APPLICABILITY OF THE WELL OPERATED CONDITIONS FOR THE SOLUTION OF MIXED PROBLEM ATTRIBUTED TO SCHRÖDINGER EQUATION

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Abstract

As a sequel of the recent works, we would like to discuss another stage for the solution of the mixed problem which considers the concepts of *well operated conditions* applicable to the solution of mixed problem, i.e. the existence and uniqueness of the solution must always conformable with some assumptions.

Introduction

It is well known in classical mathematical courses that the partial differential equations for parabolic or hyperbolic usually are Cauchy type problems, or mixed problems (i.e., Cauchy problem with boundary conditions). For an elliptical equation, the boundary conditions are considered by some other problems such as, Dirichlet and Neumann problems or in a specific case, the Poincaré problem [1-3]. They have suggested that, for an elliptical equation, conditions of the mixed problem are local boundary conditions, as petrovskii in his consideration has pointed out [4]. However, there is a possibility to apply such problems for a mixed problem with non-local boundary conditions transfering the mixed problem to the spectral problem form, [5]. The transformed boundary problem (spectral problem), under some conditions, will be in the form of the second type Fredholm's integral at a half cylinder space [6].

In the final study the existence and uniqueness of the solution for the spectral problem has been treated, and the boundary equations for domain D were defined [7].

Keywords: Well operated spectral problem; Mixed boundary conditions; Local boundary conditions

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Analytic Solution of the Schrödinger Equation

The present study considers existence and uniqueness of the solution for the Schrödinger equation at half cylinder space with some assumptions to find a well operated solution. To obtain this, it is necessary to find an asymptote for a adjoint problem [the boundary value problem correlated to mixed problem]. However, the mixed problem has been recently quoted [6]:

$$ih\frac{\partial u(x,t)}{\partial t}+\frac{h^2}{2\mu}\Delta_x u(x,t)-V(x)u(x,t)=0$$

$$x=(x_1,x_2)\in D\subset\mathbb{R}^2, t>0 \tag{1}$$

and its boundary conditions

$$u(x,0)=\psi(x),\quad x\in\overline{D} \tag{2}$$

$$\sum_{k=1}^{2}\left[\sum_{j=1}^{2}\alpha_{pj}^{(k)}(x_1)\frac{\partial u(x,t)}{\partial x_j}+\alpha_{p0}^{(k)}(x_1)u(x,t)]\right]_{|x_2=\gamma_k(x_1)}$$

$$=\alpha_p(x_1,t)$$

$$x_1\in[a_1,b_1];\quad p=1,2;\quad t\geq 0 \tag{3}$$

To solve such a problem as proposed by Courant and Vladimirov [1,2], the Laplace transformation should be carried out, using a method suggested by Rasulov [8]. The transformed form of the mixed problems could be written as follow:

$$\Delta_x\tilde{u}(x,\lambda,+\frac{2\mu}{h^2}(ih\lambda-V(x))\tilde{u}(x,\lambda)=\frac{2\mu i}{h}\psi(x),$$

$$x\in D \tag{4}$$

and the transformed boundary condition is:

$$\sum_{k=1}^{2}\left[\sum_{j=1}^{2}\alpha_{pj}^{(k)}(x_1)\frac{\partial\tilde{u}(x,\lambda)}{\partial x_j}+\alpha_{p0}^{(k)}(x_1)\tilde{u}(x,\lambda)\right]_{|x_2=\gamma_k(x_1)}$$

$$=\tilde{\alpha}_p(x_1,\lambda)$$

$$x_1\in[a_1,b_1];\quad p=1,2, \tag{5}$$

using the Helmohtz equation,

$$\Delta_x\tilde{u}(x,\lambda)+\frac{2\mu\lambda i}{h}\tilde{u}(x,\lambda)=0$$

where the general solution for the above equation is proposed in [2] as:

$$U(x-\xi,\lambda)=-\frac{i}{4}H_0^{(1)}\left(\sqrt{\frac{2\mu\lambda i}{h}}|x-\xi|\right) \tag{6}$$

where $H_0^{(1)}\left(\sqrt{\frac{2\mu\lambda i}{h}}|x-\xi|\right)$ is the Hankel function, and the asymptote of the Equation (6) obtained by [2] as following:

$$U(x-\xi,\lambda)=-\frac{i}{4\sqrt{\pi}}\sqrt[4]{\frac{h}{2\mu}}\frac{1-i}{\sqrt[4]{\lambda i}.\sqrt{|x-\xi|}}\times$$

$$\exp\left[-\sqrt{(2\mu/h)}\left(\sqrt{(|\lambda|+\lambda_2)/2}\right.\right.$$

$$\left.\left.-i\sqrt{(|\lambda|-\lambda_2)/2}\right)|x-\xi|\right]+\cdots\quad\lambda\to\infty \tag{7}$$

where λ_2 is the imaginary part of λ, for $|x-\xi|\to 0$ the Equation(6) could be written as following:

$$U(x-\xi,\lambda)=\frac{2i}{\pi}\ln|x-\xi|+\cdots \tag{8}$$

Any solution for Equation (4) at a defined region of the domain *D* would be written as below [6].

$$\tilde{u}(\xi,\lambda)=\int_\Gamma\left[\tilde{u}(x,\lambda)\frac{\partial U(x-\xi,\lambda)}{\partial x_1}-\right.$$

$$\left.-\frac{\partial\tilde{u}(x,\lambda)}{\partial x_1}U(x-\xi,\lambda)\right]\cos(v,x_1)dx+$$

$$+\int_\Gamma\left[\tilde{u}(x,\lambda)\frac{\partial U(x-\xi,\lambda)}{\partial x_2}-\right.$$

$$\left.-\frac{\partial\tilde{u}(x,\lambda)}{\partial x_2}U(x-\xi,\lambda)\right]\cos(v,x_2)dx+$$

$$+\frac{2\mu}{h^2}\int_D V(x)\tilde{u}(x,\lambda)U(x-\xi,\lambda)dx+$$

$$+\frac{2\mu i}{h}\int_D\psi(x)U(x-\xi,\lambda)dx\quad\xi\in D \tag{9}$$

The solution of (4) and (5), that is (9) should satisfy the boundary value.

$$\tilde{u}(\xi,\lambda)=2\int_\Gamma\left[\tilde{u}(x,\lambda)\frac{\partial U(x-\xi,\lambda)}{\partial x_1}-\right.$$

$$\left.-\frac{\partial\tilde{u}(x,\lambda)}{\partial x_1}U(x-\xi,\lambda)\right]\cos(v,x_1)dx$$

$$+2\int_{\Gamma}\left[\tilde{u}(x,\lambda)\frac{\partial U(x-\xi,\lambda)}{\partial x_2}-\right.$$

$$\left.-\frac{\partial\tilde{u}(x,\lambda)}{\partial x_2}U(x-\xi,\lambda)\right]\cos(v,x_2)dx+$$

$$+\frac{4\mu}{h^2}\int_{\cdots D}V(x)\tilde{u}(x,\lambda)U(x-\xi,\lambda)dx+$$

$$+\frac{4\mu i}{h}\int_D\psi(x)U(x-\xi,\lambda)dx \quad \xi\in\Gamma \quad (10)$$

The above conditions have been used to handle the problem in the previous work, [6], again, to give a solution for the mixed problem (1-3) by the reversed Laplace transformation, the condition proposed by [1,2,8] in the form of:

$$u(x,t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\tilde{u}(x,\lambda)e^{\lambda t}d\lambda \quad (11)$$

where $c>0$ and is a constant and $\tilde{u}(x,\lambda)$ is the solution of the boundary value problem, by these assumptions and using the following relation:

$$\Delta_x U(x-\xi,\lambda)+\frac{2\mu\lambda i}{h}U(x-\xi,\lambda)=\delta(x-\xi) \quad (12)$$

where $\delta(x-\xi)$ is the Dirac delta function. In this case, the last term of the right hand side in (10) can be calculated as following:

$$\frac{4\mu i}{h}\int_D\psi(x)U(x-\xi,\lambda)dx$$

$$=\frac{4\mu i}{h}\int_D\psi(x)\left[\frac{h}{2\mu\lambda i}\delta(x-\xi)-\frac{h}{2\mu\lambda i}\Delta_x U(x-\xi,\lambda)\right]dx$$

$$=\frac{1}{\lambda}\psi(\xi)-\frac{2}{\lambda}\int_D\psi(x)\Delta_x U(x-\xi,\lambda)dx=\frac{\psi(\xi)}{\lambda}$$

$$-\frac{2}{\lambda}\int_{\Gamma}\left[\psi(x)\frac{\partial U(x-\xi,\lambda)}{\partial v_x}-\frac{\partial\psi(x)}{\partial v_x}U(x-\xi,\lambda)\right]dx$$

$$-\frac{2}{\lambda}\int_D\Delta\psi(x)U(x-\xi,\lambda)dx \quad (12_1)$$

By the above conditions the following remarks can be elucidated:

Remark 1. If $\psi(x)\in C^{(4)}(D)\cap C^{(3)}(\overline{D})$ and the following equation

$$\psi(x)=\frac{\partial\psi(x)}{\partial v_x}=\Delta\psi(x)\approx 0 \qquad x\in\Gamma$$

is maintained, then, the asymptotic relation of (12_1) will be as following:

$$\frac{4\mu i}{h}\int_D\psi(x)U(x-\xi,\lambda)dx=O(|\lambda|^{-2-\frac{1}{4}}), \quad \xi\in\Gamma$$

Remark 2. According to Remark 1 for the Equation (9), Remark 2 will be elucidated:

$$\frac{2\mu i}{h}\int_D\psi(x)U(x-\xi,\lambda)dx$$

$$=\frac{\psi(\xi)}{\lambda}-\frac{h}{2\mu i\lambda^2}\Delta\psi(\xi)+O(|\lambda|^{-2-\frac{1}{4}}) \qquad \xi\in D$$

Assume that for $t>0$ the (11), its derivative and second derivative to x in term of x_1 and x_2 are converged and also (11) is finite for $t\to 0$. Now, if the (11) is substituted in Equation (1) we could drive an equation.

$$\frac{2\mu i}{h}\frac{\partial u(x,t)}{\partial t}+\Delta_x u(x,t)-\frac{2\mu}{h^2}V(x)u(x,t)$$

$$=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{\lambda t}\left\{\Delta_x\tilde{u}(x,\lambda)+\frac{2\mu}{h^2}[hi\lambda-V(x)]\tilde{u}(x,t)\right\}d\lambda$$

$$=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{\lambda t}\frac{2\mu i}{h}\psi(x)d\lambda=\frac{\mu}{\pi h}\psi(x)e^{ct}i\int_{\mathbb{R}}e^{i\rho t}d\rho$$

$$=\frac{2\mu i}{h}\psi(x)e^{ct}\delta(t)=0 \qquad t>0 \quad (13)$$

This relation concludes that (11) satisfies Equation (1). If (11) is substituted at boundary conditions (3), the following relation is possible:

$$\sum_{k=1}^{2}\left[\sum_{j=1}^{2}\alpha_{pj}^{(k)}(x_1)\frac{\partial u(x,t)}{\partial x_j}+\alpha_{p0}^{(k)}(x_1),u(x,t)\right]\Bigg|_{x_2=\gamma_k(x_1)}$$

$$=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{\lambda t}d\lambda$$

$$\times\sum_{k=1}^{2}\left[\sum_{j=1}^{2}\alpha_{pj}^{(k)}(x_1)\frac{\partial\tilde{u}(x,\lambda)}{\partial x_j}+\alpha_{p0}^{(k)}(x_1)\tilde{u}(x,\lambda)\right]\Bigg|_{x_2=\gamma_k(x_1)}$$

$$=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{\lambda t}\tilde{\alpha}_p(x_1,\lambda)d\lambda=\alpha_p(x_1,t),$$

$$p=1,2 \qquad x_1\in[a_1,b_1] \qquad t>0 \tag{14}$$

that is, (11) also satisfies the boundary conditions. However for $f(\lambda)=\lambda^{-n-1}$ which has been pointed by [9], the reversed Laplace transforniation of the above relation is:

$$f(t)=\frac{t^n}{\Gamma(n+1)} \quad \text{for} \quad n>-1$$

and

$$\frac{t^0}{\Gamma(1)}=\frac{t^0}{0}=1=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{\lambda t}}{\lambda}d\lambda \quad \text{for} \quad n=0$$

If the first term of asymptotic relation of $\tilde{u}(x,\lambda)$ is on the form of the above relation, then (11) can be written as following:

$$u(x,0)=\frac{1}{2\pi i}Lim_{t\to 0}\int_{c-i\infty}^{c+i\infty}e^{\lambda t}\tilde{u}(x,\lambda)d\lambda$$

$$=\frac{1}{2\pi i}Lim_{t\to 0}\int_{c-i\infty}^{c+i\infty}e^{\lambda t}\frac{\psi(x)}{\lambda}d\lambda=\psi(x) \tag{15}$$

i.e., (11) satisfies the initial condition (i.e. (2)). As pointed in [8], from the asymptotic relation of (7) it is clear that for $\lambda_2>0$, exponential terms in$U(x-\xi,\lambda)$ tends to zero, and when $|\lambda|\to\infty$, (11) shows that the upper part of the Laplace asymptotic line bends towards the left side of imaginary axis, so that, the variable λ on this carvture moves:

if $Rel\,\lambda<0$ $|\lambda|\to\infty$ and $t>0$ then $|e^{\lambda t}|=e^{t\,Rel\lambda}\to 0$.

This treatment is not possible for the lower part of the Laplace asymptote line as shown in asymptotic relation of (7) if $\lambda_2<0$ then, the exponential term of the $U(x-\xi,\lambda)$ will not tend to zero. This means that, there exists a possible specific value just close to imaginary axis on the lower part (spectral problem). If $|\lambda|\to 0$ towards the lower part of the Laplace asymptote line, the $U(x-\xi,\lambda)$ tends to zero gently, in other words, behaves as Fourier coefficient.

Substituting relation (12) at (9) and (10) with $x\in\Gamma$ and $\xi\in D$, and considering that, the dependent term to δ is zero we obtain a solution for the $U(x-\xi,\lambda)$. The other terms of Laplace operator, which $U(x-\xi,\lambda)$ has those terms, non of its derivative could be written using partial integral. To obtain the asymptotic relation for the last term of Equation (9) in the form of Remark 1, we should repeat the operation as necessary. The required asymptotic relation from second term on the right side of Equation (9) can be obtained by substitution of the $\tilde{u}(x,\lambda)$ and repeat the same stages, to maintain the asymptotic relation.

Conclusion

The problem has been considered, by taking an advantage of Ferdholem's Integral [6]. In the present study, the transformed form of the boundary value problem can be written as relations (4) and (5). The solution for the tranformed problem proposed as Equation (9). With the conditions of Remarks 1 and 2 a well operated solution for the mixed problem would be elucidated, that is a conformable and unique solution of the problem.

References

1. Courant R. and Hilbert D. *Methods of Mathematical Physics*, John Wiley & Sons, (1953).
2. Vladimirov V. S. *Equations of Mathematical Physics,* Mir Publisher Moscow, (1984).
3. Tikhonov, A. N. and Samarskii A. A. *Equations of Mathematical Physics*, Nauka, Moscow, (English Translation, Pergamon Press, Oxford), (1972).
4. Petrovskii, I. G. *Lectures on Partial Differential Equations*, Inter Science Publisher, New York, (1954).
5. Kavei, G. and Aliev, N. Danshvar, Shahed University Pub., Iran, Nos. 11-12, pp.15-20, (1996).
6. Kavei, G. and Aliev, N. *Bul. of Pure and Appl. Sci.*, **16**(2): 253-363, (1997).
7. Kavei, G. and Aliev, N. *J. Sci. I. R. Iran*, **10**(4): 252-57, (1999).
8. Rasulov, M. L. *Methods of Contour Integration*, North Holland Pub. Co., Amsterdam, (1967).
9. Tranter, C. J. *Integrat Transforms in Mathematical Physics*, John Wiley & Sons, (1949).