On the Convergence Rate of the Law of Large Numbers for Sums of Dependent Random Variables

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Abstract

In this paper, we generalize some results of Chandra and Goswami [4] for pairwise negatively dependent random variables (henceforth r.v.’s). Furthermore, we give Baum and Katz’s [1] type results on estimate for the rate of convergence in these laws.

Keywords: Negatively dependent random variables; Complete convergence; Strong law of large numbers

1. Introduction and Preliminaries

Let \( \{X_n, n \geq 1\} \) be a sequence of integrable r.v.’s defined on the same probability space.

Chandra and Goswami [4] have proved the following theorem from the arguments of Csorgo et al. [5].

**Theorem CG1.** Let \( \{X_n, n \geq 1\} \) be a sequence of non-negative r.v.’s with finite \( Var(X_n) \) and \( f(n) \) be an increasing sequence such that \( f(n) > 0 \) for each \( n \) and \( f(n) \to \infty \) as \( n \to \infty \). Put \( S_n = \sum_{i=1}^{n} X_i \). Assume that

\[
\sup_{\alpha>0} \frac{1}{\alpha} \sum_{i=1}^{n} X_i = A(\text{say}) < \infty; \tag{1.1}
\]

and there is a double sequence \( \{\rho_{ij}\} \) of nonnegative reals such that

\[
Var(S_n) \leq \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij} \quad \text{for each} \quad n \geq 1; \tag{1.2}
\]

and

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij} (f(i \vee j))^2 < \infty, \quad (i \vee j) = \max(i,j). \tag{1.3}
\]

Then \( (f(n))^{-1} [S_n - E(S_n)] \to 0 \) a.s. as \( n \to \infty \).

Nili and Bozorgnia [11] generalized (and corrected) Theorem CG1 for an array of r.v.’s and obtained the following result:

**Theorem NB.** Let \( \{X_{ni}, n \geq 1, i \geq 1\} \) be an array of non-negative r.v.’s with finite \( Var(X_{ni}) \) and \( [\log_{\alpha} f(n)] \), \( \alpha > 1 \) be an increasing sequence. Put

\[
S_n = \sum_{i=1}^{n} a_n X_{ni}, \quad \text{where} \quad l(x) \quad \text{stands for a nondecreasing continuous function with inverse} \quad l^{-1};
\]

such that \( l(n) \) is a natural sequence and \( l(n) \to \infty \). Assume that there is a double sequence of nonnegative reals \( \{\rho_{ij}\} \) such that

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\[ \text{Var}(S_n) \leq \sum_{i=1}^{n} \sum_{j=1}^{i} \rho_{ij} \text{ for each } n \geq 1; \]  
(1.4)

and
\[ \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij} f^{-1}(l^{-1}(i) \lor l^{-1}(j)) < \infty. \]  
(1.5)

Then \((f(n))^{-1} \left[ S_n - E(S_n) \right] \to 0\) completely as \( n \to \infty \), in the sense of Hsu and Robbins [6] (see also page 225 of Stout [12]), and hence, a.s. \( n \to \infty \).

The question underlying the present work is how one may refine Theorem CG1 to give more information on the law of \( \{X_n\} \). We recall the classical answer, the strong law of large numbers Baum and Katz [1] for \((\text{see } [2])\). In Section 3 we generalize Theorem CG1 and give Baum-Katz’s [1] type results on estimate for the rate of convergence in these laws.

Chandra and Goswami [4], also proved Theorem CG1, and extended the results of Landers and Rogge [8] for pairwise independent r.v.’s.

Theorem CG2. Let \( \{X_n, n \geq 1\} \) be a sequence of pairwise independent random variables such that there is a sequence \( \{B_n\} \) of Borel subsets of \( R^1\) satisfying the following conditions

\begin{itemize}
  \item[(a)] \( \sum_{n=1}^{\infty} P(X_n \in B_n^c) < \infty; \)
  \item[(b)] \( \sum_{j=1}^{\infty} E(X_j I(X_j \in B_n^c)) = o(f(n)); \)
  \item[(c)] \( \sum_{n=1}^{\infty} (f^{-1}(n) \text{Var}(X_n I(X_n \in B_n^c))) < \infty; \)
\end{itemize}

and

\begin{itemize}
  \item[(d)] \( \sup_{n=1}^{\infty} \sum_{i=1}^{n} E[I(X_i \in B_n^c)](f(n)) < \infty; \)
\end{itemize}

here \( B_n^c \) is the complement of \( B_n \). Then \((f(n))^{-1}\left[ S_n - E(S_n) \right] \to 0\) almost surly as \( n \to \infty \).

In Section 3 we also extend Theorem CG2 to negative dependence r.v.’s.

2. Negative Dependence

Definition 1. ([9]). Random variables \( X_1, \ldots, X_n(n \geq 2) \) are said to be pairwise negatively dependent (henceforth pairwise ND) if
\[ P(X_i > x_i, X_j > x_j) \leq P(X_i > x_i)P(X_j > x_j), \quad (2.1) \]
holds for all \( x_i, x_j \in R, i \neq j \). It can be shown that (2.1) is equivalent to
\[ P(X_i \leq x_i, X_j \leq x_j) \leq P(X_i \leq x_i)P(X_j \leq x_j), \quad (2.2) \]
for all \( x_i, x_j \in R, i \neq j \).

Events \( \{E_n\} \) are said to be pairwise negatively dependent if their indicator functions are pairwise negatively dependent.

Example 1. Let \( X + Y = c \), \( c \in R \). It is easy to see that \( X \) and \( Y \) are negatively dependent.

An infinite collection of \( \{X_n, n \geq 1\} \) is said to be pairwise ND if every finite subcollection is pairwise ND. We will need the following results [3,7,10].

Proposition 1. Let \( \{X_n, n \geq 1\} \) be a sequence of pairwise ND r.v’s. Then the following are true:

(i) \( \text{Cov}(X_i, X_j) < 0, \quad i \neq j \),
(ii) \( \{f_n, n \geq 1\} \) is a sequence of Borel functions all of which are monotone increasing (or all monotone decreasing) then \( \{f_n(X_n), n \geq 1\} \) is a sequence of pairwise ND r.v’s.
(iii) The Borel-Cantelli lemma holds for pairwise ND events.

3. Main Results

In the following theorems \( \alpha \geq 1/2 \) and \( r \) is an integer such that \( r = 2\alpha - 2 \) when \( 2\alpha - 2 \) is integer and \( r = [2\alpha - 2] + 1 \) (\( [x] \) is integer part of \( x \)) otherwise. Also in this paper, \( C \) stands for a generic constant, not necessarily the same at each appearance. Put
\[ S_n = \sum_{i=1}^{n} X_i. \]

Theorem 1. Let \( \{X_n, n \geq 1\} \) be a sequence of r.v.’s and \( \{f(n), n \geq 1\} \) be a sequence of positive reals such that for some \( \beta > 1, \log_{\beta}f(n) \) is an increasing sequence. Assume that there is a double sequence \( \{\rho_{ij}\} \) of non-negative reals such that \( \rho_{ii} \) is upper bound for \( \text{Var}(X_i) \) and
\[ \text{Var}(S_n) \leq \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij}. \]  
(3.1)

If for some \( \xi < 2\alpha \)
\[
\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\rho_{i,j} (i \lor j)^{-1/2}}{\beta^{i+j}} < \infty ,
\]  
(3.2)

then for every \( \varepsilon > 0 \)
\[
\sum_{i=1}^{n} n^{2a-2} P(|S_n - E(S_n)| > \varepsilon f^a(n)) < \infty.
\]  
(3.3)

Remark. If \( \alpha = 1 \) we can use theorem NB for \( X_n = X_1, l(n) = n \) and \( a_n = 1 \), it is sufficient to replace (3.2) by (1.5), then (3.3) holds.

Proof. We use sub-sequence method. Replacing \( X_i \) by \( X_i - E(X_i) \) we may use \( E(X_i) = 0 \). It is easy to show that
\[
\sum_{i=1}^{n} n^{2a-2} P(|S_n| > \varepsilon f^a(n) )
\]
\[
\leq \sum_{i=1}^{n} n^{2a-2} P(|S_n| > \varepsilon f^a(n^2)) + \frac{\sum_{i=1}^{n} k^{(2a-2)} P(D_n > \varepsilon / 2 f^a(k)) }{\beta^{2a(n^2)}} 
\]
where \( D_n = \max_{\alpha' < k < \alpha + 1} |S_k - S_{\alpha'}| \). It is sufficient to show that each of three above series is convergent.

\[
\sum_{i=1}^{n} n^{2a-2} P(|S_n| > \varepsilon f^a(n^2))
\]
\[
\leq C \sum_{i=j}^{n} \rho_{ij} \sum_{\alpha' < k < \alpha + 1} n^{2a-2} \beta^{2a(n^2)}
\]
\[
\leq C \sum_{i=j}^{n} \rho_{ij} \sum_{\alpha' < k < \alpha + 1} \frac{n^{2a-2}}{\beta^{2a(n^2)}}
\]
\[
\leq C \sum_{i=j}^{n} \rho_{ij} \sum_{\alpha' < k < \alpha + 1} \frac{n^{2a-2}}{\beta^{2a(n^2)}}
\]
for a fix \( \beta \) the second sum include one statement and we have
\[
\leq C \sum_{i=1}^{n} \rho_{ii} \frac{\sqrt{n} \beta^{2a(n^2+1)}}{\beta^{2a(n^2)}}
\]
Note that \( \frac{\sqrt{n} \beta^{2a(n^2+1)}}{\beta^{2a(n^2)}} \leq C \frac{j^{r-1/2}}{\beta^{2a(n^2)}} \), if \( j \) is sufficiently large, thus
\[
\sum_{i=1}^{n} \rho_{ii} \frac{\sqrt{n} \beta^{2a(n^2+1)}}{\beta^{2a(n^2)}} \leq C \sum_{i=1}^{n} \rho_{ii} \frac{i^{r-1/2}}{\beta^{2a(n^2)} < \infty.
\]
In the next theorems we relax the condition that for some $\beta > 1$, $[\log_p f(n)]$ is an increasing sequence. The proofs follow the same lines as the proof of Theorem 1.

**Theorem 2.** Let \{X_n, n \geq 1\} and \{\rho_j\} be as in Theorem 1 such that

$$\text{Var}(\sum_{i=1}^{n} X_i) \leq \sum_{j=1}^{n} \sum_{i=j}^{n} \rho_j / (i \lor j)^{3/2}, \quad (3.4)$$

then for every $\varepsilon > 0$

$$\sum_{n=1}^{\infty} n^{2\alpha-2} P(|S_n - E(S_n)| > \varepsilon f^\alpha(n)) < \infty.$$

**Proof.** The Chebyshev’s inequality, condition (3.4) and a change of order of summation imply that

$$\sum_{n=1}^{\infty} n^{2\alpha-2} P(|S_n| > \varepsilon f^\alpha(n^2)) \leq C \sum_{n=1}^{\infty} n^{2\alpha-2} E(S_n^2) \leq C \sum_{n=1}^{\infty} \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_j / n^4$$

$$\leq C \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \rho_j \sum_{n=1}^{\min(i, j)} 1/n^4 \leq C \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \rho_j / (i \lor j)^{3/2} < \infty.$$

For the second series we have

$$\sum_{n=1}^{\infty} n^{2\alpha-2} P(|S_n| > \varepsilon f^\alpha(n^2)) \leq C \sum_{n=1}^{\infty} k^{2\alpha-2} E(D_n^2)$$

$$\leq C \sum_{n=1}^{\infty} \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_j / (i \lor j)^{3/2} < \infty.$$

And finally we show that the third series is convergent

$$\sum_{n=1}^{\infty} n^{2\alpha-2} P(D_n > \varepsilon / 2 f^\alpha(k)) \leq C \sum_{n=1}^{\infty} k^{2\alpha-2} E(D_n^2)$$

$$\leq C \sum_{n=1}^{\infty} n^{2\alpha-2} E(S_n^2)$$

$$\leq C \sum_{n=1}^{\infty} \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_j / (i \lor j)^{3/2} < \infty.$$

**Theorem 3.** Let \{X_n, n \geq 1\} be a sequence of r.v.’s and \{\rho_j\} be a double sequence of nonnegative reals such that

$$\text{Var}(S_n) \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \rho_j \quad \text{for each } n \geq 1; \quad (3.5)$$

Assume that \{f(n)\} is an increasing sequence such that $n^\beta \leq f(n) \leq (n+1)^\beta$ for some $0 < \beta \leq 1$ and for each $n \geq 1$. If

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} f^{-1}(i \lor j) \rho_j < \infty,$$

where $\gamma = (3 + 4\alpha \beta - 4\alpha) / 2\beta$ and $\alpha < 3 / (4 - \beta)$, then for every $\varepsilon > 0$

$$\sum_{n=1}^{\infty} n^{2\alpha-2} P(|S_n - E(S_n)| > \varepsilon f^\alpha(n)) < \infty.$$

**Proof.** Again we are going to use subsequence method. Replacing $X_i$ by $X_i - E(X_i)$, we may assume $E(X_1) = 0$.

$$\sum_{n=1}^{\infty} n^{2\alpha-2} P(|S_n| > \varepsilon f^\alpha(n^2)) \leq C \sum_{n=1}^{\infty} n^{2\alpha-2} E(S_n^2)$$

$$\leq C \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \rho_j / (i \lor j)^{3/2} < \infty.$$
by Chebyshev’s inequality and (3.5). For the second sum we have

\[ \sum_{i=1}^{n} \sum_{j=1}^{n} k^{2a-2} P(|S_n| > \varepsilon / 2f^{\alpha}(k)) \]

\[ \leq C \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{k^{2a-2} f^{\alpha}(k)}{k^{2a-2} + 1} E(X_i^+) \]

\[ \leq C \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{k^{2a-2} + 1} \]

\[ \leq C \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{k^{2a-2} + 1} < \infty. \]

Thus it remains to verify that the third sum is convergent

\[ \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{k^{2a-2} f^{\alpha}(i)}{k^{2a-2} + 1} \]

\[ \leq C \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{k^{2a-2} + 1} \]

\[ \leq C \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{k^{2a-2} + 1} < \infty. \]

**Theorem 4.** Let \( \alpha, \beta, \xi, r \) and \( f(n) \) be as in Theorem 1. Also let \( \{X_n, n \geq 1\} \) be a sequence of pairwise \( ND \) r.v’s such that there is a sequence \( \{B_n, n \geq 1\} \) of semi intervals \( (-\infty, x_n] \) \((-\infty, x_n)\), \( [x_n, \infty) \) or \( (x_n, \infty) \), \( x_n \in R \), satisfying in the following conditions:

(a) \( \sum_{n=1}^{\infty} P(X_n \in B_n^c) < \infty \)

where

\[ C_n = 1 + \left( \frac{\sqrt{2}}{\beta} \right)^{\frac{r-0.5}{2}} \]

(b) \( \sum_{n=1}^{\infty} \frac{n^{r-0.5}}{\beta^{r/2}} E(X_i^2 I(X_i \in B_n)) < \infty \)

(c) \( |X_n - x_n| I(X_n \in B_n^c) \) is uniformly integrable;

here \( B_n^c \) is the complement of \( B_n \). Then for every \( \varepsilon > 0 \)

\[ \sum_{n=1}^{\infty} n^{2a-2} P(|S_n - E(S_n)| > \varepsilon f^{\alpha}(n)) < \infty. \]

**Proof.** Put \( Y_n = X_n I(X_n \in B_n) + x_n I(X_n \notin B_n) \),

\( Z_n = X_n - Y_n, S_n = \sum_{i=1}^{n} Y_i, S_n^* = \sum_{i=1}^{n} Y_i \) and \( S_n = S_n - S_n^* = \sum_{i=1}^{n} Z_i, n \geq 1 \). It is obvious that \( \{Y_n, n \geq 1\} \) and \( \{Z_n, n \geq 1\} \) are two sequences of pairwise \( ND \) r.v.’s. It is sufficient to show that

\[ \sum_{n=1}^{\infty} n^{2a-2} P(|S_n - E(S_n)| > \varepsilon f^{\alpha}(n)) < \infty, \]

\[ \sum_{n=1}^{\infty} n^{2a-2} P(|S_n^* - E(S_n^*)| > \varepsilon f^{\alpha}(n)) < \infty. \]

By Theorem 1, conditions (a) and (b) and Proposition 1 applied to \( \{Y_n, n \geq 1\} \) yields

\[ \sum_{n=1}^{\infty} n^{2a-2} P(|S_n^* - E(S_n^*)| > \varepsilon f^{\alpha}(n)) \]

\[ \leq C \sum_{n=1}^{\infty} n^{2a-2} \sum_{i=1}^{n} E(X_i^2) \]

\[ \leq C \sum_{n=1}^{\infty} n^{2a-2} \sum_{i=1}^{n} E(X_i^2 | X_i \in B_i) \]

\[ + \sum_{i=1}^{n} E(X_i^2 | X_i \in B_i) \]

Hence, it is sufficient to prove the first sentence. Since

263
\[
\sum_{s=1}^{\infty} P(X_s \neq Y_s) = \sum_{n=1}^{\infty} P(X_n \in B'_n) < \infty,
\]

\{X_n\} and \{Y_n\} are equivalent and
\[
\sum_{s=1}^{\infty} n^{2s-2} P(|S'_s - E(S'_s)| > \varepsilon f^{-n}(n))
\]
\[
\leq C \sum_{s=1}^{\infty} n^{2s-2} E(|S'_s - E(S'_s)|) < \infty,
\]
by (c) and the first Borel Cantelli lemma, the desired result follows.

**References**


