TRANSFORMATION SEMIGROUPS AND TRANSFORMED DIMENSIONS

M. Sabbaghan* and F. Ayatollah Zadeh Shirazi

Department of Mathematics, Faculty of Science, University of Tehran, Tehran, Islamic Republic of Iran

Abstract

In the transformation semigroup \((X, S)\) we introduce the height of a closed nonempty invariant subset of \(X\), define the transformed dimension of nonempty subset \(S\) of \(X\) and obtain some results and relations.

Keywords: \(\alpha\)-minimal set; Enveloping semigroup; Height; Transformation semigroup; Transformed dimension

1991 AMS Classification Subject: 54H20

*E-mail: sabbagh@khayam.ut.ac.ir

Preliminaries

By a transformation semigroup \((X, S, \pi)\) (or simply \((X, S)\)) we mean a compact Hausdorff topological space \(X\), a discrete topological semigroup \(S\) with identity \(e\) and a continuous map \(\pi: X \times S \rightarrow X(\pi(x,s)=xs(\forall x \in X, \forall s \in S))\) such that:

- \(\forall x \in X\) \(xe=x\),
- \(\forall x \in X\) \(\forall s, t \in S\) \(x(st)=(xs)t\).

In the transformation semigroup \((X, S)\) we have the following definitions:

1. For each \(s \in S\), define the continuous map \(\pi^s: X \rightarrow X\) by \(x \pi^s = xs(\forall x \in X)\), then the closure of \(\{\pi^s \mid s \in S\}\) in \(X^X\) with pointwise convergence, is called the enveloping semigroup (or Ellis semigroup) of \((X, S)\).

We shall denote the enveloping semigroup by \(E(X, S)\) or simply by \(E(X)\) if there is no possibly confusion [1, Definition 6.4]. \(E(X, S)\) has a semigroup structure [2, Chapter 3]. An element \(u\) of \(E(X, S)\) is called idempotent if \(u^2=u\). Every closed nonempty subsemigroup of \(E(X, S)\) has an idempotent element [3, Chapter I, Proposition 2.1].

A nonempty subset \(K\) of \(E(X, S)\) is called a right ideal if \(KE(X, S) \subseteq K\), and it is called a minimal right ideal if none of the right ideals of \(E(X, S)\) be a proper subset of \(K\). For each right ideal \(K\) of \(E(X, S)\) and each \(p \in K\), \(L_p: K \rightarrow K\) is defined by \(L_p(q)=pq(\forall q \in K)\).

2. A nonempty subset \(Z\) of \(X\) is called invariant if \(ZS \subseteq Z\), moreover it is called minimal if it be closed and none of the closed invariant subsets of \(X\) be a proper subset of \(Z\). The element \(a\) of \(X\) is called almost periodic if the orbit closure of \(a\) i.e. \(aS=aE(X, S)\) is a minimal subset of \(X\).

3. Let \(a \in X\), \(A\) be a nonempty subset of \(X\), \(C\) be a nonempty subset of \(E(X, S)\) and \(I\) be a right ideal of \(E(X)\), then we introduce the following sets [6, Notation 5]:

\[
\begin{align*}
F(a, C) &= \{p \in C \mid ap = p\}, \\
F(A, C) &= \{p \in C \mid \forall b \in A \quad bp = p\}, \\
\overline{F}(A, C) &= \{p \in C \mid Ap = A\}, \\
J(C) &= \{p \in C \mid p^2 = p\}, \\
S(I) &= \{p \in C \mid L_p : I \rightarrow I\ \text{is surjective}\}.
\end{align*}
\]

4. Let \((Y, S)\) be a transformation semigroup, the continuous map \(\phi: (X, S) \rightarrow (Y, S)\) is a homomorphism if \(\phi(xs) = \phi(x)s(\forall x \in X, \forall s \in S)\). If \(\phi: (X, S) \rightarrow (Y, S)\) is an onto homomorphism, then there exists an onto
homomorphism $\phi : (E(X, S), S) \to (E(Y, S), S)$ (which is also a semigroup homomorphism) such that for each $x \in X$ and $p \in E(X, S)$ we have $\phi(\varphi(p)) = \varphi(x)\phi(p)$ [2, Proposition 3.8].

5. Let $a \in X, A$ be a nonempty subset of $X$ and $K$ be a closed right ideal of $E(X, S)$, then [5, Definition 1]:

- $K \in M_{(X,S)}(a)$ (or simply $M(a)$) if:
  - $aK = aE(X, S)$,
  - $K$ does not have any proper subset like $L$, such that $L$ is a closed right ideal of $E(X, S)$ such that $aL = aE(X, S)$,
- $K \in \overline{M}_{(X,S)}(A)$ (or simply $\overline{M}(A)$) if:
  - $\exists b \in A, bK = bE(X, S)$,
  - $K$ does not have any proper subset like $L$ such that $L$ is a closed right ideal of $E(X, S)$ such that $bL = bE(X, S)$ for all $b \in A$,
- $K \in \overline{M}_{(X,S)}(A)$ (or simply $\overline{M}(A)$) if:
  - $AK = AE(X, S)$,
  - $K$ does not have any proper subset like $L$ such that $L$ is a closed right ideal of $E(X, S)$ such that $AL = AE(X, S)$.

The elements of $M_{(X,S)}(a)$ (resp. $\overline{M}_{(X,S)}(a)$ and $\overline{M}_{(X,S)}(A)$) are called $a$-minimal (resp. $A$-minimal and $A$-minimal) sets. $\overline{M}_{(X,S)}(A)$ and $M_{(X,S)}(a)$ are nonempty ([5, Theorem 2], and [6]).

6. Define $\overline{M}_{(X,S)}(A) = \{0 \neq A \subseteq X | \forall K \in \overline{M}_{(X,S)}(A) J(F(A, K)) \neq \emptyset \}$ and $\overline{M}_{(X,S)}(a) = \{0 \neq A \subseteq X | \overline{M}_{(X,S)}(A) \neq \emptyset \}$.

7. Let $a \in X$ and $A$ be a nonempty subset of $X$, then [5, Definition 13]:

- $(X, S)$ is called $a$-distal if $E(X, S) \subseteq M_{(X,S)}(a)$,
- $(X, S)$ is called $A$-distal if $(X, S)$ be $b$-distal for each $b \in A$,
- $(X, S)$ is called $A\overline{M}$-distal if $E(X, S) \subseteq \overline{M}_{(X,S)}(A)$,
- $(X, S)$ is called $A\overline{M}$-distal if $E(X, S) \subseteq \overline{M}_{(X,S)}(A)$.

8. Let $A, B$ be nonempty subsets of $X$ and $R, Q \subseteq \{M_{(X,S)}(A), \overline{M}_{(X,S)}(A)\}$, then [5, Definition 13]:

- $B$ is called is $A$-almost periodic if:
  $\forall a \in A \forall K \subseteq M_{(X,S)}(a) \forall b \in B \exists L \in M_{(X,S)}(b) L \subseteq K$,

- $B$ is called $A\overline{(R)}$-almost periodic if:
  $\forall a \in A \forall K \subseteq M_{(X,S)}(a) \exists L \in R(B) L \subseteq K$,

- $B$ is called $A\overline{(M)}$-almost periodic if $Q(A) \neq \emptyset$ and:
  $\forall K \subseteq Q(A) \forall b \in B \exists L \in M_{(X,S)}(b) L \subseteq K$,

- $B$ is called $A\overline{(\overline{M})}$-almost periodic if $Q(A) \neq \emptyset$ and:
  $\forall K \subseteq Q(A) \exists L \in R(B) L \subseteq K$.

**Definition 1.** Let $A$ be a nonempty subset of $X, a \in A$ and $Z$ be a closed invariant subset of $X$, we introduce height of $Z$ by:

$h_{(X,S)}(Z) = \{n \in N \cup \{0\} | \exists i \in \{0, \ldots, n\} \forall j \in \{0, \ldots, n\} - \{i\} Z_j \neq Z_i \}$

and transformed dimension of $a$, transformed dimension of $A$, $\overline{M}$-transformed dimension of $A$ and $\overline{M}$-transformed dimension of $A$ is defined respectively by:

- $\text{Dim} \overline{T_{(X,S)}}(a) = \sup \{h_{(E(X,S),I)}(I) | I \in M_{(X,S)}(a)\}$ (or simply $\text{Dim} \overline{T}(a)$),
- $\text{Dim} \overline{T_{(X,S)}}(A) = \sup \{\text{Dim} \overline{T_{(X,S)}}(x) | x \in A\}$ (or simply $\text{Dim} \overline{T}(A)$),
- $\text{Dim} \overline{T_{(X,S)}}(A) = \sup \{h_{(E(X,S),I)}(I) | I \in \overline{M}_{(X,S)}(A)\}$ (or simply $\text{Dim} \overline{T}(A)$),
- $\text{Dim} \overline{T_{(X,S)}}(A) = \sup \{h_{(E(X,S),I)}(I) | I \in \overline{M}_{(X,S)}(A)\}$ (or simply $\text{Dim} \overline{T}(A)$).

If $h_{(aE(X))} = n \in N \cup \{0, \infty\}$, then $a$ is called $n$-almost periodic (or almost periodic of degree $n$).

**Lemma 2.** In the transformation semigroup $(X, S)$, let $I$
and \( J \) be two (closed) right ideals of \( E(X) \), and suppose there exist \( u \in J(S(I)) \) and \( v \in J(S(J)) \) such that \( uv = u \) and \( vu = v \), then for each chain of (closed) right ideals of \( E(X) \) in \( J \), there exists a chain of (closed) right ideals of \( E(X) \) in \( J \) with the same cardinal. So if \( I \) and \( J \) be closed, then \( h(I) = h(J) \).

**Proof.** Let \( \{J_\alpha \mid \alpha \in \Gamma\} \) be a chain of (closed) right ideals of \( E(X) \) in \( I \), then \( \{vJ_\alpha \mid \alpha \in \Gamma\} \) is a chain of (closed) right ideals of \( E(X) \) in \( J \) and we have card \( \{J_\alpha \mid \alpha \in \Gamma\} \gtrless \{vJ_\alpha \mid \alpha \in \Gamma\} \). Using \( \nu v = u \) and the fact that \( L_{\nu J} = \text{id}_I \), we have card \( \{J_\alpha \mid \alpha \in \Gamma\} = \text{card} \{vJ_\alpha \mid \alpha \in \Gamma\} \).

**Corollary 3.** In the transformation semigroup \((X, S)\) let \( A \) be a nonempty subset of \( X \), then we have:

- a. \( \forall a \in X \quad \forall K \in M(a) \quad \text{Dim}^T(a) = h(K) \),
- b. if \( A \in \overline{M}(X, S) \) then for each \( K \in \overline{M}(A) \), \( \text{Dim}^T(M)(A) = h(K) \),
- c. if \( A \in \overline{M}(X, S) \) then for each \( K \in \overline{M}(A) \), \( \text{Dim}^T(M)(A) = h(K) \).

**Proof.** Let \( a \in X \) (resp. \( A \in \overline{M}(X, S) \), \( A \in \overline{M}(X, S) \)) and \( I, J \in M(a) \) (resp. \( I, J \in \overline{M}(A) \), \( I, J \in \overline{M}(A) \)), then there exist \( u \in J(F(a, I)) \) (resp. \( u \in J(F(A, I)) \)) such that \( uv = u \). Using \( \nu v = v \) and the fact that \( L_{\nu J} = \text{id}_I \), we have card \( \{J_\alpha \mid \alpha \in \Gamma\} = \text{card} \{vJ_\alpha \mid \alpha \in \Gamma\} \).

**Note 4.** In the transformation semigroup \((X, S)\) we have:

1. If \( Z \) and \( W \) be nonempty closed invariant subsets of \( X \), then:
   - a. \( Z \subseteq W \Rightarrow h_{(X, S)}(Z) = h_{(W, S)}(Z) \),
   - b. \( Z \subseteq W \Rightarrow h(Z) \leq h(W) \),
   - c. \( (Z \subseteq W \land h(Z) = h(W) < +\infty) \Rightarrow Z = W \),
   - d. \( h(Z) = 0 \Leftrightarrow Z \) is a minimal subset of \((X, S)\).
2. If \( 0 \neq B \subseteq A \subseteq X \), then:
   - a. \( \text{Dim}^T(B) \leq \text{Dim}^T(A) \),
   - b. if \( B \in \overline{M}(X, S) \), then \( \text{Dim}^T(M)(B) \leq \text{Dim}^T(M)(A) \),
   - c. \( \max \{\text{Dim}^T(B) \leq \text{Dim}^T(M)(A), \text{Dim}^T(M)(A)\} \),
   - d. \( \text{Dim}^T(B) \leq \text{Dim}^T(M)(A) \).
3. if \( \emptyset \neq A \subseteq X \), then the following statements are equivalent:
   - a. for each \( a \in A \), \( a \) is almost periodic,
   - b. \( \text{Dim}^T(A) = 0 \),
   - c. \( \text{Dim}^T(M)(A) = 0 \),
   - d. \( \text{Dim}^T(M)(A) = 0 \).
4. if \( a \in A \subseteq X \) and \( h(E(X)) < +\infty \), then:
   - a. \( (X, S) \) is distal if and only if \( h(E(X)) = 0 \),
   - b. \( (X, S) \) is \(-a\)-distal if and only if \( h(E(X)) = \text{Dim}^T(a) \),
   - c. \( (X, S) \) is \( A \overline{M} \) distal if and only if \( h(E(X)) = \text{Dim}^T(M)(A) \),
   - d. \( (X, S) \) is \( A \overline{M} \) distal if and only if \( h(E(X)) = \text{Dim}^T(M)(A) \).

**Proof.**

2. We have:
   - a. \( B \subseteq A \Rightarrow \sup \{\text{Dim}^T(x) \mid x \in B\} \leq \sup \{\text{Dim}^T(x) \mid x \in A\} \Rightarrow \text{Dim}^T(B) \leq \text{Dim}^T(A) \)
   - b. \( \forall K \in \overline{M}(A) \quad \forall b \in A \quad ak = abE(X) \)
   - c. \( \forall K \in \overline{M}(A) \quad \forall b \in B \quad bK = bE(X) \)
   - d. \( \forall K \in \overline{M}(A) \quad \exists L \in \overline{M}(B) \quad L \subseteq K \)
   - e. \( \forall K \in \overline{M}(A) \quad \exists L \in \overline{M}(B) \quad h(L) \leq h(K) \)
   - f. \( \forall K \in \overline{M}(A) \quad \text{Dim}^T(M)(B) \leq h(K) \)
   - g. \( \text{Dim}^T(M)(B) \leq \sup \{h(K) \mid K \in \overline{M}(A)\} = \text{Dim}^T(M)(A) \)
   - h. For each \( x \in A \) we have (by b):
3. Use the following facts:
- for each closed right ideal $K$ of $E(X)$, $h(K)=0$ if and only if $K$ is a minimal right ideal of $E(X)$,
- if $\text{Min}(E(X))$ denotes the set of all minimal right ideals of $E(X)$, then the following statements are equivalent [5, Note 12]:
  * for each $a \in A$, $a$ is almost periodic,
  * $\forall a \in A \ (M(a) \cap \text{Min}(E(X)) \neq \emptyset$,
  * $\forall a \in A \ (M(a) = \text{Min}(E(X))$,
  * $\overline{M}(A) \cap \text{Min}(E(X)) \neq \emptyset$,
  * $\overline{M}(A) = \text{Min}(E(X))$.

4. Use the fact that for each closed right ideal $K$ of $E(X)$, $h(K)=h(E(X))$ if and only if $K=E(X)$ (whenever $h(E(X))=+\infty$ the above statement may be false).

**Theorem 5.** In the transformation semigroup $(X, S)$, let $A, B \in (\overline{M}(X, S) \cap \overline{M}(X, S))$. We have the following table:

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\pi(B, A, \alpha)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\neg, \neg, \overline{M}, \neg)$</td>
<td>$\forall a \in A \ (\text{Dim}^{\overline{M}}(B) \leq \text{Dim}^{\overline{M}}(A))$</td>
</tr>
<tr>
<td>$(\neg, \overline{M}, \overline{M}, \overline{M})$</td>
<td>$\text{Dim}^{\overline{M}}(B) \leq \text{Dim}^{\overline{M}}(A)$</td>
</tr>
<tr>
<td>$(\neg, \overline{M}, \overline{M}, \overline{M})$</td>
<td>$\text{Dim}^{\overline{M}}(B) \leq \text{Dim}^{\overline{M}}(A)$</td>
</tr>
<tr>
<td>$(\overline{M}, \neg)$</td>
<td>$\forall a \in A \ (\text{Dim}^{\overline{M}}(B) \leq \text{Dim}^{\overline{M}}(A))$</td>
</tr>
<tr>
<td>$(\overline{M}, \overline{M})$</td>
<td>$\text{Dim}^{\overline{M}}(B) \leq \text{Dim}^{\overline{M}}(A)$</td>
</tr>
</tbody>
</table>

4. In the above table we have:

"If $B$ is $A^{\overline{M}}$ almost periodic, then $\pi(B, A, \alpha)$ is true."

So if $B$ is $A^{(P, Q)}$ almost periodic (where $P, Q \in \{-M, M\}$), then $\text{Dim}^{(P)}(B) \leq \text{Dim}^{(Q)}(A)$.

**Proof.** Let $P, Q \in \{-M, M\}$, then:

$B$ is $A^{(P, Q)}$ almost periodic

$\Rightarrow \forall K \in Q(A) \ \exists L \in P(B) \ L \subseteq K$

$\Rightarrow \forall K \in Q(A) \ \exists L \in P(B) \ h(L) \leq h(K)$

$\Rightarrow \text{Dim}^{(P)}(B) \leq \text{Dim}^{(Q)}(A)$ (by Corollary 3).

In the other cases use a similar method.

**Lemma 6.**

1. Let $(X_1, S_1), \ldots, (X_n, S_n)$ be transformation semigroup. $I$ be a closed right ideal of $E(\prod_{i=1}^{n} X_i, \prod_{i=1}^{n} S_i)$ such that $S(I) \neq \emptyset$ and for each $i \in \{1, \ldots, n\}$, $0 \neq A_i \subseteq X_i$, where $\prod_{i=1}^{n} S_i$ is a semigroup under coordinate multiplication, then:

a. $h(I) \geq \sum_{i=1}^{n} h(\pi_i(I)) \ (\pi_i$ is the projection map on $i$th coordinate),

b. $\text{Dim}^{(-)}(\prod_{i=1}^{n} A_i) \geq \sum_{i=1}^{n} \text{Dim}^{(-)}(A_i)$,

c. $\prod_{i=1}^{n} A_i \in \bar{M}(\prod_{i=1}^{n} X_i, \prod_{i=1}^{n} S_i) \Rightarrow \text{Dim}^{\bar{M}}(\prod_{i=1}^{n} A_i) \geq \sum_{i=1}^{n} \text{Dim}^{\bar{M}}(A_i)$.

2. In the transformation semigroup $(X, S)$ if $Z$ is a nonempty closed invariant subset of $X, 0 \neq A \subseteq Z$ and $I$ is a closed right ideal of $E(X)$, then:

a. $h_{E(Z, S)}(I | Z) \leq h_{E(Z, S)}(I)$,

b. $\text{Dim}^{(Z, S)}(A) \leq \text{Dim}^{(Z, S)}(A)$,

c. $\text{Dim}^{\bar{M}}(A) \leq \text{Dim}^{\bar{M}}(A)$. 
Proof.

1. (a) If \( p \in S(I) \), then \( I \subseteq pE((\prod_{i=1}^{n} X_i, \prod_{i=1}^{n} S_i)) \subseteq I \), so

\[
I = pE((\prod_{i=1}^{n} X_i, \prod_{i=1}^{n} S_i)) = (p_1, \ldots, p_n) \prod E(X_i, S_i)
\]

\[
= \prod_{i=1}^{n} p_i E(X_i, S_i) = \prod_{i=1}^{n} \pi_i(I).
\]

Let \( n = 2 \), and \( I_0 \supseteq I_1 \subseteq \cdots \subseteq I_n = \pi_i(I) \) be a chain of closed distinct right ideals in \( E(X_i, S_i)(i = 1, 2) \), then \( I_0 \times I_0 \subseteq I_1 \times I_1 \subseteq \cdots \subseteq I_{n-1} \times I_{n-1} \subseteq I_n \times I_n \), where \( \pi_1(I) \times \pi_2(I) = I \) is a chain of closed distinct right ideals in \( E(\prod_{i=1}^{n} X_i, \prod_{i=1}^{n} S_i) \), thus \( h(I) \geq n_1 + n_2 \) and \( h(I) \geq h(\pi_i(I)) + h(\pi_i(I)) \). Now by induction for \( n \geq 2 \) we have:

\[
h(I) = h(\prod_{i=1}^{n} \pi_i(I)) \geq h(\pi_i(I)) + h(\prod_{i=2}^{n} \pi_i(I)) \geq \sum_{i=1}^{n} h(\pi_i(I)).
\]

Which completes the proof.

(b) For each \( (a_1, \ldots, a_n) \in \prod_{i=1}^{n} X_i \) we have \( \bar{M}(a_1, \ldots, a_n) = \prod_{i=1}^{n} M(a_i) \). Moreover if \( K \in M(a_1, \ldots, a_n) \), then \( \emptyset \neq F((a_1, \ldots, a_n), K) \subseteq S(K) \), so by (a) and Corollary 3, \( \dim^{(\cdot)}(a_1, \ldots, a_n) = h(K) \geq \sum_{i=1}^{n} h(\pi_i(K)) \).

\[
\sum_{i=1}^{n} h(\pi_i(K)) \geq \sum_{i=1}^{n} \dim^{(\cdot)}(a_i).
\]

Therefore:

\[
\dim^{(\cdot)}(\prod_{i=1}^{n} A_i) = \sup \{ \dim^{(\cdot)}(a_1, \ldots, a_n) \mid \forall i \in \{1, \ldots, n\}, a_i \in A_i \}
\]

\[
\quad \geq \sup \{ \sum_{i=1}^{n} \dim^{(\cdot)}(a_i) \mid \forall i \in \{1, \ldots, n\}, a_i \in A_i \}
\]

\[
= \sum_{i=1}^{n} \sup \{ \dim^{(\cdot)}(a_i) \mid a_i \in A_i \}
\]

Note 7. In the transformation semigroups \((X_1, S)\) and \((X_2, S)\), let \( n \in \mathbb{N} \cup \{0, +\infty\} \), \( Z \) be a nonempty closed invariant subset of \( X_1 \), \( a \in A \subseteq Z \), \( \phi : (X_1, S) \to (X_2, S) \) be a homomorphism and \( \phi : (X_1, S) \to (X_2, S) \) be an onto homomorphism, then:

1. \( h(\phi(Z)) \leq h(Z) \) (\( i = 1, 2 \)),

2. \( \dim^{(\cdot)}(\phi(A)) \leq \dim^{(\cdot)}(A) \) (\( i = 1, 2 \)),

3. \( \phi(A) \in M(X_1, S) \Rightarrow M(X_1, S) \Rightarrow \dim^{(\cdot)}(\phi(A)) \leq \dim^{(\cdot)}(\phi(A)) \) (\( i = 1, 2 \)),

4. \( \phi(a) \) is an \( n \)-almost periodic point,

\[
(\exists m \leq n) \ (\phi(a) \text{ is an } m \text{-almost periodic point})
\]

(i = 1, 2),

5. \( \phi(a) \) is an \( n \)-almost periodic point in \((X_1, S)\)

\[
\Rightarrow (a \text{ is an } n \text{-almost periodic point in } (Z, S)).
\]

Lemma 8. In the transformation semigroup \((X, S)\), we
have:

1. If \( Z \) be a nonempty closed invariant subset of \( X \), then:
\[
h(Z) = \sup \{ n \in \mathbb{N} \cup \{0\} \mid \exists z_0, \ldots, z_n \in Z \ \forall i \in \{1, \ldots, n\} \ z_j \notin \bigcup_{k=0}^{i-1} z_k E(X) \}
\]

2. If \( Z_1, \ldots, Z_n \) be nonempty closed invariant subsets of \( X \), then:
\[
h(\bigcup_{i=1}^{n} Z_i) \leq \sum_{i=1}^{n} h(Z_i) + (n-1).
\]

3. If \( Z_1, \ldots, Z_n \) be nonempty closed invariant disjoint subset of \( X \), then:
   a. \( h(\bigcup_{i=1}^{n} Z_i) = \sum_{i=1}^{n} h(Z_i) + (n-1) \),
   b. \( h(\bigcup_{i=1}^{n} Z_i) = n-1 \) if and only if for each \( i \in \{1, \ldots, n\}, Z_i \) is minimal.

4. If \( Z_1, \ldots, Z_n \) be minimal subsets of \( X \), then \( \text{card}\{Z_1, \ldots, Z_n\} = h(\bigcup_{i=1}^{n} Z_i) + 1 \).

**Proof.**

1. Let \( n \in \mathbb{N} \cup \{0\} \) and \( z_0, \ldots, z_n \in Z \) be such that for each \( i \in \{1, \ldots, n\} \), \( z_i \notin \bigcup_{k=0}^{i-1} z_k E(X) \). For each \( i \in \{0, \ldots, n\} \), \( Z_i = \bigcup_{k=0}^{i} z_k E(X) \) is a closed nonempty invariant subset of \( Z \), moreover for each \( i \in \{0, \ldots, n-1\} \), \( Z_i \) is a proper subset of \( Z_{i+1} \), therefore \( n \leq h(Z) \). Thus \( \sup \{ n \in \mathbb{N} \cup \{0\} \mid \exists z_0, \ldots, z_n \in Z \ \forall i \in \{1, \ldots, n\} \ z_j \notin \bigcup_{k=0}^{i-1} z_k E(X) \} \leq h(Z) \). On the other hand let \( n \in \mathbb{N} \cup \{0\} \) and \( Z_0, \ldots, Z_n \) be closed nonempty invariant subsets of \( Z \) such that for each \( i \in \{0, \ldots, n-1\} \), \( Z_i \) is a proper subset of \( Z_{i+1} \), then for \( z_0 \in Z_0 \) and \( z_i \in Z_i - Z_{i-1} (1 \leq i \leq n) \), we have \( z_j \notin \bigcup_{k=0}^{i-1} z_k E(X) (1 \leq i \leq n) \). Therefore \( h(Z) \leq \sup \{ n \in \mathbb{N} \cup \{0\} \mid \exists z_0, \ldots, z_n \in Z \ \forall i \in \{1, \ldots, n\} \ z_j \notin \bigcup_{k=0}^{i-1} z_k E(X) \} \), which completes the proof.

2. It is clear for \( n=1 \). For \( n=2 \), let \( m \in \mathbb{N} \cup \{0\} \) and \( x_0, \ldots, x_m \in Z_1 \cup Z_2 \) be such that for each \( i \in \{1, \ldots, m\} \), \( x_j \notin \bigcup_{k=0}^{i-1} z_k E(X) \), then by (1) we have \( m-1 \leq (\text{card}(\{x_0, \ldots, x_m\} \cap Z_1)) - 1 + (\text{card}(\{x_0, \ldots, x_m\} \cap Z_2)) - 1 \leq h(Z_1) + h(Z_2) \), thus \( h(Z_1 \cup Z_2) \leq h(Z_1) + h(Z_2) + 1 \). Let \( m \in \mathbb{N} \) and suppose for \( n=m \) the above statement holds, for \( n=m+1 \) we have:
\[
h(\bigcup_{i=1}^{m+1} Z_i) = h(\bigcup_{i=1}^{m} Z_{i+1}) \leq h(\bigcup_{i=1}^{m} Z_i) + h(Z_{m+1}) + 1
\]
\[
\leq \sum_{i=1}^{m} h(Z_i) + (m-1) + h(Z_{m+1}) + 1
\]
\[
= \sum_{i=1}^{m+1} h(Z_i) + m,
\]
which will give the result by induction on \( n \).

3.
   a. For each \( i \in \{1, \ldots, n\} \), let \( Z_1, \ldots, Z_n \) be nonempty closed invariant subsets of \( Z \) such that for each \( j \in \{0, \ldots, m_i-1\} \), \( Z'_j \) is a proper subset of \( Z'_{j+1} \), then:
\[
Z'_0 \subset Z'_1 \subset \cdots \subset Z'_{m_1}
\]
\[
\subset Z'_0 \cup Z'_1 \subset \cdots \subset Z'_0 \cup Z'_1 \cup Z'_2 \subset \cdots \subset Z'_0 \cup Z'_1 \cup \cdots \cup Z'_n
\]
is a chain of distinct nonempty closed invariant subsets of \( Z \), therefore
(2) we have \( h(\bigcup_{i=1}^{n} Z_i) = \sum_{i=1}^{n} h(Z_i) + (n-1) \).

b. By (a), \( h(\bigcup_{i=1}^{n} Z_i) = n-1 \) if and only if for each \( i \in \{1, \ldots, n\} \), \( h(Z_i) = 0 \). Now (1) in Note 4 completes the proof.

4. It is clear by (3).
**Theorem 9.** In the transformation semigroup \((X, S)\), if \(Z\) be a nonempty closed invariant subset of \(X\), then the following statements are equivalent:

i. \(h(Z) = 1\),

ii. “\(Z\) is the union of two disjoint minimal subset of \(X\)” or “there exists a 1-almost periodic point \(z \in X\) such that \(zE(X) = Z\)”.

**Proof.**

\((i) \Rightarrow (ii))\: By (1) Lemma 8, there exist \(z_0, z_1 \in Z\) such that \(z_0E(X)\) is minimal and \(z_1 \notin z_0E(X)\). Since \(h(z_0E(X)) \leq h(Z) = 1\), we have \(h(z_1E(X)) = 1\) or \(h(z_1E(X)) = 0\). If \(h(z_1E(X)) = 1\), then \(z_1\) is 1-almost periodic and \(z_1E(X) = Z\), if \(h(z_1E(X))) = 0\), then \(z_1E(X)\) is minimal, \(h(z_1E(X)) \cup z_0E(X) = 1\) and \(Z\) is the union of two disjoint minimal set \(z_1E(X)\) and \(z_0E(X)\).

\((ii) \Rightarrow (i))\: Use Lemma 8.

**Lemma 10.** In the transformation semigroup \((X, S)\), if \(r \in E(X)\) be such that \(r^2E(X) = rE(X)\) and \(h(rE(X)) < +\infty\), then:

1. \(rE(X) \in M(r)\),

2. \(r\) is \(n\)-almost periodic if and only if \(\text{Dim}^T(r) = n\).

Therefore if \(I\) is a closed right ideal of \(E(X)\) and \(p \in S(I)\), then \(\text{Dim}^T(p) = h(I)\) (since \(I = pE(X) = p^2E(X)\)).

**Proof.** Use \(E(E(X), S) = E(X, S)\).

**Corollary 11.** In the transformation semigroup \((X, S)\), let \(n \in \mathbb{N} \cup \{0\}\) and \(0 \neq A \subseteq X\).

1. If \(A \in \overline{M}(X, S)\), then the following statements are equivalent:

a. \(\text{Dim}^T(A) = n\),

b. \(\exists u \in J(F(A, E(X)))\)

\((uE(X) \in \overline{M}(A) \land \text{Dim}^T(u) = n)\),

c. \(\exists p \in F(A, E(X))\)

\((pE(X) \in \overline{M}(A) \land \text{Dim}^T(p) = n)\),

d. \(\min \{\text{Dim}^T(p) \mid p \in F(A, E(X))\} = n\),

e. \(\min \{\text{Dim}^T(u) \mid u \in J(F(A, E(X)))\} = n\),

f. \(\overline{M}(A) = \{uE(X) \mid \text{Dim}^T(u) = n \land u \in J(F(A, E(X)))\}\),

g. \(\overline{M}(A) = \{pE(X) \mid \text{Dim}^T(p) = n \land p \in F(A, E(X))\}\).

2. If \(A \in \overline{M}(X, S)\), then the following statements are equivalent:

a. \(\text{Dim}^{\overline{M}}(A) = n\),

b. \(\exists u \in J(F(A, E(X)))\)

\((uE(X) \in \overline{M}(A) \land \text{Dim}^T(u) = n)\),

c. \(\exists p \in F(A, E(X))\)

\((pE(X) \in \overline{M}(A) \land \text{Dim}^T(p) = n)\),

d. \(\exists q \in F(A, E(X))\)

\((qE(X) \in \overline{M}(A) \land \text{Dim}^T(q) = n)\),

e. \(\min \{\text{Dim}^T(q) \mid q \in F(A, E(X))\} = n\),

f. \(\min \{\text{Dim}^T(p) \mid p \in F(A, E(X))\} = n\),

g. \(\min \{\text{Dim}^T(u) \mid u \in J(F(A, E(X)))\} = n\),

h. \(\overline{M}(A) = \{uE(X) \mid \text{Dim}^T(u) = n \land u \in J(F(A, E(X)))\}\),

i. \(\overline{M}(A) = \{pE(X) \mid \text{Dim}^T(p) = n \land p \in F(A, E(X))\}\),

j. \(\overline{M}(A) = \{qE(X) \mid \text{Dim}^T(q) = n \land q \in F(A, E(X))\}\).

3. If \(A \in \overline{M}(X, S)\) and \(\text{Dim}^{\overline{M}}(A) = n\), then each of the following sets is equal to \(\overline{M}(A)\):

\(\{pE(X) \mid \forall q \in F(A, pE(X))(\text{Dim}^T(q) = n \land p \in F(A, E(X)))\}\),

\(\{uE(X) \mid \forall v \in J(F(A, uE(X)))(\text{Dim}^T(v) = n \land v \in J(F(A, E(X)))\}\),

\(\{pE(X) \mid \forall q \in F(A, pE(X))(\text{Dim}^T(q) = n \land p \in F(A, E(X)))\}\),

\(\{uE(X) \mid \forall v \in J(F(A, uE(X)))(\text{Dim}^T(v) = n \land v \in J(F(A, E(X)))\}\).
Proof.

1. 

\( ((a) \Rightarrow (b)) \): Let \( \dim^{T}(\overline{M}(A)) = n \). By Definition 1, there exists \( K \subseteq \overline{M}(A) \) such that \( h(K) = n \). Choose \( u \in J(F(A, K)) \), since \( J(F(A, K)) \subseteq S(K) \), so \( uE(X) = K = \overline{M}(A) \) and by Lemma 10 \( \dim^{T}(u) = h(K) = n \).

\( ((b) \Rightarrow (c)) \): It is clear by \( J(F(A, K)) \subseteq F(A, K) \).

\( ((c) \Rightarrow (d)) \): By (c), \( \min \{ \dim^{T}(p) \mid p \in F(A, E(X)) \} \leq n \) and by Corollary 3, \( \dim^{T}(\overline{M}(A)) = n \). Let \( q \in F(A, E(X)) \), \( \dim^{T}(q) = m \) and \( K \in M(q) \). By Corollary 3 (and \( E(E(X)) = E(X) \)) \( m = h(K) \), on the other hand for each \( a \in A \) we have: \( aK = aqK = aqE(X) = aE(X) \) thus there exists \( L \subseteq \overline{M}(A) \) such that \( L \subseteq K \), so by Corollary 3 we have: \( n = h(L) \leq h(K) = m \). Thus \( \min \{ \dim^{T}(p) \mid p \in F(A, E(X)) \} \geq n \).

\( ((d) \Rightarrow (e)) \): By (d), \( \min \{ \dim^{T}(u) \mid u \in J(F(A, E(X))) \} \geq n \). Let \( p \in F(A, E(X)) \), \( \dim^{T}(p) = n \) and \( K \in M(p) \), for each \( a \in A \), \( apK = apE(X) = aE(X) \) so there exists \( L \subseteq \overline{M}(A) \) such that \( L \subseteq K \). Choose \( v \in J(F(A, L)) \). By Corollary 3, Note 4, Note 7 and Lemma 10, \( \dim^{T}(v) = h(vE(X)) = h(L) \leq h(pK) \leq h(K) = \dim^{T}(p) = n \), so \( \min \{ \dim^{T}(u) \mid u \in J(F(A, E(X))) \} \leq n \). Let \( v \in J(F(A, E(X))) \) and \( \dim^{T}(v) = n \). There exists \( L \subseteq \overline{M}(A) \) such that \( L \subseteq \overline{M}(A) \). Choose \( w \in J(F(A, L)) \), by Lemma 10 \( n = \dim^{T}(w) = h(L) \leq h(vE(X)) = \dim^{T}(v) = n \) so \( h(L) = h(vE(X)) = \dim^{T}(v) = n \). By (f), \( \dim^{T}(v) = n \) and \( \dim^{T}(u) = n \). Thus \( \dim^{T}(u) = n \) and \( \dim^{T}(v) = n \).

\( ((e) \Rightarrow (f)) \): Let \( v \in J(F(A, E(X))) \) and \( \dim^{T}(v) = n \). There exists \( L \subseteq \overline{M}(A) \) such that \( L \subseteq \overline{M}(A) \). Choose \( w \in J(F(A, L)) \), by Lemma 10 \( n \leq \dim^{T}(w) = h(L) \leq h(vE(X)) = \dim^{T}(v) = n \) so \( h(L) = h(vE(X)) = \dim^{T}(v) = n \). By (f), \( \dim^{T}(v) = n \) and \( \dim^{T}(u) = n \). Thus \( \dim^{T}(u) = n \) and \( \dim^{T}(v) = n \).

\( ((f) \Rightarrow (g)) \): By (f), \( \overline{M}(A) \subseteq \{ pE(X) \mid \dim^{T}(p) = n \wedge p \in F(A, E(X)) \} \). Let \( q \in F(A, E(X)) \), \( \dim^{T}(q) = n \) and \( K \in M(q) \). For each \( a \in A \), \( aK = apK = apE(X) = aE(X) \) so there exists \( L \subseteq \overline{M}(A) \) such that \( L \subseteq K \), by (f) there exists \( u \in J(F(A, E(X))) \) such that \( \dim^{T}(u) = n \). Let \( L = uE(X) \), so by Lemma 10 we have: \( n = \dim^{T}(u) = h(uE(X)) = h(L) \leq h(K) = \dim^{T}(p) = n \). Thus \( h(L) = h(K) < +\infty \) and \( L = K \), therefore \( pE(X) = pK = pL = puE(X) \), now there exists \( M \subseteq pE(X) \) such that \( M \subseteq \overline{M}(A) \) and by (f), Corollary 3 and Corollary 10 we have: \( \dim^{T}(M) = n \). By (f), \( \dim^{T}(p) = n \) and \( \dim^{T}(u) = n \). Thus \( \dim^{T}(u) = n \) and \( \dim^{T}(v) = n \).

\( ((g) \Rightarrow (a)) \): Use Lemma 10 and Corollary 3.

2. Like (1) use Lemma 10 and \( \forall K \subseteq M(A) \) \( \overline{F(A, K)} \subseteq S(K) \).

3. Use (1).

4. Use (2).

**Theorem 12.** In the transformation semigroup \((X, S)\),
let $A_1,\ldots, A_n$ be nonempty subsets of $X$, we have:

1. $\overline{\text{M}}(\bigcup_{i=1}^{n} A_i) = \min\{\bigcup_{i=1}^{n} K_i \mid \forall i \in \{1,\ldots, n\} K_i \in \overline{\text{M}}(A_i)\}, \subseteq$.

2. $\text{Dim}^{\overline{\text{M}}}(\bigcup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} h(A_i) + (n-1)$.

3. $\forall K \in \overline{\text{M}}(\bigcup_{i=1}^{n} A_i) \ (S(K) \neq \emptyset \Rightarrow K \in \bigcup_{i=1}^{n} \overline{\text{M}}(A_i))$.

4. If $\bigcup_{i=1}^{n} A_i \in \overline{\text{M}}(X, S)$, then:

   a. $\overline{\text{M}}(\bigcup_{i=1}^{n} A_i)$ is a subset of:

   $\bigcup\{\overline{\text{M}}(A_i) \mid 1 \leq i \leq n, \text{Dim}^{\overline{\text{M}}}(A_i) = \text{Dim}^{\overline{\text{M}}}(\bigcup_{i=1}^{n} A_i)\}$

   $A_i \in \overline{\text{M}}(X, S)$

   b. $\text{Dim}^{\overline{\text{M}}}(\bigcup_{i=1}^{n} A_i) = \max\{\text{Dim}^{\overline{\text{M}}}(A_i) \mid 1 \leq i \leq n, A_i \in \overline{\text{M}}(X, S)\}$

   c. $\text{Dim}^{\overline{\text{M}}}(\bigcup_{i=1}^{n} A_i) = +\infty$ if and only if there exists $i \in \{1,\ldots, n\}$ such that $\text{Dim}^{\overline{\text{M}}}(A_i) = +\infty$.

5. If $\bigcup_{i=1}^{n} A_i \in \overline{\text{M}}(X, S)$ and $\text{Dim}^{\overline{\text{M}}}(\bigcup_{i=1}^{n} A_i) < \infty$, then $\overline{\text{M}}(\bigcup_{i=1}^{n} A_i)$ is a subset of:

   $\bigcap\{\overline{\text{M}}(A_i) \mid 1 \leq i \leq n, \text{Dim}^{\overline{\text{M}}}(A_i) = \text{Dim}^{\overline{\text{M}}}(\bigcup_{i=1}^{n} A_i)\}$

   $A_i \in \overline{\text{M}}(X, S)$

6. If $\max\{\text{Dim}^{\overline{\text{M}}}(A_i) \mid 1 \leq i \leq n, 1 \leq n \leq 1 \} = \lambda \in \{0,1\}$, then the following statements are equivalent:

   a. $\text{Dim}^{\overline{\text{M}}}(\bigcup_{i=1}^{n} A_i) = \lambda$,

   b. $\max\{\text{Dim}^{\overline{\text{M}}}(A_i) \mid 1 \leq i \leq n\} = \lambda$ and for each $K \in \overline{\text{M}}(\bigcup_{i=1}^{n} A_i), S(K) \neq \emptyset$,

   c. $\max\{\text{Dim}^{\overline{\text{M}}}(A_i) \mid 1 \leq i \leq n\} = \lambda$ and $\overline{\text{M}}(\bigcup_{i=1}^{n} A_i) \subseteq \bigcup_{i=1}^{n} \overline{\text{M}}(A_i)$,

   d. $\overline{\text{M}}(\bigcup_{i=1}^{n} A_i) \subseteq \bigcap\{\overline{\text{M}}(A_i) \mid 1 \leq i \leq n, \text{Dim}^{\overline{\text{M}}}(A_i) = \lambda\}$,

   e. $\overline{\text{M}}(\bigcup_{i=1}^{n} A_i) = \bigcap\{\overline{\text{M}}(A_i) \mid 1 \leq i \leq n, \text{Dim}^{\overline{\text{M}}}(A_i) = \lambda\}$.

**Proof.**

1. Let $K \in \overline{\text{M}}(\bigcup_{i=1}^{n} A_i)$:

   $K \in \overline{\text{M}}(\bigcup_{i=1}^{n} A_i) \Rightarrow \forall a \in \bigcup_{i=1}^{n} A_i \ aK = aE(X) \Rightarrow \forall i \in \{1,\ldots, n\} \ \forall a_i \in A_i \ a_i K = a_i E(X)$

   $\Rightarrow \forall i \in \{1,\ldots, n\} \ \exists K_i \in \overline{\text{M}}(A_i) \ K_i \subseteq K \Rightarrow \forall i \in \{1,\ldots, n\} \ \exists K_i \in \overline{\text{M}}(A_i) \ \bigcup_{j=1}^{n} K_j \subseteq K$

   $\Rightarrow \forall i \in \{1,\ldots, n\} \ \exists K_i \in \overline{\text{M}}(A_i) \ \bigcup_{j=1}^{n} K_j = K$

   (since $\bigcup_{i=1}^{n} K_i$ is a closed ideal of $E(X)$ and for each $a \in \bigcup_{i=1}^{n} A_i$, $a \bigcup_{i=1}^{n} K_i = aE(X)$).

Therefore $\overline{\text{M}}(\bigcup_{i=1}^{n} A_i) \subseteq \{\bigcup_{i=1}^{n} K_i \mid \forall i \in \{1,\ldots, n\} K_i \in \overline{\text{M}}(A_i)\}$, and:

$\overline{\text{M}}(\bigcup_{i=1}^{n} A_i) \subseteq \min\{\bigcup_{i=1}^{n} K_i \mid \forall i \in \{1,\ldots, n\} K_i \in \overline{\text{M}}(A_i)\}$.

On the other hand let $L_1 \in \overline{\text{M}}(A_1), \ldots, L_n \in \overline{\text{M}}(A_n)$ be such that $\bigcup_{i=1}^{n} L_i \in \min\{\bigcup_{i=1}^{n} K_i \mid \forall i \in \{1,\ldots, n\} K_i \in \overline{\text{M}}(A_i)\}$, then:
\[ \forall i \in \{1, \ldots, n\} \quad L_i \in \overline{M}(A_i) \]
\[ \Rightarrow \forall a \in \bigcup_{i=1}^{n} A_i \quad \sum_{i=1}^{n} L_i = a E(X) \]
\[ \Rightarrow \exists K \in \overline{M} \left( \bigcup_{i=1}^{n} A_i \right) \quad K \subseteq \bigcup_{i=1}^{n} L_i \]

3. Let \( K \in \overline{M} \left( \bigcup_{i=1}^{n} A_i \right) \) and \( p \in S(K) \), for each \( i \in \{1, \ldots, n\} \), choose \( K_i \in \overline{M}(A_i) \) such that \( K = \bigcup_{i=1}^{n} K_i \). Choose \( i_0 \in \{1, \ldots, n\} \) such that \( p \in K_{i_0} \). By \( K_{i_0} \subseteq K = p K \subseteq K K_{i_0} \subseteq K_{i_0} \), we have \( K = K_{i_0} \in \overline{M}(A_j) \).

4. \( \bigcup_{i=1}^{n} A_i \in \overline{M}(X, S) \), then for all \( K \in \overline{M} \left( \bigcup_{i=1}^{n} A_i \right) \),
\[ 0 \neq F \left( \bigcup_{i=1}^{n} A_i, K \right) \subseteq S(K) \], and for each \( j \in \{1, \ldots, n\} \), \( F \left( \bigcup_{i=1}^{n} A_i, K \right) \subseteq F(A_j, K) \), now use (3).

5. Use Corollary 3 and a similar method described for (3) and (4).

6. Use the above items and (3) in Note 4.

**Corollary 13.** In the transformation semigroup \((X, S)\), let \( A \) be a nonempty subset of \( X \), we have:

1. For each \( a \in A \), if \( \text{Dim}^{\overline{M}}(A) = \text{Dim}^{T}(a) < +\infty \), then \( \overline{M}(A) \subseteq M(a) \).

2. “\( \text{Dim}^{\overline{M}}(A) = 1 \)” if and only if \( \max \{\text{Dim}^{T}(a) | a \in A\} = 1 \) and \( \overline{M}(A) = \bigcap \{M(a) | a \in A, \text{Dim}^{T}(a) = 1\} \).

**Proof.**

1. Let \( a \in A \), \( \text{Dim}^{\overline{M}}(A) = \text{Dim}^{T}(a) < +\infty \), and \( K \in \overline{M}(A) \). We have:
\[ K \in \overline{M}(A) \]
\[ \Rightarrow aK = aE(X, S) \]
\[ \Rightarrow \exists L \in M(a) \quad L \subseteq K \quad ([5, Corollary 3]) \]
\[ \Rightarrow \exists L \in M(a) \quad (L \subseteq K \wedge \text{Dim}^{T}(a) = h(L) \leq h(K) \leq \text{Dim}^{\overline{M}}(A)) \]
\[ \Rightarrow \exists L \in M(a) \quad (L \subseteq K \wedge h(L) \leq h(K) < +\infty) \]
\[ \Rightarrow K = M(a) \]

2. Let \( \text{Dim}^{\overline{M}}(A) = 1 \). For each \( a \in A \), \( \text{Dim}^{T}(a) \leq \text{Dim}^{\overline{M}}(A) = 1 \).
Dim^{T(M)}(A)=1$, thus by (2) in Note 4 we have $\max \{\text{Dim}^T(a) \mid a \in A\} \leq 1$. If $\max \{\text{Dim}^T(a) \mid a \in A\} < 1$, then $\text{Dim}^{T(-)}(A)=0$ and by (3) in Note 4 we have $\text{Dim}^{T(M)}(A)=0$ which is a contraction, so $\max \{\text{Dim}^T(a) \mid a \in A\}=1$. By (1), $\overline{M}(A) \subseteq \bigcap \{M(a) \mid a \in A\}$, $\text{Dim}^{T(a)}=1$, moreover if $K \subseteq \bigcap \{M(a) \mid a \in A\}$, $\text{Dim}^{T(a)}=1$, then for each $a \in A$ such that $\text{Dim}^{T(a)}=1$, $aK=aE(X)$ and for each $b \in A$ such that $\text{Dim}^{T(b)}=0$, $bK=bE(X)$ (since $b$ is almost periodic by (3) in Note 4), so there exists $L \in \overline{M}(A) \subseteq \bigcap \{M(a) \mid a \in A\}$ such that $L \subseteq K$ [5, Corollary 3]. Suppose $a_0 \in A$ be such that $\text{Dim}^{T(a)}=1$, by $L \subseteq K$ and $K, L \in \overline{M}(a_0)$ we have $K=L$ and $K \in \overline{M}(A)$, so $\overline{M}(A) = \bigcap \{M(a) \mid a \in A\}$, $\text{Dim}^{T(a)}=1$.

Acknowledgements
The authors are grateful to the Research Division of AZAD University of Lahijan for the grant which supported this research.

References