Journal of Sciences, Islamic Republic of Iran 17(3): 253-258 (2006) http://jsciences.ut.ac.ir
University of Tehran, ISSN 1016-1104

Minimax Estimation of the Scale Parameter in a Family of Transformed Chi-Square Distributions under Asymmetric Squared Log Error and MLINEX Loss Functions

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Abstract

This paper is concerned with the problem of finding the minimax estimators of the scale parameter θ in a family of transformed chi-square distributions, under asymmetric squared log error (SLE) and modified linear exponential (MLINEX) loss functions, using the Lehmann Theorem [2]. Also we show that the results of Podder *et al.* [4] for Pareto distribution are a special case of our results for this family of distributions.

Keywords: Minimaxity; MLINEX loss; Squared log error loss; Transformed chi-square distributions

1. Introduction

In this paper, the minimax estimators of the scale parameter θ in a family of transformed chi-square distributions are derived under two asymmetric loss functions, square log error loss (SLE) and modified linear exponential (MLINEX). We use the following Theorem, due to Lehmann [2] to show that the estimators for scale parameter θ in this family of distributions are minimax.

Brown [1] proposed a new loss function for scale parameter estimation. See also Pal and Ling [3]. This loss that is called squared log error loss is

$$L(\theta,\delta)=(\ln\delta-\ln\theta)^2=\left(\ln\frac{\delta}{\theta}\right)^2, \tag{1.1}$$

which is balanced and $\lim L(\theta,\delta)=\infty$ as $\delta\to 0$ or ∞ [1]. This loss is not always convex, it is convex for $\frac{\delta}{\theta}\le e$ and concave otherwise, but its risk function has a unique minimum w.r.t. δ.

Podder [5] introduced a modified linear exponential (MLINEX) loss function as

$$L(\theta,\delta)=\omega\left[\left(\frac{\delta}{\theta}\right)^c-c\ln\left(\frac{\delta}{\theta}\right)-1\right];\quad c\neq 0,\omega>0, \tag{1.2}$$

which is asymmetric one. If $\frac{\delta}{\theta}=1$, then $L(\theta,\delta)=0$, writing $R=\frac{\delta}{\theta}$, the relative error $L(R)$ is minimized at $R=1$. If we write $D=\ln R=\ln\delta-\ln\theta$, then $L(R)$ can be expressed as the same form of LINEX loss

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$$L(\theta,\delta)=\kappa\left[e^{\lambda(\delta-\theta)}-\lambda(\delta-\theta)-1\right];\quad \lambda\neq 0,\kappa>0. \tag{1.3}$$

Theorem 1.1. ([2]) Let $\tau=\{F_\theta;\theta\in\Theta\}$ be a family of distribution functions and D be a class of estimators of θ. Suppose that $d^*\in D$ is a Bayes estimator concerning a prior distribution $\pi(\theta)$ on the parameter space Θ. If the risk function $R(d^*,\theta)=const$ on Θ, then d^* is a minimax estimator for θ.

2. Family of Transformed Chi-Square Distributions

Let $X_1,X_2,...$ be a sequence of independent and identically distributed random variables from a one parameter exponential family

$$f(x,\eta)=e^{a(x)b(\eta)+c(\eta)+h(x)}. \tag{2.1}$$

Rahman and Gupta [6] proved the following Theorem for the family of distributions (2.1).

Theorem 2.1. ([6]) In family (2.1), the function $-2a(X)b(\eta)$ has $Gamma(\frac{j}{2},2)$ distribution if and only if

$$\frac{2c'(\eta)b(\eta)}{b'(\eta)}=j, \tag{2.2}$$

where j is positive and free from η. The one parameter exponential family of form (2.1) satisfying (2.2) is called the family of transformed chi-square distributions, provided j is a positive integer.

For example the density of $Gamma(\alpha,\beta)$, for known α, belongs to this family with

$$a(X)=X,\ b(\beta)=-\frac{1}{\beta},\ c(\beta)=-\alpha\ln\beta,$$

$$-2a(X)b(\beta)=\frac{2X}{\beta},\ j=2\alpha,$$

and also the $Pareto(\alpha)$ distribution belongs to this family with

$$a(X)=\ln X,\ b(\alpha)=-\alpha,\ c(\alpha)=\ln\alpha,$$

$$-2a(X)b(\alpha)=2\alpha\ln X,\ j=2.$$

From condition (2.2) we get

$$c(\eta)=\frac{j}{2}\ln|b(\eta)|+k_1. \tag{2.3}$$

Let $\theta=-b(\eta)>0$ and $v=\frac{j}{2}$, then (2.3) reduces to

$$e^{c(\eta)}=|-b(\eta)|^{\frac{1}{2}}e^{k_1}-\theta^{v}e^{k_1}.$$

Hence, the family (2.1) can be written as

$$e^{\frac{-a(x)}{\left(-\frac{1}{b(\eta)}\right)}c(\eta)h(x)}=e^{h(x)}e^{c(\eta)}e^{\frac{-a(x)}{\left(-\frac{1}{b(\eta)}\right)}},$$

i.e.

$$f(x;\theta)=c(x)\theta^{v}e^{-\theta a(x)}, \tag{2.4}$$

where $c(x)=e^{h(x)+k_1}$, $\theta=-b(\eta)>0$ and $v=\frac{j}{2}>0$. Also note that $-2a(X)b(\eta)=2\theta a(X)$ has $Gamma(v,\frac{1}{2})$ distribution or $a(X)\approx Gamma(v,\frac{1}{\theta})$.

Now, if $X_1,X_2,...,X_n$ is a sample of size n from distribution (2.4), then the joint density of X_1, $X_2,...,X_n$ is given by

$$f(\underline{x};\theta)=c(\underline{x},n)\theta^{nv}e^{-\theta\sum_{i=1}^{n}a(x_i)},\theta>0 \tag{2.5}$$

where $c(\underline{x},n)=\prod_{i=1}^{n}c(x_i)$ and $S_n(\underline{X})=\sum_{i=1}^{n}a(X_i)$ $\approx Gamma(nv,\frac{1}{\theta})$.

3. Main Results

Theorem 3.1. Let $X_1,X_2,...,X_n$ be a sample of size n from distribution (2.4). If θ has Jeffrey's non-informative prior density $\pi(\theta)\propto\frac{1}{\theta};\theta>0$, then

(a) $\delta^{\pi}_{MSLE}=\frac{e^{\Psi(nv)}}{S_n}$ is the minimax estimator of parameter θ under squared log error loss of the type $L(\theta,\delta)=(\ln\delta-\ln\theta)^2=\left(\ln\frac{\delta}{\theta}\right)^2$, where $\Psi(\alpha)=\frac{\Gamma'(\alpha)}{\Gamma(\alpha)}$,

(b) $\delta^{\pi}_{MMl}=\left(\frac{\Gamma(nv)}{\Gamma(nv-c)}\right)^{\frac{1}{c}}\frac{1}{S_n}$, is the minimax estimator of parameter θ under MLINEX loss function of the type $L(\theta,\delta)=\omega\left[\left(\frac{\delta}{\theta}\right)^c - c\ln\left(\frac{\delta}{\theta}\right)-1\right];\omega>0, c\neq 0$,

(c) $\delta^{\pi}_{MQl}=\frac{nv-2}{S_n}$ is the minimax estimator of parameter θ under quadratic loss function of the type $L(\theta,\delta)=\left(\frac{\theta-\delta}{\theta}\right)^2$, where $S_n=\sum_{i=1}^{n}a(X_i)$.

Remark 3.1. In order to derive the results for *Pareto* distribution by putting $a(X)=\ln X$, $b(.)=-\alpha$, $-2a(X)b(.)=2\alpha\ln X$, $v=\frac{j}{2}=1$, the minimax estimators for $\theta=\alpha$ under the above loss functions are

i. $\delta^{\pi}_{MSLE}=\frac{e^{\Psi(n)}}{\sum_{i=1}^{n}\ln X_i}$, where $\Psi(n)=\frac{\Gamma'(n)}{\Gamma(n)}=\frac{d}{dn}\ln\Gamma(n)=\int_0^{\infty}\frac{(\ln y)y^{n-1}e^{-y}}{\Gamma(n)}dy$,

ii. $\delta^{\pi}_{MMl}=\left(\frac{\Gamma(n)}{\Gamma(n-c)}\right)^{\frac{1}{c}}\frac{1}{\sum_{i=1}^{n}\ln X_i}$ and $\delta^{\pi}_{MQl}=\frac{n-2}{\sum_{i=1}^{n}\ln X_i}$.

These results for MLINEX and quadratic loss functions are conformed to the results of Podder *et al.* [4]. Hence, our results for these three loss functions in the family of transformed chi-square are general and containing the former works in this field.

Remark 3.2. Putting $a(X)=X, b(.)=-\frac{1}{\beta}$, $-2a(X)b(.)=\frac{2X}{\beta}$, $v=\frac{j}{2}=\alpha$, the minimax estimators for $\theta=\frac{1}{\beta}$ in *Gamma* distribution with $f(x;\beta)=\frac{1}{\beta^{\alpha}\Gamma(\alpha)}x^{\alpha-1}\exp\left(\frac{-x}{\beta}\right)$ under above loss functions are

i. $\delta^{\pi}_{MSLE}=\frac{e^{\Psi(n\alpha)}}{\sum_{i=1}^{n}X_i}=\frac{e^{\Psi(n\alpha)}}{n}\frac{1}{\bar{X}_n}$, and $\delta^{\pi}_{MQl}=\frac{n\alpha-2}{\sum_{i=1}^{n}X_i}=\frac{n\alpha-2}{n}\frac{1}{\bar{X}_n}$,

ii. $\delta^{\pi}_{MMl}=\left(\frac{\Gamma(n\alpha)}{\Gamma(n\alpha-c)}\right)^{\frac{1}{c}}\frac{1}{\sum_{i=1}^{n}X_i}=\frac{\left(\frac{\Gamma(n\alpha)}{\Gamma(n\alpha-c)}\right)^{\frac{1}{c}}}{n}\frac{1}{\bar{X}_n}$.

Proof. Part (a): It is enough to show that the estimator $\delta^{\pi}_{MSLE}=\frac{e^{\Psi(nv)}}{S_n}$ is the Bayes estimator for parameter θ, in transformed chi-square family as the form (2.5), with constant risk under the prior distribution $\pi(\theta)\propto\frac{1}{\theta}$; $\theta>0$.

The posterior distribution of θ given $\underline{X}=(X_1,\dots,X_n)$ is

$$\pi\left(\theta|\underline{x}\right)=\frac{(S_n)^{nv}\ \theta^{nv-1}e^{-\theta S_n}}{\Gamma(nv)};\theta\geq 0, \tag{3.1}$$

which is $Gamma\left(nv,\frac{1}{S_n}\right)$. The Bayes estimator for θ under squared log error (1.1) is

$$\delta^{\pi}_{BSLE}=\exp\left[E\left(\ln\theta|\underline{X}\right)\right], \tag{3.2}$$

where

$$E\left[\ln\theta|\underline{X}\right]=\int_0^{\infty}\ln\theta\frac{(S_n)^{nv}\ \theta^{nv-1}e^{-\theta S_n}}{\Gamma(nv)}d\theta$$

$$=\frac{1}{\Gamma(nv)}\left[\int_0^{\infty}\left(\ln y-\ln S_n\right)y^{nv-1}e^{-y}dy\right]$$

$$(\text{using } \theta S_n=y) \tag{3.3}$$

$$=\frac{1}{\Gamma(nv)}\left[\int_0^{\infty}(\ln y)y^{nv-1}e^{-y}dy-\ln S_n\int_0^{\infty}y^{nv-1}e^{-y}dy\right]$$

$$=\frac{1}{\Gamma(nv)}\left[\Gamma'(nv)-\ln S_n\Gamma(nv)\right]=\Psi(nv)-\ln S_n.$$

Hence, we have

$$\begin{aligned}\delta^{\pi}_{BSLE}&=\exp\left[E\left(\ln\theta|\underline{X}\right)\right]\\&=\exp\left(\Psi(nv)-\ln S_n\right)\\&=(S_n)^{-1}e^{\Psi(nv)}=\frac{e^{\Psi(nv)}}{S_n}.\end{aligned} \tag{3.4}$$

Therefore, it is enough to show that the risk of δ^{π}_{BSLE} is constant.

$$R_{BSLE}(\theta) = E\left[\ln \delta^{\pi}_{BSLE} - \ln \theta\right]^2$$
$$= E\left[\ln \delta^{\pi}_{BSLE}\right]^2 - 2\ln\theta E\left[\ln \delta^{\pi}_{BSLE}\right] + \left[\ln\theta\right]. \quad (3.5)$$

Since, the distribution of $S_n(\underline{X}) = \sum_{i=1}^{n} a(X_i)$ is $Gamma(nv, \frac{1}{\theta})$, then

$$E\left[\ln \delta^{\pi}_{BSLE}\right] = E\left[\Psi(nv) - \ln S_n\right]$$
$$= \Psi(nv) - \left(\Psi(nv) - \ln\theta\right) = \ln\theta, \quad (3.6)$$

using the relation $\theta S_n = y$, we have

$$E\left[\ln S_n\right] = \int_0^{\infty} \ln S_n \frac{\theta^{nv}}{\Gamma(nv)} (S_n)^{nv-1} e^{-\theta S_n} dS_n$$
$$= \int_0^{\infty} \left[\ln y - \ln\theta\right] \frac{y^{nv-1} e^{-y}}{\Gamma(nv)} dy = \Psi(nv) - \ln\theta, \quad (3.7)$$

and also we get

$$E\left[\ln \delta^{\pi}_{BSLE}\right]^2 = E\left[\Psi(nv) - \ln S_n\right]^2$$
$$= \Psi^2(nv) - 2\Psi(nv)E\left[\ln S_n\right] + E\left[\ln S_n\right]^2. \quad (3.8)$$

But

$$\Psi'(nv) = \int_0^{\infty} \frac{(\ln y)^2 y^{nv-1} e^{-y}}{\Gamma(nv)} dy$$
$$- \int_0^{\infty} \frac{(\ln y)^2 y^{nv-1} e^{-y}}{\Gamma(nv)} \Psi(nv) dy \quad (3.9)$$
$$= E\left[\ln Y\right]^2 - \Psi^2(nv).$$

This implies that $E\left[\ln Y\right]^2 = \Psi^2(nv) + \Psi'(nv)$. Using this fact we have

$$E\left[\ln S_n\right]^2 = \Psi'(nv) + \Psi^2(nv)$$
$$-2\ln\theta\Psi(nv) + (\ln\theta)^2. \quad (3.10)$$

Substituting the relations (3.7) and (3.10) in equation (3.8) we have

$$E\left[\ln \delta^{\pi}_{BSLE}\right]^2 = \Psi'(nv) + (\ln\theta)^2.$$

Also, by replacing the relations (3.7) and (3.10) in (3.8) we have gotten

$$R_{BSLE}(\theta) = E\left[\ln \delta^{\pi}_{BSLE} - \ln\theta\right]^2 = \Psi'(nv), \quad (3.11)$$

which is constant w.r.t. θ, as v and n are known and independent of θ. So from the Lehmann's Theorem it follows that $\delta^{\pi}_{MSLE} = \frac{e^{\Psi(nv)}}{S_n}$, is the minimax estimator of the scale parameter θ in a family of transformed chi-square distributions under the square log error loss function (1.1).

Part (b): The Bayes estimator for θ under the MLINEX loss function (1.2) is

$$\delta^{\pi}_{BML} = \left[E(\theta^{-c} | \underline{X})\right]^{-\frac{1}{c}}, \quad (3.12)$$

where

$$E(\theta^{-c} | \underline{X}) = \int_0^{\infty} \theta^{-c} \frac{(S_n)^{nv} \theta^{nv-1} e^{-\theta S_n}}{\Gamma(nv)} d\theta$$
$$= \frac{\Gamma(nv-c)}{\Gamma(nv)} \left(S_n\right)^c.$$

Using the relation (3.12) gives

$$\delta^{\pi}_{BML} = \left(\frac{\Gamma(nv)}{\Gamma(nv-c)}\right)^{\frac{1}{c}} \frac{1}{S_n} = \frac{K}{S_n}, \quad (3.13)$$

where $K = \left(\frac{\Gamma(nv)}{\Gamma(nv-c)}\right)^{\frac{1}{c}}$ and $S_n = \sum_{i=1}^{n} a(x_i)$ is the complete sufficient statistics for θ. Now the risk function of δ^{π}_{BML} under the MLINEX (1.2) is given by

$$R_{BML}(\theta) = E\left[L\left(\theta, \delta^{\pi}_{BML}\right)\right]$$
$$= \omega\left\{\frac{1}{\theta^c} E\left(\delta^{\pi}_{BML}\right)^c - cE\left(\ln \delta^{\pi}_{BML}\right) + c\ln\theta - 1\right\}. \quad (3.14)$$

But

$$E\left[\delta^{\pi}_{BML}\right]^c = E\left[\frac{K}{S_n}\right]^c = K^c E\left(S_n\right)^{-c}$$
$$= K^c \frac{\Gamma(nv-c)}{\Gamma(nv)} \theta^c = \theta^c, \quad (3.15)$$

and using the relation (3.13) gives

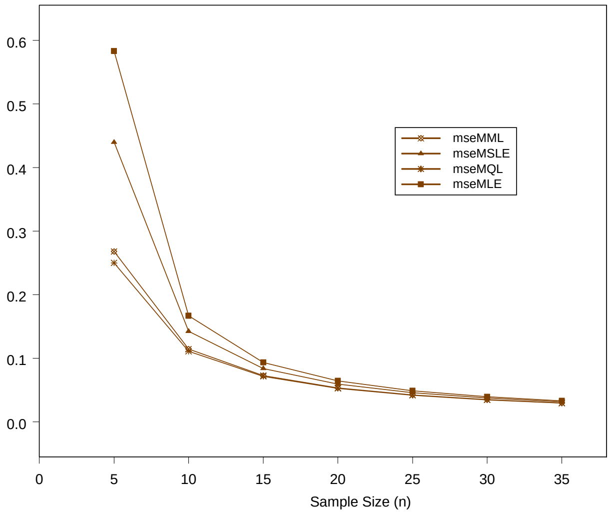

Figure 1. Graphs of MSEs for different values of n under MML, MSLE, MQL and MLE when $\theta = 1$, $c = 2$ for known values $\beta = 1$ & 2.

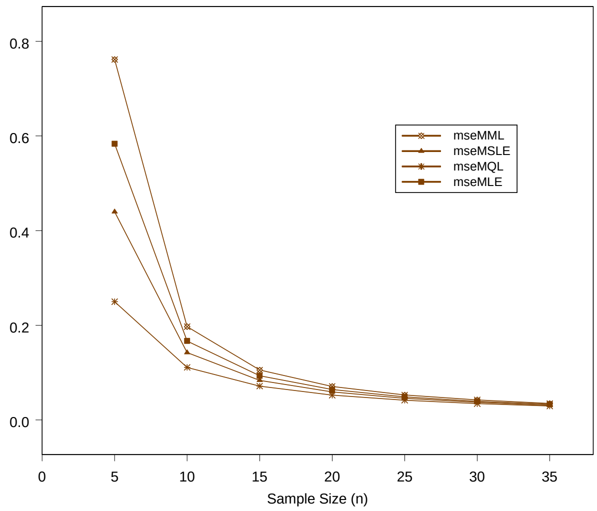

Figure 2. Graphs of MSEs for different values of n under MML, MSLE, MQL and MLE when $\theta = 1$, $c = -2$ for known values $\beta = 1$ & 2.

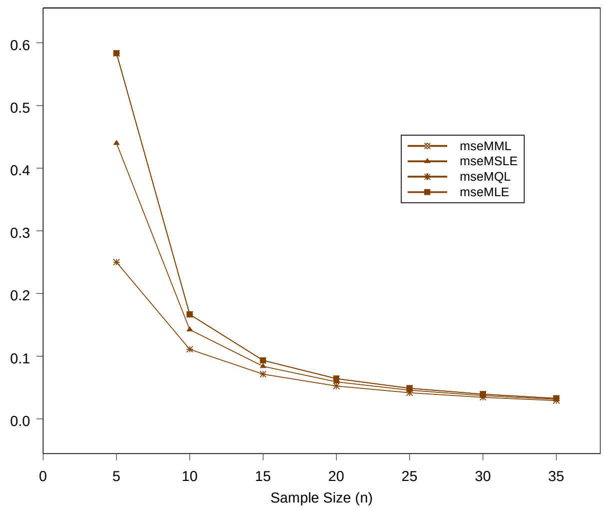

Figure 3. Graphs of MSEs for different values of n under MML, MSLE, MQL and MLE when $\theta = 1$, $c = -1$ for known values $\beta = 1$ & 2.

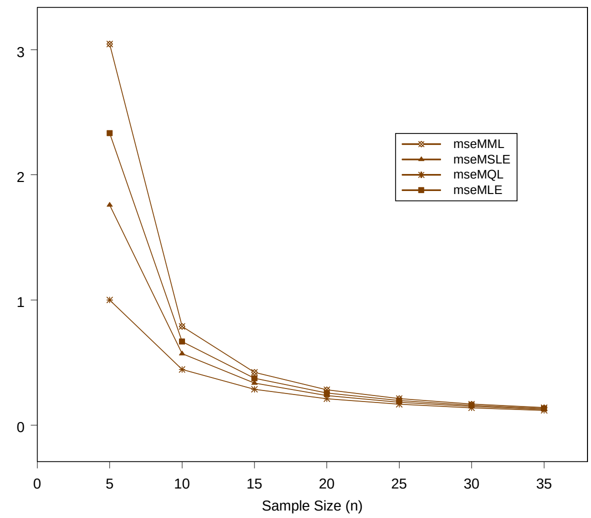

Figure 4. Graphs of MSEs for different values of n under MML, MSLE, MQL and MLE when $\theta = 2$, $c = -2$ for known values $\beta = 1$ & 2.

$$E\left[\ln \delta_{BML}^{\pi}\right] = E\left[\ln \frac{K}{S_n}\right] = \ln K - E\left(\ln S_n\right) \tag{3.16}$$

$$= \ln K + \ln \theta - \Psi(nv).$$

By substituting the relations (3.15) and (3.16) in (3.14) we have

$$R_{BML}(\theta) = \omega\left[\ln K^{-c} + c\Psi(nv)\right], \tag{3.17}$$

which is constant w.r.t. θ, as v and n are known and independent of θ.

So from the Lehmann's Theorem it follows that $\delta_{MML}^{\pi} = \left(\frac{\Gamma(nv)}{\Gamma(nv-c)}\right)^{\frac{1}{c}} \frac{1}{S_n} = \frac{K}{S_n}$ is the minimax estimator of the scale parameter θ in this family under MLINEX loss function.

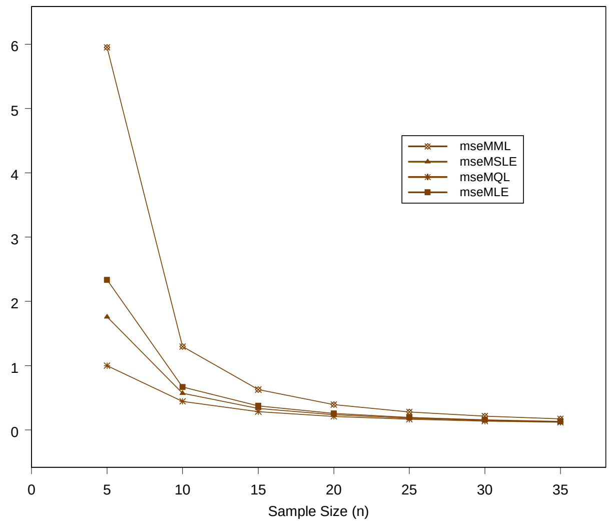

Figure 5. Graphs of MSEs for different values of n under MML, MSLE, MQL and MLE when $\theta = 2$, $c = -5$ for known values $\beta = 1$ & 2.

Part (c): Proof of this part is very similar to parts (a) and (b) and it's omitted.

4. Empirical Study

Mean Square Errors (MSEs) are considered to compare the different estimators of the parameter $\theta = \alpha^{-\beta}$ (β known), in Weibull distribution $f(x,\alpha) = \beta\alpha^{-\beta}x^{\beta-1}\exp\left(-\left(\frac{x}{\alpha}\right)^{\beta}\right)$, obtained by the method of maximum likelihood and method of minimax for squared log error, MLINEX and quadratic loss functions.

The estimated values of the parameter θ and MSEs of the estimators are computed by the Monte-Carlo simulation method using the Weibull distribution. It is seen that for small sample size $n < 25$ and $c > 0$, minimax estimators for quadratic loss appear to be better than the minimax estimators under MLINEX and squared log error loss functions, and the minimax estimator under MLINEX loss is better than the minimax estimator under squared log error loss in terms of MSE. But for $n < 25$ and $c < 0$, the minimax estimator for θ under the squared log error is better than the minimax estimator under MLINEX loss function. In both cases the minimax estimator under quadratic loss has the least MSE.

For the large sample size $n > 25$, estimators have approximately the same MSEs.

The results for Weibull distribution with different values $\theta = 1, 2$ and $c = -5, -1, -2, 2$ are demonstrated in Figures 1-5.

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