Research Note

STRESS INTENSITY FACTORS FOR AN INTERIOR GRIFFITH CRACK OPENED BY HEATED WEDGE IN A STRIP WHOSE EDGES ARE NORMAL TO CRACK AXIS

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Abstract

After second world war, we have seen a very rapid development of Thermoelasticity stimulated by various engineering sciences. As we know, structures behave in different fashion under climate variation and the most responsible factor for these variations is heat which falls through and causes deformation of the body. A considerable progress in the field of aircraft and machine structures, mainly with gas and steam turbine and the emergence of new topics in chemical and nuclear engineering, have given rise to numerous problems in which thermal stresses play an important and frequently even a primary role. The concern of the present paper is the problem of an interior Griffith crack opened by heated wedge in a strip whose edges are normal to crack axis and the medium is assumed to be homogenous and isotropic. Under plane strain condition the closed form expressions for the stress intensity factors and the crack shape are obtained by use of Fourier transform technique. Two special cases of heat distributions are discussed in the end when wedge geometry is prescribed.

1. Introduction

A very rapid development of Thermoelasticity has been considered after second world war by various engineering sciences. Griffith who was known as father of fracture mechanics considered the mathematical theory of elasticity of crack problems in his two pioneering papers published in 1920 and 1924 [11,12].

Keywords: Griffith crack; Fourier transform; Heated wedge; Isotropic medium

A problem of an interior Griffith crack opened by a heated wedge in an infinite strip whose edges are parallel to crack axis has been recently published by Saraj [1]. Kushwaha has been introduced a new approach in investigating the problem of stress field in the neighborhood of Griffith crack [2]. The problem of stress intensity factors for a Griffith crack opened by

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thermal stresses in an infinite strip is discussed by Kushwaha and Umesh [3]. Problem of crack opening due to stresses on crack faces as well as the neighborhood of the crack, by Lowengrub [4,6]. Two exterior Griffith cracks opened by heated wedge in an infinite and isotropic medium is discussed by Kushwaha and Saraj [10].

A note on Griffith cracks is investigated by Lowengrub [5]. An excellent survey of the crack problems in the theory of elasticity can be seen by Sneddon and Lowengrub [7].

The title problem can also be assumed as an infinite number of Griffith cracks which are equally spaced along y-axis, see Figure 1. We reduce the above problem to the problem of an interior crack in an elastic medium (Fig. 2). While the Griffith crack occupies the spaces $b x \; c$ ($y = 0$) with the following boundary conditions.

$$\sigma_{xy}(\pm a, y) = 0, \quad 0 \le |y| < \infty \tag{1.1}$$

$$U_x(\pm a, y) = 0, \quad 0 \le |y| < \infty \tag{1.2}$$

$$\sigma_{xy}(x, 0) = 0, \quad 0 \le |x| \le a \tag{1.3}$$

Figure 1. Infinite number of Griffith cracks which are equally spaced along Y-axis in a long vertical elastic strip.

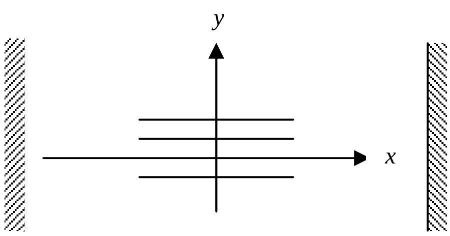
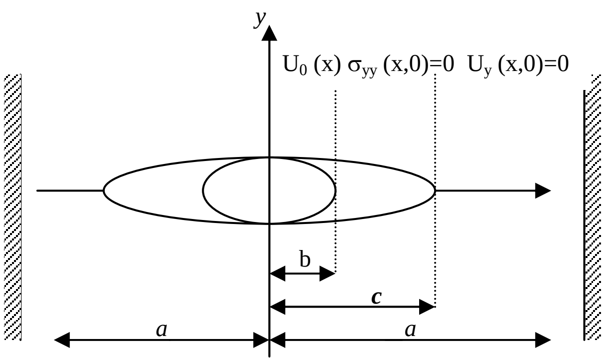


Figure 2. Crack opening due to heat with boundary conditions for elasticity problem.

and the mixed boundary conditions are



$$\sigma_{yy}(x, 0) = 0, \quad b < |x| < c \tag{1.4}$$

$$U_y(x,o) = \begin{cases} u_o(x) & o \le |x| \le b \\ 0 & c \le |x| \le a \end{cases} \tag{1.5}$$

Where (σ_{xx}, σ_{xy}, σ_{yy}) and (u_x, u_y) are components of stresses and of the displacement vector respectively,

$u_0(x)$ is a wedge shape function, and we assume that the thermal and elastic properties of the medium does not change with heat variation, and also all the physical quantities vanishes as $|x| \to \infty$.

Since the problem is linear we assume that stresses developed by temperature variation opens out the Griffith crack as it is given through the boundary conditions (1.1)-(1.5). No heat sources or sinks are assumed in the medium and the medium is assumed under plane strain condition.

Throughout the analysis it has been checked that the crack faces do not meet other than the crack tips. (see Burniston [8], $u_y(x,0) > 0 \quad b < |x| < c$.

We use the following definition for infinite Fourier sine and cosine transform

$$f_{cs}(\alpha_p \zeta) = \int_0^\infty \int_0^a f(x,y)\cos(\alpha_p x)\sin(\zeta y)\,dx\,dy \tag{1.6}$$

with

$$\alpha_p = p\frac{\pi}{a} \quad P = 0, 1, 2, 3, \ldots .$$

2. Formulation

The physical problem is reduced to the solution of the following partial differential equation in absence of body forces,

$$\frac{\partial \sigma xx}{\partial x} + \frac{\partial \sigma xy}{\partial y} = 0 \tag{2.1}$$

$$\frac{\partial \sigma xy}{\partial x} + \frac{\partial \sigma yy}{\partial y} = 0 \tag{2.2}$$

with stress-strain relations as

$$\sigma_{ij} = 2\mu\, e_{ij} + \lambda(e_{kk} - \gamma T)\delta_{ij} \quad i, j = x, y \tag{2.3}$$

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}),\ \gamma = (3\lambda+2\mu)\alpha_t \quad i, j = x, y \tag{2.4}$$

where λ and μ are Lame's constant, α_t is the coefficient of linear expansion, δ_{ij} is the kronecker delta and T satisfies the Laplace's equation

$$\nabla^2 T = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) T = 0 \tag{2.5}$$

for simplicity, μ is taken as the unit of stress. Substituting for the stress components in terms of the displacement from (2.3) and (2.4) in (2.1)-(2.2) and solve for U_x, we get.

$$\nabla^4 U_x = 0 \tag{2.6}$$

The present problem is solved in two stages. **Stage-A** (Temperature distribution problem) and **Stage-B** (Elasticity problem).

In **Stage-A** of the problem, we see that the temperature which is distributed over the surface of the crack $b < x < c$ due to the heated wedge, causes the further opening of the crack, whose solution is obtained by solving (2.5). In **stage-B** of the problem we deal with the problem of elasticity in which we apply the method of Kushwaha [2] to solve the fourth order homogenous partial differential equation in (2.6), which represents the solution of elasticity problem, while the solution of Laplace's equation in (2.5) represents the solution of temperature distribution with the following conditions.

$$T(x,0) = \begin{cases} \theta(x) & 0 < |x| < b \\ 0 & c < |x| < a \end{cases} \tag{2.7}$$

$$T_y(x, 0) = f(x) \quad b < |x| < c \tag{2.8}$$

$$T_x(\pm a, y) = 0 \quad 0 < |y| < \infty \tag{2.9}$$

where $\theta(x)$ and $f(x)$ are known functions. The other component of displacement u_y is obtained in terms of u_x through the equations (2.1)-(2.2) as

$$u_y(x, y) = \frac{1}{1-\beta^2}\left[\beta^2 \int \frac{\partial u_x}{\partial x} dy + \int \frac{\partial u_x}{\partial y} dx\right]$$

$$\beta^2 = \frac{2(1-\sigma)}{1-2\sigma} = \frac{\lambda+\mu}{\mu} \tag{2.10}$$

where σ is the Poisson's ratio of the medium.

3. Reduction to Triple Series Equations

Stage-A: (Temperature distribution problem)

We take the finite Fourier cosine transform of the Laplace's equation (2.5) w.r.t. x. Next we solve the results equation and after inverting, we get

$$T(x, y) = \frac{A_0}{a} + \frac{2}{a}\sum_{p=1}^{\infty} A_p e^{-\alpha_p y} \cos(\alpha_p x) \tag{3.1}$$

where A_0 is a constant to be determined.

Since the geometry of the problem is symmetrical, therefore the boundary conditions (2.7) and (2.8) on using (3.1) yield the result,

$$\frac{A_0}{2} + \sum_{p=1}^{\infty} A_p \cos(\alpha_p x) = \frac{a}{2}\begin{cases} \theta(x) & 0 \le |x| \le b \\ 0 & c \le |x| \le a \end{cases} \tag{3.2}$$

and

$$\sum_{p=1}^{\infty} p A_p \cos(\alpha_p x) = -\frac{a^2}{2\pi} f(x) \tag{3.3}$$

Stage-B: (Elasticity problem)

We follow the method of Kushwaha [2] by taking finite Fourier sine transform from both sides of the equation (2.6), solving the results equations and then inverting, we get

$$u_x(x, y) = \frac{2}{a}\sum_{p=1}^{\infty}\left[B_p + yE_p\right] e^{-\alpha_p y} \sin(\alpha_p, x) \tag{3.4}$$

where B and E are constants to be determined.

On solving for the boundary condition (1.3), we get

$$E_p = \frac{P\pi}{a}(1-\beta^2)B_p \tag{3.5}$$

and for the boundary conditions (1.4) and (1.5), on using (3.5) yield the following triple series equations.

$$\sum_{p=1}^{\infty} - B_P \cos(\alpha_P, x) + \frac{B}{2} = \frac{a}{2}\beta^{-2}\begin{cases} u_0(x) & 0 \le |x| \le b \\ 0 & c \le |x| \le a \end{cases} \tag{3.6}$$

$$\sum_{p=1}^{\infty} PB_P \cos(\alpha_P, x) = \frac{a^2(3\beta^2-4)\alpha_t}{4\pi(\beta^2-1)} T(x,0) \tag{3.7}$$

$$b < x < c$$

4. Solution of Triple Series Equations

The trial solution for the triple series equations of (3.2)-(3.3) and (3.6)-(3.7) is sought with the help of method of Parihar [9].

We take the trial solution of (3.2)-(3.3) as follows

$$A_P = \frac{a}{P\pi}\left[\int_b^c g_0(t)\sin(\alpha_P t)dt - \int_0^b \theta'(t)\sin(\alpha_P t)dt\right] \tag{4.1}$$

and

$$A = \frac{a}{\pi}\left[\int_b^c t g_0(t)dt - \int_0^b t\theta'(t)dt\right] \tag{4.2}$$

on using the property

$$\frac{t}{2} + \sum_{P=1}^{\infty} \frac{\sin(\alpha_P t)\cos(\alpha_P x)}{P} = \begin{cases} \frac{\pi}{2} & t > x \\ 0 & t < x \end{cases} \tag{4.3}$$

the equation (3.2) will be satisfied identically if

$$\int_b^c g_0(t)dt = \theta(b) \tag{4.4}$$

where (b) is the temperature at the tip "b" of the crack. On the substitution of (4.1) in (3.3) and using the formula

$$\sum_{P=1}^{\infty} \frac{\sin(\alpha_P x)\sin(\alpha_P y)}{P} = \frac{1}{2} Ln \left| \frac{\sin q(\frac{y+x}{2})}{\sin q(\frac{y-x}{2})} \right| \tag{4.5}$$

and then inverting the results equation and adjusting the terms properly, and using Parihar's method [9], we get

$$g_0(t) = \frac{1}{q\delta(t)}\left[\int_b^c \frac{\sin qx\delta(x)p(x)}{\cos qx - \cos qt}dx + L_0\right] \tag{4.6}$$

where

$$q = \frac{\pi}{a}$$

and

$$\delta(t) = \left[(\cos qb - \cos qt)(\cos qt - \cos qc)\right]^{1/2} \tag{4.7}$$

$$p(x) = -af(x) + \int_0^b \theta'(t)\frac{\sin(qt)}{G(t,x)} \tag{4.8}$$

with

$$G(m,n) = \cos(qm) - \cos(qn) \tag{4.9}$$

where L_0 is an arbitrary constant to be determined through the conditions (4.4) and (4.6).

In a similar fashion, we take the trial solution for (3.6)-(3.7) as follows

$$B_P = \frac{a}{p\pi}\left[\int_b^c g_1(t)\sin(\alpha_P t)dt - \beta^{-2}\int_0^b u_0'(t)\sin(\alpha_P t)dt\right] \tag{4.10}$$

$$B = \frac{a}{\pi}\left[\int_b^c t g_1(t)dt - \beta^{-2}\int_0^b u_0'(t)dt\right] \tag{4.11}$$

and then (3.6) is satisfied identically if

$$\int_b^c g_1(t)dt = \beta^{-2}u_0(b) \tag{4.12}$$

On the substitution of (4.10) in (3.7) and next solving the integrals and after inverting, we get

$$g_1(t) = \frac{\Delta_0(t)}{\delta(t)} \tag{4.13}$$

where

$$\Delta_0(t) = \frac{1}{q}\left[\int_b^c \frac{\sin(qx)\delta(x)p_1(x)}{G(x_1 t)}dx + L_1\right] \tag{4.14}$$

and L_1 an arbitrary constant to be determined through the equations of (4.12)-(4.14).

$$p_1(x) = -\frac{a(3\beta^2-4)\alpha_t}{2(\beta^2-1)}T(x,0) + \beta^{-2}\int_0^b u_0'(t)\frac{\sin(qt)}{G(t,x)}dt \tag{4.15}$$

5. Physical Quantities

The heated wedge which develops the stresses that causes the further opening of the crack for the interval $b < |x| < c$ is a very important physical quantities, thus $T(x, 0)$ can be determined by solving the series in (3.2) for the interval $b < x < c$, as

$$T(x,0) = \int_x^c g_0(t)dt \quad \text{b < x < c} \tag{5.1}$$

where $g_0(t)$ is given by (4.6).

Crack Shape

The crack opening displacement $u_y(x, 0)$ is obtained by solving the equation of (3.6) for the interval $b < x < c$ as

$$u_y(x,0) = \beta^2\int_x^c g_1(t)dt \tag{5.2}$$

where $g_1(t)$ is given by (4.13)

Normal Stress Components

$\sigma_{yy}(x,0)$ is obtained by solving the series equation of (3.7) for the intervals $0 \le |x| \le b$ and $c \le |x| \le a$ thus substitution (4.10) in (3.7) and then using (4.13), we get.

$$\sigma_{yy}(x,0) = \frac{1}{2q}\left[\pm\frac{a}{2}\frac{\Delta_0(x)}{\delta_1(x)} - \beta^{-2}\int_0^b u_0'(t)\frac{\sin(q,t)}{G(t,x)}dt\right] \tag{5.3}$$

$\pm$ signs are for the intervals $0 \le |x| \le b$ and $c \le |x| \le a$ i.e.

$$\delta_1(x) = [G(x,b)G(x,c)]^{1/2} \quad 0 \le |x| \le b \tag{5.4}$$

$$\delta_1(x) = [G(b,x)G(c,x)]^{1/2} \quad c \le |x| \le a \tag{5.5}$$

Stress Intensity Factor

A very interested physical quantity in fracture mechanics are stress intensity factors, which are very important to be calculated at the edge of the crack tips. They are defined as follows

$$K_b = \lim_{x\to b}\sqrt{b-x}\quad \sigma_{yy}(x,0) \tag{5.6}$$

$$K_c = \lim_{x\to c}\sqrt{x-c}\quad \sigma_{yy}(x,0) \tag{5.7}$$

Substituting from (5.3) in (5.6)-(5.7), next by solving and then evaluating the limits, we get

$$K_b = \Delta_0(b)\, n(b) + t_1(b) \tag{5.8}$$

$$K_c = -\Delta_0(c)\, n(c,) \tag{5.9}$$

where $\Delta_0(t)$ is given by (4.14) and

$$n(x) = -\frac{a}{4q\sqrt{q\sin(qx)G(b,c)}} \tag{5.10}$$

$$t_1(b) = \frac{\beta^{-2}}{2q}\lim_{x\to b^+}\sqrt{b-x}\int_0^b u_0'(t)\frac{\sin(qt)dt}{G(t,x)} \tag{5.11}$$

6. Special Cases

To emphasize more on the analysis which is done, we discuss two cases in which we determine the stress intensity factor and then report on the crack shape.

Case-I

In this case we assume that the flux $f(x)$ on the surface of the crack $b < x < c$, the wedge shape function u_0 (x) and the temperature distribution θ (x) to be constant. i.e.

$$f(x) = f_0 \quad \text{(Constant)}$$

$$u_0(x) = u_0 \quad \text{(Constant)} \qquad (6.1)$$

$$\theta(x) = \theta_0 \quad \text{(Constant)}$$

To determine the stress intensity factor we first obtain g_0 (t), which from (4.8) on using (6.1) we get for $p(x)$ as

$$p(x) = -a\, f_0 \qquad (6.2)$$

On substitution of (6.2) in (4.6) and next by solving the integrals we get

$$g_0(t) = \frac{1}{q\delta(t)}\left[-\frac{a^2}{2}\, f_0\{G(b,c) - 2G(t,c) + L_0\}\right] \qquad (6.3)$$

$$b < y < c$$

where $G(m, n)$ is given by (4.9). The constant L_0 can be easily determined by integrating (6.3) through (4.4), as

$$L_0 = \frac{1}{2\psi_1(\frac{\pi}{2})}\left[q^2\theta(b)\sin(qc) + a^2 f_0 G(b,c)\left\{\psi_1(\frac{\pi}{2}) - 2\psi_2(\frac{\pi}{2})\right\}\right] \qquad (6.4)$$

where

$$\psi_n(\phi) = \int_0^{\phi} \frac{\sin^{2(n-1)}\theta}{F(\theta)}\, d\theta \qquad (6.5)$$

with

$$F(\theta) = \left[1 - \frac{\alpha^2 - \beta^2}{1-\beta^2}\sin^2\theta\right]^{1/2}\left[1 + \frac{\alpha^2 - \beta^2}{\beta^2}\sin^2\theta\right]^{1/2} \qquad (6.6)$$

$$\phi = \sin^{-1}\left(\frac{\beta^2 - x^2}{\beta^2 - \alpha^2}\right)^{1/2} \qquad (6.7)$$

Now to get the crack shape we need g_1 (t), where p_1 (x) and $T(x, 0)$ in first case on using (6.1) are reduced to

$$p_1(x) = \frac{-a(3\beta^2 - 4)\alpha_t}{2(\beta^2 - 1)}\; T(x, 0) \qquad (6.8)$$

$$T(x,0) = \frac{1}{q^2\sin(qc)} \times\left[-a^2 f_0 G(b,c)\{\psi_1(\phi) - 2\psi_2(\phi)\} + 2L_0\psi_1(\phi)\right] \qquad (6.9)$$

substitution of (6.8) in (4.14) and then using (4.13) yields the result

$$g_1(t) = \frac{1}{q\delta(t)}\left[\frac{-a(3\beta^2 - 4)\alpha_t}{2(\beta^2 - 1)} \int_b^c \frac{\sin(qx)\delta(x)T(x,0)}{G(x,t)}dx + L_1\right] \qquad (6.10)$$

the constant L_1 can be easily determined by integrating (6.10) by using (4.12). The stress intensity factors in this case on using the conditions of (6.1) are reduced to

$$K_b = n(b)\,\Delta_0(b) \qquad (6.11)$$

$$K_c = n(c)\,\Delta_0(c) \qquad (6.12)$$

where $\Delta_0(b)$ and $\Delta_0(c)$ from (4.14) are reduced to

$$\Delta_0(b) = \frac{1}{q}\left[\int_b^c p_1(x)\sin(qx)\sqrt{\frac{G(c,x)}{G(x,b)}}T(x,0) + L_1\right] \qquad (6.13)$$

$$\Delta_0(c) = \frac{1}{q}\left[\int_b^c p_1(x)\sin(qx)\sqrt{\frac{G(b,x)}{G(x,c)}}T(x,o) + L_1\right] \qquad (6.14)$$

the crack shape can be easily determined by integrating (6.10) by using (5.2).

Case-II

As in case-I we assume the wedge shape function $u_0(x)$ and the temperature distribution $\theta(x)$ to be constant, but the surface of the crack to be insulated, i.e. there is no flux $f(x)$ in the interval $b < x < c$.

$$u_0(x) = u_0 \quad \text{(constant)}$$
$$\theta(x) = \theta_0 \quad \text{(constant)} \tag{6.15}$$

$$f(x) = 0 \tag{6.16}$$

using (6.16) in (6.2), we get

$$p(x) = 0 \tag{6.17}$$

the condition of (6.16) reduces $g_0(t)$ from (6.3) to

$$g_0(t) = \frac{L_0}{q\delta(t)} \tag{6.18}$$

where L_0 from (6.4) on using (6.16) is reduced to

$$L_0 = \frac{q^2\theta(b)\sin(qc)}{2\psi_1(\frac{\pi}{2})} \tag{6.19}$$

and similarly the constant L_1 can also be obtained on using (6.16) in (6.10). Hence the stress intensity factors K_b and K_c in this case are reduced to

$$K_b = n(b)\,\Delta_0(b)$$

$$K_c = n(c)\,\Delta_0(c)$$

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