ln t + (1-t)^2 \ln(1-t) - (t + \frac{1}{2})\right).$$

With exact solution $y(t) = t$. Table 1 and Figure 1 are the numerical results for Example 1.

Example 2

$$y(t) = \int_0^1 \left(-\frac{1}{3} e^{2t - \frac{5}{3}s}\right) y(s)ds + e^{2t + \frac{1}{3}}.$$

With exact solution $y(t) = e^{2t}$. Table 2 and Figure 2 are the numerical results for Example 2.

Example 3

$$y(t) = \int_0^t (s^2t^2 - st)y(s)ds - \frac{3}{4}t^6 + \frac{1}{3}t^5 - t^4 + \frac{1}{2}t^3 + 3t - 1$$

With exact solution $y(t) = 3t - 1$. Table 3 and Figure 3 are the numerical results for Example 3.

Example 4

$$y(t) = \int_0^t (t + (6(t-s) - 4(t-s)^2)y(s)ds + (3-t)e^t - 2 - t - 4t^2$$

With exact solution $y(t) = e^t$. Table 4 and Figure 4 are the numerical results for Example 4.

Table 1.

x_i	$y(x_i)$	y_i	$\|y(x_i) - y_i\|$
0.0	0.0000000000	0.0000000118	0.0000000118
0.2	0.2000000000	0.1999999965	0.0000000035
0.4	0.4000000000	0.4000000117	0.0000000117
0.6	0.6000000000	0.5999999726	0.0000000274
0.8	0.8000000000	0.8000001816	0.0000001816
1.0	1.0000000000	0.9999995534	0.0000004466

Table 2.

x_i	$y(x_i)$	y_i	$\|y(x_i) - y_i\|$
0.0	1.000000	1.012990	0.012990
0.2	1.491825	1.487708	0.004116
0.4	2.225541	2.230965	0.005424
0.6	3.320117	3.307555	0.012561
0.8	4.953032	4.962956	0.009924
1.0	7.389056	7.348320	0.040736

Table 3.

x_i	$y(x_i)$	y_i	$\lvert y(x_i) - y_i \rvert$
0.0	-1.000000	-1.000000	0.000000
0.2	-0.400000	-0.400401	0.000401
0.4	0.200000	0.201107	0.001107
0.6	0.800000	0.797021	0.002979
0.8	1.400000	1.403141	0.003141
1.0	2.000000	1.990637	0.009363

Table 4.

x_i	$y(x_i)$	y_i	$\lvert y(x_i) - y_i \rvert$
0.0	1.000000	0.965175	0.034825
0.2	1.221403	1228185	0.006782
0.4	1.491825	1.509680	0.017855
0.6	1.822119	1.786376	0.035743
0.8	2.225541	2.205723	0.019818
1.0	2.718282	2.728838	0.010556

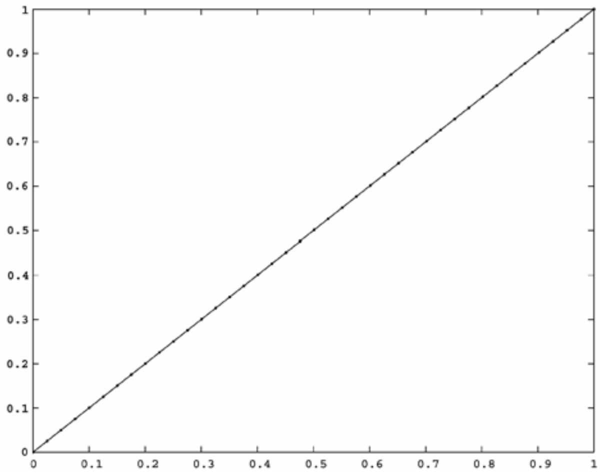

Figure 1.

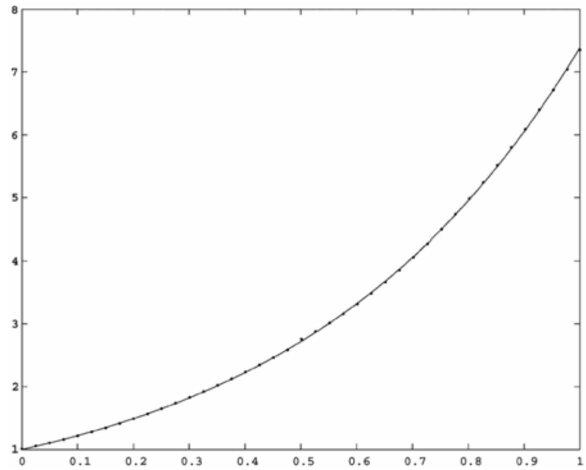

Figure 2.

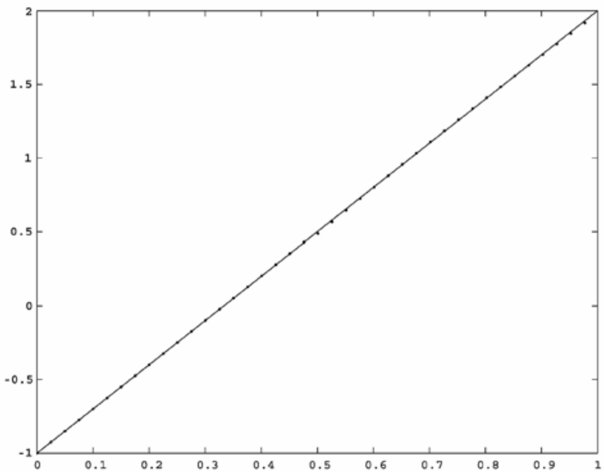

Figure 3.

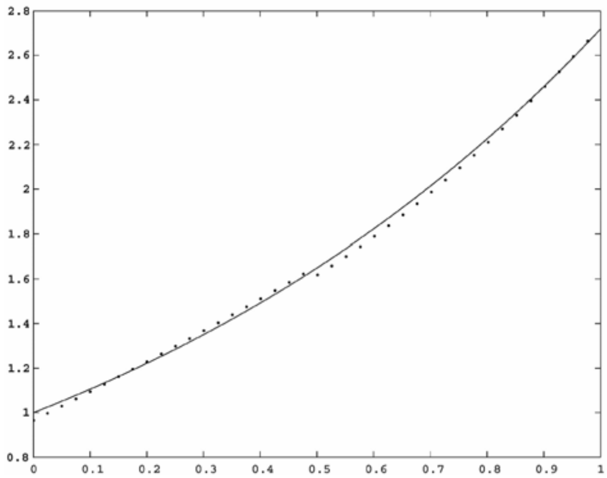

Figure 4.

Solid: true solution, Doted: numerical solution

Conclusion

The Legendre wavelet operational matrix *P*, together with the integration of the product of two Legendre wavelet vectors functions, are utilized to solve the integral equation. The present method reduces an integral equation into a set of algebraic equations. In this paper, we use the 6-base Legendre wavelets, the result for the product with quadrature solution is good. For better results, using the greater *N* is recommended.

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