TRANSFORMATION SEMIGROUPS AND TRANSFORMED DIMENSIONS

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Abstract

In the transformation semigroup (*X*, *S*) we introduce the height of a closed nonempty invariant subset of *X*, define the transformed dimension of nonempty subset *S* of *X* and obtain some results and relations.

Preliminaries

By a transformation semigroup (*X*, *S*, π) (or simply (*X*, *S*)) we mean a compact Hausdorff topological space *X*, a discrete topological semigroup *S* with identity *e* and a continuous map $\pi: X\times S\rightarrow X(\pi(x,s)=xs(\forall x\in X, \forall s\in S))$ such that:

- $\forall x\in X \quad xe=x$,
- $\forall x\in X \quad \forall s,t\in S \quad x(st)=(xs)t$,

In the transformation semigroup (*X*, *S*) we have the following definitions:

1. For each $s\in S$, define the continuous map π^s: $X\rightarrow X$ by $x\pi^s=xs(\forall x\in X)$, then the closure of $\{\pi^s \mid s\in S\}$ in X^X with pointwise convergence, is called the enveloping semigroup (or Ellis semigroup) of (*X*, *S*). We shall denote the enveloping semigroup by E(*X*, *S*) or simply by E(*X*) if there is no possibily confusion [1, Definition 6.4]. E(*X*, *S*) has a semigroup structure [2, Chapter 3]. An element *u* of E(*X*, *S*) is called idempotent if $u^2=u$. Every closed nonempty subsemigroup of E(*X*, *S*) has an idempotent element [3, Chapter I, Proposition 2.1]. A nonempty subset *K* of E(*X*, *S*) is called a right ideal if $K\mathrm{E}(X,S)\subseteq K$, and it is called a minimal right ideal if none of the right ideals of E(*X*, *S*) be a proper subset of *K*. For each right ideal *K* of E(*X*, *S*) and each $p\in K$, $L_p:K\rightarrow K$ is defined by $L_p(q)=pq(\forall q\in K)$.

2. A nonempty subset *Z* of *X* is called invariant if $ZS\subseteq Z$, moreover it is called minimal if it be closed and none of the closed invariant subsets of *X* be a proper subset of *Z*. The element *a* of *X* is called almost periodic if the orbit closure of *a* i.e. $\overline{aS}=a\mathrm{E}(X,S)$ is a minimal subset of *X*.

3. Let $a\in X$, *A* be a nonempty subset of *X*, *C* be a nonempty subset of E(*X*, *S*) and *I* be a right ideal of E(*X*), then we introduce the following sets [6, Notation 5]:

$$F(a,C)=\{p\in C \mid ap=p\},$$

$$F(A,C)=\{p\in C \mid \forall b\in A \quad bp=p\},$$

$$\overline{F}(A,C)=\{p\in C \mid Ap=A\},$$

$$J(C)=\{p\in C \mid p^2=p\},$$

$$S(I)=\{p\in C \mid L_p: I\rightarrow I \text{ is surjective}\}.$$

4. Let (Y,S) be a transformation semigroup, the continuous map $\varphi:(X,S)\rightarrow(Y,S)$ is a homomorphism if $\varphi(xs)=\varphi(x)s\ (\forall x\in X,\forall s\in S)$. If $\varphi:(X,S)\rightarrow(Y, S)$ is an onto homomorphism, then there exists an onto

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homomorphism $\hat{\varphi}:(\mathrm{E}(X,S),S)\to(\mathrm{E}(Y,S),S)$ (which is also a semigroup homomorphism) such that for each $x\in X$ and $p\in$ E(X, S) we have $\varphi(xp)=\varphi(x)\hat{\varphi}(p)$ [2, Proposition 3.8].

5. Let $a\in X, A$ be a nonempty subset of X and K be a closed right ideal of E(X, S), then [5, Definition 1]:

- $K\in M_{(X,S)}(a)$ (or simply M(a)) if:
  - $aK = a$E(X, S),
  - K does not have any proper subset like L, such that L be a closed right ideal of E(X, S) such that $aL = a$E(X, S),
- $K\in\overline{M}_{(X,S)}(A)$ (or simply $\overline{M}(A)$) if:
  - $\forall b\in A \quad bK=b\mathrm{E}(X,S)$,
  - K does not have any proper subset like L, such that L be a closed right ideal of E(X, S) such that $bL=b\mathrm{E}(X,S)$ for all $b\in A$,
- $K\in\overline{\overline{M}}_{(X,S)}(A)$ (or simply $\overline{\overline{M}}(A)$) if:
  - $AK = A$E(X, S),
  - K does not have any proper subset like L such that L be a closed right ideal of E(X, S) such that $AL = A$E(X, S).

The elements of $M_{(X,S)}(a)$ (resp. $\overline{M}_{(X,S)}(A)$ and $\overline{\overline{M}}_{(X,S)}(A)$) are called a-minimal (resp. $A-\overline{\mathrm{m}inimal}$ and $A-\overline{\overline{\mathrm{m}inimal}}$) sets. $\overline{M}_{(X,S)}(A)$ and $M_{(X,S)}(a)$ are nonempty ([5, Theorem 2], and [6]).

6. Define $\overline{\mathcal{M}}(X,S)=\{\emptyset\neq A\subseteq X \mid \forall K\in\overline{M}_{(X,S)}(A)$ $\mathrm{J}(\mathrm{F}(A,K))\neq\emptyset\}$ and $\overline{\overline{\mathrm{M}}}(X,S)=\{\emptyset\neq A\subseteq X \mid \overline{\overline{M}}_{(X,S)}(A)$ $\neq\emptyset\wedge\forall K\in\overline{\overline{M}}_{(X,S)}(A)$ $\mathrm{J}(\overline{\mathrm{F}}(A,K))\neq\emptyset\}$.

7. Let $a\in X$ and A be a nonempty subset of X, then [5, Definition 13]:

- (X, S) is called a-distal if E(X, S)$\in$M$_{(X,S)}(a)$,
- (X, S) is called A-distal if (X, S) be b-distal for each $b\in A$,
- (X, S) is called $A\frac{(\overline{\mathrm{M}})}{}$distal if $E(X,S)\in\overline{M}_{(X,S)}(A)$,
- (X, S) is called $A\frac{(\overline{\overline{\mathrm{M}}})}{}$distal if $E(X,S)\in\overline{\overline{M}}_{(X,S)}(A)$.

8. Let A, B be nonempty subsets of X and R, Q $\in\{\overline{M}_{(X,S)},\overline{\overline{M}}_{(X,S)}\}$, then [5, Definition 13]:

- B is called is A-almost periodic if: $\forall a\in A \quad \forall K\in\mathrm{M}_{(X,S)}(a) \quad \forall b\in B \quad \exists L\in\mathrm{M}_{(X,S)}(b) \quad L\subseteq K$,
- B is called $A\frac{(\mathrm{R},-)}{}$almost periodic if: $\forall a\in A \quad \forall K\in\mathrm{M}_{(X,S)}(a) \quad \exists L\in\mathrm{R}(B) \quad L\subseteq K$,
- B is called $A\frac{(-,\mathrm{Q})}{}$almost periodic if $\mathrm{Q}(A)\neq\emptyset$ and: $\forall K\in\mathrm{Q}(A) \quad \forall b\in B \quad \exists L\in\mathrm{M}_{(X,S)}(b) \quad L\subseteq K$,
- B is called $A\frac{(\mathrm{R},\mathrm{Q})}{}$almost periodic if $\mathrm{Q}(A)\neq\emptyset$ and: $\forall K\in\mathrm{Q}(A) \quad \exists L\in\mathrm{R}(B) \quad L\subseteq K$.

Definition 1. Let A be a nonempty subset of $X, a\in A$ and Z be a closed invariant subset of X, we introduce height of Z by:

$$\mathrm{h}_{(X,S)}(Z)=\{n\in\mathrm{N}\cup\{0\}\mid\exists Z_0,\ldots,\subseteq Z_n\ ((Z_0\subseteq Z_1\subseteq\cdots\subseteq Z_n)\wedge(\forall i\in\{0,\ldots,n\}\ \forall j\in\{0,\ldots,n\}-\{i\}\quad Z_i\neq Z_j)\wedge(\forall_i\in\{0,\ldots,n\}\quad Z_i \text{ is a closed invariant subset of } Z))\}$$

and transformed dimension of a, transformed dimension of A, $\overline{\mathrm{M}}-$transformed dimension of A and $\overline{\overline{\mathrm{M}}}-$transformed dimension of A is defined respectively by:

- $\mathrm{Dim}^{\mathrm{T}(-)}_{(X,S)}(a)=\sup\{\mathrm{h}_{(\mathrm{E}(X),S)}(I)\mid I\in\mathrm{M}_{(X,S)}(a)\}$ (or simply $\mathrm{Dim}^{\mathrm{T}}(a)$),
- $\mathrm{Dim}^{\mathrm{T}(-)}_{(X,S)}(A)=\sup\{\mathrm{Dim}^{\mathrm{T}(-)}_{(X,S)}(x)\mid x\in A\}$ (or simply $\mathrm{Dim}^{\mathrm{T}}(A)$),
- $\mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}_{(X,S)}(A)=\sup\{\mathrm{h}_{(\mathrm{E}(X),S)}(I)\mid I\in\overline{\mathrm{M}}_{(X,S)}(A)\}$ (or simply $\mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(A)$),
- $\mathrm{Dim}^{\mathrm{T}(\overline{\overline{\mathrm{M}}})}_{(X,S)}(A)=\sup\{\mathrm{h}_{(\mathrm{E}(X),S)}(I)\mid I\in\overline{\overline{\mathrm{M}}}_{(X,S)}(A)\}$ (or simply $\mathrm{Dim}^{\mathrm{T}(\overline{\overline{\mathrm{M}}})}(A)$).

If $\mathrm{h}(a\mathrm{E}(X))=n\in\mathrm{N}\cup\{0,+\infty\}$, then a is called n-almost periodic (or almost periodic of degree n).

Lemma 2. In the transformation semigroup (X, S), let I

and J be two (closed) right ideals of E(X), and suppose there exist $u\in$J(S(I)) and $v\in$J(S(J)) such that $uv = u$ and $vu = v$, then for each chain of (closed) right ideals of E(X) in I, there exists a chain of (closed) right ideals of E(X) in J with the same cardinal. So if I and J be closed, then h(I) = h(J).

Proof. Let $\{I_\alpha \mid \alpha\in\Gamma\}$ be a chain of (closed) right ideals of E(X) in I, then $\{vI_\alpha \mid \alpha\in\Gamma\}$ is a chain of (closed) right ideals of E(X) in J and we have card $\{I_\alpha \mid \alpha\in\Gamma\}\geq\{vI_\alpha \mid \alpha\in\Gamma\}$. Using $uv = u$ and the fact that $L_u|_I = \mathrm{id}_I$, we have card$\{I_\alpha|\alpha\in\Gamma\}$ = card$\{vI_\alpha|\alpha\in\Gamma\}$.

Corollary 3. In the transformation semigroup (X, S) let A be a nonempty subset of X, then we have:

- $\forall a\in X \quad \forall K\in \mathrm{M}(a) \quad \mathrm{Dim}^{\mathrm{T}}(a)=\mathrm{h}(K)$,
- if $A\in\overline{\mathcal{M}}(X,S)$, then for each $K\in\overline{\mathrm{M}}(A)$, $\mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(A)=\mathrm{h}(K)$,
- if $A\in\overline{\overline{\mathcal{M}}}(X,S)$, then for each $K\in\overline{\overline{\mathrm{M}}}(A)$, $\mathrm{Dim}^{\mathrm{T}(\overline{\overline{\mathrm{M}}})}(A)=\mathrm{h}(K)$.

Proof. Let $a\in X$ (resp. $A\in\overline{\mathcal{M}}(X,S)$, $A\in\overline{\overline{\mathcal{M}}}(X,S)$) and $I,J\in\mathrm{M}(a)$ (resp. $I,J\in\overline{\mathrm{M}}(A)$, $I,J\in\overline{\overline{\mathrm{M}}}(A)$), then there exist $u\in\mathrm{J}(\mathrm{F}(a,I))\ (\subseteq\mathrm{J}(\mathrm{S}(I)))$ (resp. $u\in\mathrm{J}(\mathrm{F}(A,I))\ (\subseteq\mathrm{J}(\mathrm{S}(I)))$) and $v\in\mathrm{J}(\mathrm{F}(a,J))\ (\subseteq\mathrm{J}(\mathrm{S}(J)))$ (resp. $v\in\mathrm{J}(\mathrm{F}(A,J))\ (\subseteq\mathrm{J}(\mathrm{S}(J)))$) such that $uv=u$ and $vu=v$ [5, Theorem 7]. Now Lemma 2 completes the proof [4].

Note 4. In the transformation semigroup (X, S) we have:

1. If Z and W be nonempty closed invariant subsets of X, then:

   a. $Z\subseteq W\Rightarrow \mathrm{h}_{(X,S)}(Z)=\mathrm{h}_{(W,S)}(Z)$,

   b. $Z\subseteq W\Rightarrow \mathrm{h}(Z)\leq\mathrm{h}(W)$,

   c. $(Z\subseteq W\wedge \mathrm{h}(Z)=\mathrm{h}(W)<+\infty)\Rightarrow Z=W$,

   d. $\mathrm{h}(Z)=0\Leftrightarrow Z$ is a minimal subset of (X, S).

2. if $\emptyset\neq B\subseteq A\subseteq X$, then:

   a. $\mathrm{Dim}^{\mathrm{T}(-)}(B)\leq\mathrm{Dim}^{\mathrm{T}(-)}(A)$,

   b. if $B\in\overline{\mathcal{M}}(X,S)$, then $\mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(B)\leq \mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(A)$,

   c. $\max\{\mathrm{Dim}^{\mathrm{T}(-)}(A)\leq\mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(A),\mathrm{Dim}^{\mathrm{T}(\overline{\overline{\mathrm{M}}})}(A)\}\leq\mathrm{h}(\mathrm{E}(X))$,

   d. $\mathrm{Dim}^{\mathrm{T}(-)}(A)\leq\mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(A)$.

3. if $\emptyset\neq A\subseteq X$, then the following statements are equivalent:

   a. for each $a\in A$, a is almost periodic,

   b. $\mathrm{Dim}^{\mathrm{T}(-)}(A)=0$,

   c. $\mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(A)=0$,

   d. $\mathrm{Dim}^{\mathrm{T}(\overline{\overline{\mathrm{M}}})}(A)=0$.

4. if $a\in A\subseteq X$ and $\mathrm{h}(\mathrm{E}(X))<+\infty$, then:

   a. (X, S) is distal if and only if $\mathrm{h}(\mathrm{E}(X))=0$,

   b. (X, S) is $a-$distal if and only if $\mathrm{h}(\mathrm{E}(X))=\mathrm{Dim}^{\mathrm{T}}(a)$,

   c. (X, S) is $A\,\underline{\overline{\mathrm{M}}}$ distal if and only if $\mathrm{h}(\mathrm{E}(X))=\mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(A)$,

   d. (X, S) is $A\,\underline{\overline{\overline{\mathrm{M}}}}$ distal if and only if $\mathrm{h}(\mathrm{E}(X))=\mathrm{Dim}^{\mathrm{T}(\overline{\overline{\mathrm{M}}})}(A)$.

Proof.

2. We have:

   a. $B\subseteq A\Rightarrow\sup\{\mathrm{Dim}^{\mathrm{T}}(x)\mid x\in B\}$

   $\leq\sup\{\mathrm{Dim}^{\mathrm{T}}(x)\mid x\in A\}$

   $\Rightarrow\mathrm{Dim}^{\mathrm{T}(-)}(B)\leq\mathrm{Dim}^{\mathrm{T}(-)}(A)$

   b. $\forall K\in\overline{\mathrm{M}}(A)\quad\forall b\in A\quad aK=a\mathrm{E}(X)$

   $\Rightarrow\forall K\in\overline{\mathrm{M}}(A)\quad\forall b\in B\quad bK=b\mathrm{E}(X)$

   $\Rightarrow\forall K\in\overline{\mathrm{M}}(A)\quad\exists L\in\overline{\mathrm{M}}(B)\quad L\subseteq K$

   $\Rightarrow\forall K\in\overline{\mathrm{M}}(A)\quad\exists L\in\overline{\mathrm{M}}(B)\quad \mathrm{h}(L)\leq\mathrm{h}(K)$

   $\Rightarrow\forall K\in\overline{\mathrm{M}}(A)\quad\mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(B))\leq\mathrm{h}(K)$

   (by Corollary 3)

   $\Rightarrow\mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(B))\leq\sup\{\mathrm{h}(K)\mid K\in\overline{\mathrm{M}}(A)\}$

   $=\mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(A)$

   d. For each $x\in A$ we have (by (b)):

$\mathrm{Dim}^{\mathrm{T}}(x)=\mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(\{x\})\leq\mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(A)$.

So $\mathrm{Dim}^{\mathrm{T}(-)}(A)=\sup\{\mathrm{Dim}^{\mathrm{T}}(x)\mid x\in A\}$
$\leq\mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(A)$.

3. Use the following facts:

- for each closed right ideal K of E(X), h(K)=0 if and only if K is a minimal right ideal of E(X),
- if Min(E(X)) denotes the set of all minimal right ideals of E(X), then the following statements are equivalent [5, Note 12]:

* for each $a\in A$, a is almost periodic,

* $\forall a\in A\quad \mathrm{M}(a)\cap\mathrm{Min}(\mathrm{E}(X))\neq\emptyset$,

* $\forall a\in A\quad \mathrm{M}(a)=\mathrm{Min}(\mathrm{E}(X))$,

* $\overline{\mathrm{M}}(A)\cap\mathrm{Min}(\mathrm{E}(X))\neq\emptyset$,

* $\overline{\mathrm{M}}(A)=\mathrm{Min}(\mathrm{E}(X))$,

* $\overline{\overline{\mathrm{M}}}(A)\cap\mathrm{Min}(\mathrm{E}(X))\neq\emptyset$,

* $\overline{\overline{\mathrm{M}}}(A)=\mathrm{Min}(\mathrm{E}(X))$.

4. Use the fact that for each closed right ideal K of E(X), h(K)=h(E(X)) if and only if K=E(X) (whenever h(E(X)=$+\infty$ the above statement may be false).

Theorem 5. In the transformation semigroup (X, S), let $A,B\in(\overline{\mathrm{M}}(X,S)\cap\overline{\overline{\mathrm{M}}}(X,S))$. We have the following table:

α	$\pi(B,A,\alpha)$
$(-,-),(\overline{\mathrm{M}},-)$	$\forall a\in A\quad \mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(B)\leq\mathrm{Dim}^{\mathrm{T}}(a)$
$(-,\overline{\mathrm{M}}),(\overline{\mathrm{M}},\overline{\mathrm{M}})$	$\mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(B)\leq\mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(A)$
$(-,\overline{\overline{\mathrm{M}}}),(\overline{\mathrm{M}},\overline{\overline{\mathrm{M}}})$	$\mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(B)\leq\mathrm{Dim}^{\mathrm{T}(\overline{\overline{\mathrm{M}}})}(A)$
$(\overline{\overline{\mathrm{M}}},-)$	$\forall a\in A\quad \mathrm{Dim}^{\mathrm{T}(\overline{\overline{\mathrm{M}}})}(B)\leq\mathrm{Dim}^{\mathrm{T}}(a)$
$(\overline{\overline{\mathrm{M}}},\overline{\mathrm{M}})$	$\mathrm{Dim}^{\mathrm{T}(\overline{\overline{\mathrm{M}}})}(B)\leq\mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(A)$
$(\overline{\overline{\mathrm{M}}},\overline{\overline{\mathrm{M}}})$	$\mathrm{Dim}^{\mathrm{T}(\overline{\overline{\mathrm{M}}})}(B)\leq\mathrm{Dim}^{\mathrm{T}(\overline{\overline{\mathrm{M}}})}(A)$

In the above table we have:

"If B is $A\underline{\ \alpha\ }$ almost periodic, then $\pi(B,A,\alpha)$ is true."

So if B is $A\underline{\ (P,Q)\ }$ almost periodic (where $P,Q\in\{-,\overline{\mathrm{M}},\overline{\overline{\mathrm{M}}}\}$), then $\mathrm{Dim}^{\mathrm{T(P)}}(B)\leq\mathrm{Dim}^{\mathrm{T(Q)}}(A)$.

Proof. Let $P,Q\in\{\overline{\mathrm{M}},\overline{\overline{\mathrm{M}}}\}$, then:

B is $A\underline{\ (P,Q)\ }$ almost periodic

$$\Rightarrow\forall K\in Q(A)\quad\exists L\in P(B)\quad L\subseteq K$$

$$\Rightarrow\forall K\in Q(A)\quad\exists L\in P(B)\quad \mathrm{h}(L)\leq\mathrm{h}(K)$$

$$\Rightarrow\mathrm{Dim}^{\mathrm{T(P)}}(B)\leq\mathrm{Dim}^{\mathrm{T(Q)}}(A)\quad\text{(by Corollary 3).}$$

In the other cases use a similar method.

Lemma 6.

1. Let $(X_1,S_1),\ldots,(X_n,S_n)$ be transformation semigroup. I be a closed right ideal of $\mathrm{E}(\prod_{i=1}^{n}X_i,\prod_{i=1}^{n}S_i)$ such that $S(I)\neq\emptyset$ and for each $i\in(1,\ldots,n\}$, $\emptyset\neq A_i\subseteq X_i$, where $\prod_{i=1}^{n}S_i$ is a semigroup under coordinate multiplication, then:

a. $\mathrm{h}(I)\geq\sum_{i=1}^{n}\mathrm{h}(\pi_i(I))$ (π_i is the projection map on ith coordinate),

b. $\mathrm{Dim}^{\mathrm{T}(-)}(\prod_{i=1}^{n}A_i)\geq\sum_{i=1}^{n}\mathrm{Dim}^{\mathrm{T}(-)}(A_i)$,

c. $\prod_{i=1}^{n}A_i\in\overline{\mathcal{M}}(\prod_{i=1}^{n}X_i,\prod_{i=1}^{n}S_i)\Rightarrow\mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(\prod_{i=1}^{n}A_i)$
$\geq\sum_{i=1}^{n}\mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(A_i)$.

2. In the transformation semigroup (X,S) if Z is a nonempty closed invariant subset of $X,\emptyset\neq A\subseteq Z$ and I is a closed right ideal of E(X), then:

a. $\mathrm{h}_{(\mathrm{E}(Z),S)}(I\mid_Z)\leq\mathrm{h}_{(\mathrm{E}(Z),S)}(I)$,

b. $\mathrm{Dim}^{\mathrm{T}}_{(Z,S)}(A)\leq\mathrm{Dim}^{\mathrm{T}}_{(X,S)}(A)$,

c. $\mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}_{(Z,S)}(A)\leq\mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}_{(X,S)}(A)$.

Proof.

1.

a. If $p \in S(I)$, then $I \subseteq p\mathrm{E}(\prod_{i=1}^{n} X_i, \prod_{i=1}^{n} S_i) \subseteq I$, so $I = p\mathrm{E}(\prod_{i=1}^{n} X_i, \prod_{i=1}^{n} S_i)$. For each $i \in (1,\dots,n\}$, let $p_i \in \mathrm{E}(X_i, S_i)$ be such that $p = (p_1,\dots,p_n)$, thus $I = p\mathrm{E}(\prod_{i=1}^{n} X_i, \prod_{i=1}^{n} S_i) = (p_1,\dots,p_n)\prod_{i=1}^{n}\mathrm{E}(X_i,S_i)$ $= \prod_{i=1}^{n} p_i\mathrm{E}(X_i,S_i) = \prod_{i=1}^{n}\pi_i(I)$. Let n=2, and $I_0^i \subseteq I_1^i \subseteq \cdots \subseteq I_n^i = \pi_i(I)$ be a chain of closed distinct right ideals in $E(X_i,S_i)(i=1,2)$, then $I_0^1 \times I_0^2 \subseteq I_1^1 \times I_0^2 \subseteq \cdots \subseteq I_{n_1}^1 \times I_0^2 \subseteq I_{n_1}^1 \times I_1^2 \subseteq \cdots \subseteq I_{n_1}^1 \times I_{n_2}^2 = \pi_1(I)\times\pi_2(I) = I$ is a chain of closed distinct right ideals in $\mathrm{E}(\prod_{i=1}^{n} X_i, \prod_{i=1}^{n} S_i)$, thus $\mathrm{h}(I) \geq n_1 + n_2$ and $\mathrm{h}(I) \geq \mathrm{h}(\pi_1(I)) + \mathrm{h}(\pi_2(I))$. Now by induction for $n \geq 2$ we have:

$$\mathrm{h}(I) = \mathrm{h}(\prod_{i=1}^{n}\pi_i(I)) \geq \mathrm{h}(\pi_1(I)) + \mathrm{h}(\prod_{i=2}^{n}\pi_i(I)) \geq \sum_{i=1}^{n}\mathrm{h}(\pi_i(I)).$$

Which completes the proof.

b. For each $(a_1,\dots,a_n) \in \prod_{i=1}^{n} X_i$ we have $\overline{\mathrm{M}}(a_1,\dots,a_n) = \prod_{i=1}^{n}\overline{\mathrm{M}}(a_i)$. Moreover if $K \in \overline{\mathrm{M}}(a_1,\dots,a_n)$, then $\emptyset \neq \mathrm{F}((a_1,\dots,a_n),K) \subseteq \mathrm{S}(K)$, so (by (a) and Corollary 3) $\mathrm{Dim}^{\mathrm{T}}(a_1,\dots,a_n) = \mathrm{h}(K) \geq \sum_{i=1}^{n}\mathrm{h}(\pi_i(K))$ $\sum_{i=1}^{n}\mathrm{h}(\pi_i(K)) = \sum_{i=1}^{n}\mathrm{Dim}^{\mathrm{T}}(a_i)$. Therefore:

$$\begin{aligned}\mathrm{Dim}^{\mathrm{T}(-)}(\prod_{i=1}^{n} A_i) &= \sup\{\mathrm{Dim}^{\mathrm{T}}(a_1,\dots,a_n) \mid \forall i \in \{1,\dots,n\} \quad a_i \in A_i\} \\ &\geq \sup\{\sum_{i=1}^{n}\mathrm{Dim}^{\mathrm{T}}(a_i) \mid \forall i \in \{1,\dots,n\} \quad a_i \in A_i\} \\ &= \sum_{i=1}^{n}\sup\{\mathrm{Dim}^{\mathrm{T}}(a_i) \mid a_i \in A_i\} \\ &= \sum_{i=1}^{n}\mathrm{Dim}^{\mathrm{T}(-)}(A_i)\end{aligned}$$

c. Let $\prod_{i=1}^{n} A_i \in \overline{\mathcal{M}}(\prod_{i=1}^{n} X_i, \prod_{i=1}^{n} S_i)$, and let $K \in \overline{\mathcal{M}}(\prod_{i=1}^{n} A_i)$, then $F(\prod_{i=1}^{n} A_i, K)$ is a nonempty subset of S(K), $\overline{\mathrm{M}}(\prod_{i=1}^{n} A_i) = \prod_{i=1}^{n}\overline{\mathrm{M}}(A_i)$ and for each $i \in \{1,\dots,n\}, A_i \in \overline{\mathcal{M}}(\prod_{i=1}^{n} X_i, \prod_{i=1}^{n} S_i)$, so (by (a) and Corollary 3) $\mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(\prod_{i=1}^{n} A_i) = \mathrm{h}(K) \geq \sum_{i=1}^{n}\mathrm{h}(\pi_i(K)) = \sum_{i=1}^{n}\mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(A_i)$.

2. Use the following facts:

- $\mathrm{E}(Z,S) = \{p \mid Z \mid p \in \mathrm{E}(X,S)\}(= \mathrm{E}(X,S) \mid Z)$,
- L is a closed right ideal of $\mathrm{E}(Z,S)$ if and only if there exists a closed right ideal K of $\mathrm{E}(X,S)$ such that $L = \{p \mid Z \mid p \in K\}(= K \mid Z)$,
- $\overline{\mathrm{M}}_{(Z,S)}(A) = \{K \mid Z : K \in \overline{\mathrm{M}}_{(Z,S)}(A)\}$,

Note 7. In the transformation semigroups (X_1, S) and (X_2, S), let $n \in \mathrm{N} \cup \{0,+\infty\}$, Z be a nonempty closed invariant subset of X_1 $a \in A \subseteq Z, \phi_1 : (X_1,S) \to (X_1,S)$ be a homomorphism and $\phi_2 : (X_1,S) \to (X_2,S)$ be an onto homomorphism, then:

1. $\mathrm{h}(\phi_i(Z)) \leq \mathrm{h}(Z) \quad (i = 1,2)$,
2. $\mathrm{Dim}^{\mathrm{T}}(\phi_i(A)) \leq \mathrm{Dim}^{\mathrm{T}}(A) \quad (i = 1,2)$,
3. $\phi_i(A) \in \overline{\mathcal{M}}(X_i,S) \Rightarrow \mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(\phi_i(A)) \leq \mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(A) \quad (i = 1,2)$,
4. (a is an n–almost periodic point) $\Rightarrow$ ($\exists m \leq n$ ($\phi_i(a)$ is an m–almost periodic point)) $(i = 1,2)$,
5. (a is an n–almost periodic point in (X_1, S)) $\Leftrightarrow$ (a is an n–almost periodic point in (Z, S)).

Lemma 8. In the transformation semigroup (X, S), we have:

1. If Z be a nonempty closed invariant subset of X, then:
   $\mathrm{h}(Z) = \sup\{n \in N \cup \{0\} \mid \exists z_0, \ldots, z_n \in Z \; \forall i \in \{1, \ldots, n\} \quad z_i \notin \bigcup_{k=0}^{i-1} z_k \mathrm{E}(X)\}$
2. If $Z_1, \ldots, Z_n$ be nonempty closed invariant subsets of X, then:
   $\mathrm{h}(\bigcup_{i=1}^{n} Z_i) \leq \sum_{i=1}^{n} \mathrm{h}(Z_i) + (n-1)$.
3. If $Z_1, \ldots, Z_n$ be nonempty closed invariant disjoint subset of X, then:
   a. $\mathrm{h}(\bigcup_{i=1}^{n} Z_i) = \sum_{i=1}^{n} \mathrm{h}(Z_i) + (n-1)$,
   b. $\mathrm{h}(\bigcup_{i=1}^{n} Z_i) = n-1$ if and only if for each $i \in \{1, \ldots, n\}, Z_i$ is minimal.
4. If $Z_1, \ldots, Z_n$ be minimal subsets of X, then card$\{Z_i \mid 1 \leq i \leq n\} = \mathrm{h}(\bigcup_{i=1}^{n} Z_i) + 1$.

Proof.

1. Let $n \in \mathrm{N} \cup \{0\}$ and $z_0, \ldots, z_n \in Z$ be such that for each $i \in \{1, \ldots, n\}$, $z_i \notin \bigcup_{k=0}^{i-1} z_k \mathrm{E}(X)$. For each $i \in \{0, \ldots, n\}$, $Z_i = \bigcup_{k=0}^{i} z_k \mathrm{E}(X)$ is a closed nonempty invariant subset of Z, moreover for each $i \in \{0, \ldots, n-1\}$, Z_i is a proper subset of Z_{i+1}, therefore $n \leq \mathrm{h}(Z)$. Thus $\sup\{n \in \mathrm{N} \cup \{0\} \mid \exists z_0, \ldots, z_n \in Z \; \forall i \in \{1, \ldots, n\} \quad z_i \notin \bigcup_{k=0}^{i-1} z_k \mathrm{E}(X)\} \leq \mathrm{h}(Z)$. On the other hand let $n \in \mathrm{N} \cup \{0\}$ and $Z_0, \ldots, Z_n$ be closed nonempty invariant subsets of Z such that for each $i \in \{0, \ldots, n-1\}$, Z_i is a proper subset of Z_{i+1}, then for $z_0 \in Z_0$ and $z_i \in Z_i - Z_{i-1}$ $(1 \leq i \leq n)$, we have $z_i \notin \bigcup_{k=0}^{i-1} z_k \mathrm{E}(X)$ $(1 \leq i \leq n)$. Therefore $\mathrm{h}(Z) \leq \sup\{n \in \mathrm{N} \cup \{0\} \mid \exists z_0, \ldots, z_n \in Z \; \forall i \in \{1, \ldots, n\} \quad z_i \notin \bigcup_{k=0}^{i-1} z_k \mathrm{E}(X)\}$, which completes the proof.

2. It is clear for n=1. For n=2, let $m \in \mathrm{N} \cup \{0\}$ and $x_0, \ldots, x_m \in Z_1 \cup Z_2$ be such that for each $i \in \{1, \ldots, m\}$, $x_i \notin \bigcup_{k=0}^{i-1} z_k \mathrm{E}(X)$, then by (1) we have $m-1 \leq (\mathrm{card}(\{x_0, \ldots, x_m\} \cap Z_1) - 1) + (\mathrm{card}(\{x_0, \ldots, x_m\} \cap Z_2) - 1) \leq \mathrm{h}(Z_1) + \mathrm{h}(Z_2)$, thus $\mathrm{h}(Z_1 \cup Z_2) \leq \mathrm{h}(Z_1) + \mathrm{h}(Z_2) + 1$. Let $m \in \mathrm{N}$ and suppose for $n = m$ the above statement holds, for $n = m+1$ we have:

$$\begin{aligned}\mathrm{h}(\bigcup_{i=1}^{m+1} Z_i) &= \mathrm{h}(\bigcup_{i=1}^{m} Z_i \cup Z_{m+1}) \leq \mathrm{h}(\bigcup_{i=1}^{m} Z_i) + \mathrm{h}(Z_{m+1}) + 1 \\ &\leq \sum_{i=1}^{m} \mathrm{h}(Z_i) + (m-1) + \mathrm{h}(Z_{m+1}) + 1 \\ &= \sum_{i=1}^{m+1} \mathrm{h}(Z_i) + m,\end{aligned}$$

which will give the result by induction on n.

3.

a. For each $i \in \{1, \ldots, n\}$, let $Z_0^i, \ldots, Z_m^i$, be nonempty closed invariant subsets of Z_i such that for each $j \in \{0, \ldots, m_i - 1\}$, Z_j^i is a proper subset of Z_{j+1}^i, then:

$$\begin{array}{lll} Z_0^1 & \subset Z_1^1 & \subset \cdots \subset Z_{m_1}^1 \\ \subset Z_{m_1}^1 \cup Z_0^2 & \subset Z_{m_1}^1 \cup Z_1^2 & \subset \cdots \subset Z_{m_1}^1 \cup Z_{m_2}^2 \\ \vdots & & \\ \subset \bigcup_{i=11}^{n-1} Z_{m_i}^i \cup Z_0^n & \subset \bigcup_{i=11}^{n-1} Z_{m_i}^i \cup Z_1^n & \subset \cdots \subset \bigcup_{i=11}^{n-1} Z_{m_i}^i \cup Z_{m_n}^n \end{array}$$

is a chain of distinct nonempty closed invariant subsets of $\bigcup_{i=1}^{n} Z_i$, therefore $\mathrm{h}(\bigcup_{i=1}^{n} Z_i) \geq \sum_{i=1}^{n} m_i + (n-1)$ and $\mathrm{h}(\bigcup_{i=1}^{n} Z_i) \geq \sum_{i=1}^{n} \mathrm{h}(Z_i) + (n-1)$. Considering (2) we have $\mathrm{h}(\bigcup_{i=1}^{n} Z_i) = \sum_{i=1}^{n} \mathrm{h}(Z_i) + (n-1)$.

b. By (a), $\mathrm{h}(\bigcup_{i=1}^{n} Z_i) = n-1$ if and only if for each $i \in \{1, \ldots, n\}$, $\mathrm{h}(Z_i) = 0$. Now (1) in Note 4 completes the proof.

4. It is clear by (3).

Theorem 9. In the transformation semigroup (*X*, *S*), if *Z* be a nonempty closed invariant sunset of *X*, then the following statements are equivalent:

i. h(*Z*)=1,

ii. "*Z* is the union of two disjoint minimal subset of *X*" or "there exists a 1-almost periodic point $z \in X$ such that $z\mathrm{E}(X) = Z$ ".

Proof.
((i) $\Rightarrow$ (ii)): By (1) Lemma 8, there exist $z_0, z_1 \in Z$ such that $z_0\mathrm{E}(X)$ is minimal and $z_1 \notin z_0\mathrm{E}(X)$. Since $\mathrm{h}(z_1\mathrm{E}(X)) \leq \mathrm{h}(Z) = 1$, we have $\mathrm{h}(z_1\mathrm{E}(X)) = 1$ or $\mathrm{h}(z_1\mathrm{E}(X)) = 0$. If $\mathrm{h}(z_1\mathrm{E}(X)) = 1$, then z_1 is 1-almost periodic and $z_1\mathrm{E}(X) = Z$, if $\mathrm{h}(z_1\mathrm{E}(X)) = 0$, then $z_1\mathrm{E}(X)$ is minimal, $\mathrm{h}(z_1\mathrm{E}(X) \cup z_0\mathrm{E}(X)) = 1$ and Z is the union of two disjoint minimal set $z_1\mathrm{E}(X)$ and $z_0\mathrm{E}(X)$.
((ii) $\Rightarrow$ (i)): Use Lemma 8.

Lemma 10. In the transformation semigroup (*X*, *S*), if $r \in \mathrm{E}(X)$ be such that $r^2\mathrm{E}(X) = r\mathrm{E}(X)$ and $\mathrm{h}(r\mathrm{E}(X)) < +\infty$, then:

1. $r\mathrm{E}(X) \in \mathrm{M}(r)$,

2. r is n–almost periodic if and only if $\mathrm{Dim}^{\mathrm{T}}(r) = n$.

Therefore if I is a closed right ideal of $\mathrm{E}(X)$ and $p \in \mathrm{S}(I)$, then $\mathrm{Dim}^{\mathrm{T}}(p) = \mathrm{h}(I)$ (since $I = p\mathrm{E}(X) = p^2\mathrm{E}(X)$).

Proof. Use E(E(*X*, *S*),*S*) = E(*X*, *S*).

Corollary 11. In the transformation semigroup (*X*, *S*), let $n \in \mathrm{N} \cup \{0\}$ and $\emptyset \neq A \subseteq X$.

1. If $A \in \overline{\mathcal{M}}(X,S)$, then the following statements are equivalent:

a. $\mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(A) = n$,

b. $\exists u \in \mathrm{J}(\mathrm{F}(A, \mathrm{E}(X)))$
$(uE(X) \in \overline{M}(A) \wedge \mathrm{Dim}^{\mathrm{T}}(u) = n)$,

c. $\exists p \in \mathrm{F}(A, \mathrm{E}(X))$
$(pE(X) \in \overline{M}(A) \wedge \mathrm{Dim}^{\mathrm{T}}(p) = n)$,

d. $\min\{\mathrm{Dim}^{\mathrm{T}}(p) \mid p \in \mathrm{F}(A, \mathrm{E}(X))\} = n$,

e. $\min\{\mathrm{Dim}^{\mathrm{T}}(u) \mid u \in \mathrm{J}(\mathrm{F}(A, \mathrm{E}(X)))\} = n$,

f. $\overline{\mathrm{M}}(A) = \{u\mathrm{E}(X) \mid \mathrm{Dim}^{\mathrm{T}}(u) = n \wedge u \in \mathrm{J}(\mathrm{F}(A, \mathrm{E}(X)))\}$,

g. $\overline{\mathrm{M}}(A) = \{p\mathrm{E}(X) \mid \mathrm{Dim}^{\mathrm{T}}(p) = n \wedge p \in \mathrm{F}(A, \mathrm{E}(X))\}$.

2. If $A \in \overline{\overline{\mathcal{M}}}(X,S)$, then the following statements are equivalent:

a. $\mathrm{Dim}^{\mathrm{T}(\overline{\overline{\mathrm{M}}})}(A) = n$,

b. $\exists u \in \mathrm{J}(\mathrm{F}(A, \mathrm{E}(X)))$
$(uE(X) \in \overline{\overline{\mathrm{M}}}(A) \wedge \mathrm{Dim}^{\mathrm{T}}(u) = n)$,

c. $\exists p \in \mathrm{F}(A, \mathrm{E}(X))$
$(pE(X) \in \overline{\overline{\mathrm{M}}}(A) \wedge \mathrm{Dim}^{\mathrm{T}}(p) = n)$,

d. $\exists q \in \overline{\mathrm{F}}(A, \mathrm{E}(X))$
$(qE(X) \in \overline{\overline{\mathrm{M}}}(A) \wedge \mathrm{Dim}^{\mathrm{T}}(q) = n)$,

e. $\min\{\mathrm{Dim}^{\mathrm{T}}(q) \mid q \in \mathrm{F}(A, \mathrm{E}(X))\} = n$

f. $\min\{\mathrm{Dim}^{\mathrm{T}}(p) \mid p \in \mathrm{F}(A, \mathrm{E}(X))\} = n$,

g. $\min\{\mathrm{Dim}^{\mathrm{T}}(u) \mid u \in \mathrm{J}(\mathrm{F}(A, \mathrm{E}(X)))\} = n$,

h. $\overline{\overline{\mathrm{M}}}(A) = \{u\mathrm{E}(X) \mid \mathrm{Dim}^{\mathrm{T}}(u) = n \wedge u \in \mathrm{J}(\mathrm{F}(A, \mathrm{E}(X)))\}$,

i. $\overline{\overline{\mathrm{M}}}(A) = \{p\mathrm{E}(X) \mid \mathrm{Dim}^{\mathrm{T}}(p) = n \wedge p \in \mathrm{F}(A, \mathrm{E}(X))\}$,

j. $\overline{\overline{\mathrm{M}}}(A) = \{q\mathrm{E}(X) \mid \mathrm{Dim}^{\mathrm{T}}(q) = n \wedge q \in \overline{\mathrm{F}}(A, \mathrm{E}(X))\}$.

3. If $A \in \overline{\mathcal{M}}(X,S)$ and $\mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(A) = n$, then each of the following sets is equal to $\overline{\mathrm{M}}(A)$:

– $\{p\mathrm{E}(X) \mid \forall q \in \mathrm{F}(A, p\mathrm{E}(X))(\mathrm{Dim}^{\mathrm{T}}(q)$
$= n \wedge p \in \mathrm{F}(A, \mathrm{E}(X)))\}$,

– $\{u\mathrm{E}(X) \mid \forall v \in \mathrm{J}(\mathrm{F}(A, u\mathrm{E}(X)))(\mathrm{Dim}^{\mathrm{T}}(v)$
$= n \wedge v \in \mathrm{J}(\mathrm{F}(A, \mathrm{E}(X))))\}$,

– $\{p\mathrm{E}(X) \mid \forall q \in \mathrm{F}(A, p\mathrm{E}(X))(\mathrm{Dim}^{\mathrm{T}}(q)$
$= \mathrm{Dim}^{\mathrm{T}}(p) \wedge p \in \mathrm{F}(A, \mathrm{E}(X)))\}$,

– $\{u\mathrm{E}(X) \mid \forall v \in \mathrm{J}(\mathrm{F}(A, u\mathrm{E}(X)))(\mathrm{Dim}^{\mathrm{T}}(v)$
$= \mathrm{Dim}^{\mathrm{T}}(u) \wedge v \in \mathrm{J}(\mathrm{F}(A, \mathrm{E}(X))))\}$.

4. If $A \in \overline{\overline{\mathcal{M}}}(X,S)$ and $\mathrm{Dim}^{\mathrm{T}(\overline{\overline{\mathrm{M}}})}(A) = n$, then each of the following sets is equal to $\overline{\overline{\mathrm{M}}}(A)$:

– $\{p\mathrm{E}(X) \mid \forall q \in \overline{\mathrm{F}}(A, p\mathrm{E}(X))(\mathrm{Dim}^{\mathrm{T}}(q)$
$= n \wedge p \in \overline{\mathrm{F}}(A, \mathrm{E}(X)))\}$,

– $\{p\mathrm{E}(X) \mid \forall q \in \mathrm{F}(A, p\mathrm{E}(X))(\mathrm{Dim}^{\mathrm{T}}(q) = n \wedge p \in \mathrm{F}(A, \mathrm{E}(X)))\}$,

– $\{u\mathrm{E}(X) \mid \forall v \in \mathrm{J}(\mathrm{F}(A, u\mathrm{E}(X)))(\mathrm{Dim}^{\mathrm{T}}(v) = n \wedge v \in \mathrm{J}(\mathrm{F}(A, \mathrm{E}(X))))\}$,

– $\{p\mathrm{E}(X) \mid \forall q \in \overline{\mathrm{F}}(A, p\mathrm{E}(X))(\mathrm{Dim}^{\mathrm{T}}(q) = \mathrm{Dim}^{\mathrm{T}}(p) < +\infty \wedge p \in \overline{\mathrm{F}}(A, \mathrm{E}(X)))\}$,

– $\{p\mathrm{E}(X) \mid \forall q \in \mathrm{F}(A, p\mathrm{E}(X))(\mathrm{Dim}^{\mathrm{T}}(q) = \mathrm{Dim}^{\mathrm{T}}(p) < +\infty \wedge p \in \mathrm{F}(A, \mathrm{E}(X)))\}$,

– $\{u\mathrm{E}(X) \mid \forall v \in \mathrm{J}(\mathrm{F}(A, u\mathrm{E}(X)))(\mathrm{Dim}^{\mathrm{T}}(v) = \mathrm{Dim}^{\mathrm{T}}(u) < +\infty \wedge v \in \mathrm{J}(\mathrm{F}(A, \mathrm{E}(X))))\}$.

Proof.

1.

- ((a) $\Rightarrow$ (b)): Let $\mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(A) = n$. By Definition 1, there exists $K \in \overline{\mathrm{M}}(A)$ such that $\mathrm{h}(K) = n$. Choose $u \in \mathrm{J}(\mathrm{F}(A, K))$, since $\mathrm{J}(\mathrm{F}(A, K)) \subseteq \mathrm{S}(K)$, so $u\mathrm{E}(X) = K \in \overline{\mathrm{M}}(A)$ and by Lemma 10 $\mathrm{Dim}^{\mathrm{T}}(u) = \mathrm{h}(K) = n$.

- ((b) $\Rightarrow$ (c)): It is clear by $\mathrm{J}(\mathrm{F}(A, K)) \subseteq \mathrm{F}(A, K)$.

- ((c) $\Rightarrow$ (d)): By (c), $\min\{\mathrm{Dim}^{\mathrm{T}}(p) | p \in \mathrm{F}(A, \mathrm{E}(X))\} \le n$ and by Corollary 3, $\mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(A) = n$. Let $q \in \mathrm{F}(A, \mathrm{E}(X))$, $\mathrm{Dim}^{\mathrm{T}}(q) = m$ and $K \in \mathrm{M}(q)$. By Corollary 3 (and $\mathrm{E}(\mathrm{E}(X)) = \mathrm{E}(X)$) $m = \mathrm{h}(K)$, on the other hand for each $a \in A$ we have: $aK = aqK = aq\mathrm{E}(X) = a\mathrm{E}(X)$ thus there exists $L \in \overline{\mathrm{M}}(A)$ such that $L \subseteq K$, so by Corollary 3 we have: $n = \mathrm{h}(L) \le \mathrm{h}(K) = m$. Thus $\min\{\mathrm{Dim}^{\mathrm{T}}(p) | p \in \mathrm{F}(A, \mathrm{E}(X))\} \ge n$ and $\min\{\mathrm{Dim}^{\mathrm{T}}(p) \mid p \in \mathrm{F}(A, \mathrm{E}(X))\} = n$.

- ((d) $\Rightarrow$ (e)): By (d), $\min\{\mathrm{Dim}^{\mathrm{T}}(u) \mid u \in \mathrm{J}(\mathrm{F}(A, \mathrm{E}(X)))\} \ge n$. Let $p \in \mathrm{F}(A, \mathrm{E}(X))$, $\mathrm{Dim}^{\mathrm{T}}(p) = n$ and $K \in \mathrm{M}(p)$, for each $a \in A$, $apK = ap\mathrm{E}(X) = a\mathrm{E}(X)$ so there exists $L \in \overline{\mathrm{M}}(A)$ such that $L \subseteq K$. Choose $v \in \mathrm{J}(\mathrm{F}(A, L))$. By Corollary 3, Note 4, Note 7 and Lemma 10, $\mathrm{Dim}^{\mathrm{T}}(v) = \mathrm{h}(v\mathrm{E}(X)) = \mathrm{h}(L) \le \mathrm{h}(pK) \le \mathrm{h}(K) = \mathrm{Dim}^{\mathrm{T}}(p) = n$, so $\min\{\mathrm{Dim}^{\mathrm{T}}(u) | u \in \mathrm{J}(\mathrm{F}(A, \mathrm{E}(X)))\} \le n$ and $\min\{\mathrm{Dim}^{\mathrm{T}}(u) \mid u \in \mathrm{J}(\mathrm{F}(A, \mathrm{E}(X)))\} = n$.

- ((e) $\Rightarrow$ (f)): Let $v \in \mathrm{J}(\mathrm{F}(A, \mathrm{E}(X)))$ and $\mathrm{Dim}^{\mathrm{T}}(v) = n$. There exists $L \in \overline{\mathrm{M}}(A)$ such that $L \subseteq v\mathrm{E}(X)$. Choose $w \in \mathrm{J}(\mathrm{F}(A, L))$, by Lemma 10 $n \le \mathrm{Dim}^{\mathrm{T}}(w) = \mathrm{h}(L) \le \mathrm{h}(v\mathrm{E}(X)) = \mathrm{Dim}^{\mathrm{T}}(v) = n$ so $\mathrm{h}(L) = \mathrm{h}(v\mathrm{E}(X)) < +\infty$, thus by Note 4, $v\mathrm{E}(X) = L \in \overline{\mathrm{M}}(A)$ and $\{uE(X) \mid \mathrm{Dim}^{\mathrm{T}}(u) = n \wedge u \in \subseteq \mathrm{J}(\mathrm{F}(A, \mathrm{E}(X)))\} \subseteq \overline{\mathrm{M}}(A)$ and by Corollary 3, $\mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(A) = n$. On the other hand let $L' \in \overline{\mathrm{M}}(A)$, there exists $v' \in \mathrm{J}(\mathrm{F}(A, L'))$ and $v'\mathrm{E}(X) = L'$. By Corollary 3 and Corollary 10, $\mathrm{Dim}^{\mathrm{T}}(v') = \mathrm{h}(L') = \mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(A) = n$, so $\{u\mathrm{E}(X) \mid \mathrm{Dim}^{\mathrm{T}}(u) = n \wedge u \in \mathrm{J}(\mathrm{F}(A, \mathrm{E}(X)))\} \supseteq \overline{\mathrm{M}}(A)$ and $\{u\mathrm{E}(X) \mid \mathrm{Dim}^{\mathrm{T}}(u) = n \wedge u \in \mathrm{J}(\mathrm{F}(A, \mathrm{E}(X)))\} = \mathrm{M}(A)$

- ((f) $\Rightarrow$ (g)): By (f), $\overline{\mathrm{M}}(A) \subseteq \{p\mathrm{E}(X) \mid \mathrm{Dim}^{\mathrm{T}}(p) = n \wedge p \in \mathrm{F}(A, \mathrm{E}(X))\}$. Let $q \in \mathrm{F}(A, \mathrm{E}(X))$, $\mathrm{Dim}^{\mathrm{T}}(q) = n$ and $K \in \mathrm{M}(q)$. For each $a \in A$, $aK = apK = apE(X) = aE(X)$ so there exists $L \in \overline{\mathrm{M}}(A)$ such that $L \subseteq K$, by (f) there exists $u \in \mathrm{J}(\mathrm{F}(A, \mathrm{E}(X)))$ such that $\mathrm{Dim}^{\mathrm{T}}(u) = n$ and $L = u\mathrm{E}(X)$, so by Lemma 10 we have: $n = \mathrm{Dim}^{\mathrm{T}}(u) = \mathrm{h}(u\mathrm{E}(X)) = \mathrm{h}(L) \le \mathrm{h}(K) = \mathrm{Dim}^{\mathrm{T}}(p) = n$, thus $\mathrm{h}(L) = \mathrm{h}(K) < +\infty$ and $L = K$, therefore $p\mathrm{E}(X) = p\mathrm{K} = p\mathrm{L} = pu\mathrm{E}(X)$, now there exists $M \subseteq p\mathrm{E}(X)$ such that $M \in \overline{\mathrm{M}}(A)$ so by (f), Corollary 3 and Corollary 10 we have: $\mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(A) = n = \mathrm{h}(M) \le h(p\mathrm{E}(X)) = \mathrm{h}(pu\mathrm{E}(X)) \le h(u\mathrm{E}(X) = \mathrm{Dim}^{\mathrm{T}}(u) = n$, therefore $\mathrm{h}(M) = \mathrm{h}(p\mathrm{E}(X)) < +\infty$ and $p\mathrm{E}(X) = M \in \overline{\mathrm{M}}(A)$. So $\{p\mathrm{E}(X) \mid \mathrm{Dim}^{\mathrm{T}}(p) = n \wedge p \in \mathrm{F}(A, \mathrm{E}(X))\} \subseteq \overline{\mathrm{M}}(A)$ and $\{p\mathrm{E}(X) \mid \mathrm{Dim}^{\mathrm{T}}(p) = n \wedge p \in \mathrm{F}(A, \mathrm{E}(X))\} = \overline{\mathrm{M}}(A)$.

- ((g) $\Rightarrow$ (a)): Use Lemma 10 and Corollary 3.

2. Like (1) use Lemma 10 and $\forall K \in \overline{\overline{\mathrm{M}}}(A)$ $\overline{\mathrm{F}}(A, K) \subseteq \mathrm{S}(K)$.

3. Use (1).

4. Use (2).

Theorem 12. In the transformation semigroup (X, S), let $A_1,\dots,A_n$ be nonempty subsets of X, we have:

1. $\overline{\mathrm{M}}(\bigcup_{i=1}^{n} A_i) = \min(\{\bigcup_{i=1}^{n} K_i \mid \forall i \in \{1,\dots,n\}$ $K_i \in \overline{M}(A_i)\}, \subseteq)$.

2. $\mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})} \bigcup_{i=1}^{n} A_i \le \sum_{i=1}^{n} \mathrm{h}(A_i) + (n-1)$.

3. $\forall K \in \overline{\mathrm{M}}(\bigcup_{i=1}^{n} A_i) \quad (\mathrm{S}(K) \ne \emptyset \Rightarrow K \in \bigcup_{i=1}^{n} \overline{\mathrm{M}}(A_i))$.

4. If $\bigcup_{i=1}^{n} A_i \in \overline{\mathcal{M}}(X,S)$, then:

 a. $\overline{\mathrm{M}}(\bigcup_{i=1}^{n} A_i)$ is a subset of:

 $\bigcup\{\overline{\mathrm{M}}(A_i) | 1 \le i \le n, \mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(A_i) = \mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(\bigcup_{i=1}^{n} A_i),$ $A_i \in \overline{\mathcal{M}}(X,S)\}$

 b. $\mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(\bigcup_{i=1}^{n} A_i) = \max\{\mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(A_i) \mid 1 \le i \le n,$ $A_i \in \overline{\mathcal{M}}(X,S)\}$

 c. $\mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(\bigcup_{i=1}^{n} A_i) = +\infty$ if and only if there exists $i \in \{1,\dots,n\}$ such that $\mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(A_i) = +\infty$.

5. If $\bigcup_{i=1}^{n} A_i \in \overline{\mathcal{M}}(X,S)$ and $\mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(\bigcup_{i=1}^{n} A_i) < \infty$, then $\overline{\mathrm{M}}(\bigcup_{i=1}^{n} A_i)$ is a subset of:

 $\bigcap\{\overline{\mathrm{M}}(A_i) | 1 \le i \le n, \mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(A_i) = \mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(\bigcup_{i=1}^{n} A_i),$ $A_i \in \overline{\mathcal{M}}(X,S)\}$.

6. If $\max\{\mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(A_i) \mid 1 \le i \le n \le 1$ and $\lambda \in \{0,1\}$, then the following statements are equivalent:

 a. $\mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(\bigcup_{i=1}^{n} A_i) = \lambda$,

 b. $\max\{\mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(A_i) \mid 1 \le i \le n\} = \lambda$ and for each $K \in \overline{\mathrm{M}}(\bigcup_{i=1}^{n} A_i)$, $\mathrm{S}(K) \ne \emptyset$,

 c. $\max\{\mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(A_i) \mid 1 \le i \le n\} = \lambda$ and $\overline{\mathrm{M}}(\bigcup_{i=1}^{n} A_i) \subseteq \bigcup_{i=1}^{n} \overline{\mathrm{M}}(A_i)$,

 d. $\overline{\mathrm{M}}(\bigcup_{i=1}^{n} A_i) \subseteq \bigcap\{\overline{\mathrm{M}}(A_i) | 1 \le i \le n, \mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(A_i) = \lambda\}$,

 e. $\overline{\mathrm{M}}(\bigcup_{i=1}^{n} A_i) = \bigcap\{\overline{\mathrm{M}}(A_i) | 1 \le i \le n, \mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(A_i) = \lambda\}$.

Proof.

1. Let $K \in \overline{\mathrm{M}}(\bigcup_{i=1}^{n} A_i)$:

$$K \in \overline{\mathrm{M}}(\bigcup_{i=1}^{n} A_i) \Rightarrow \forall a \in \bigcup_{i=1}^{n} A_i \quad aK = a\mathrm{E}(X)$$

$$\Rightarrow \forall i \in \{1,\dots,n\} \quad \forall a_i \in A_i \quad a_i K = a_i E(X)$$

$$\Rightarrow \forall i \in \{1,\dots,n\} \quad \exists K_i \in \overline{\mathrm{M}}(A_i) \quad K_i \subseteq K$$

$$\Rightarrow \forall i \in \{1,\dots,n\} \quad \exists K_i \in \overline{\mathrm{M}}(A_i) \quad \bigcup_{j=1}^{n} K_j \subseteq K$$

$$\Rightarrow \forall i \in \{1,\dots,n\} \quad \exists K_i \in \overline{\mathrm{M}}(A_i) \quad \bigcup_{j=1}^{n} K_j = K$$

(since $\bigcup_{i=1}^{n} K_i$ is a closed ideal of E(X) and for each $a \in \bigcup_{i=1}^{n} A_i$, $a\bigcup_{i=1}^{n} K_i = a\mathrm{E}(X)$) .

Therefore $\overline{\mathrm{M}}(\bigcup_{i=1}^{n} A_i) \subseteq \{\bigcup_{i=1}^{n} K_i \mid \forall i \in \{1,\dots,n\}$ $K_i \in \overline{M}(A_i)\}$, and:

$\overline{\mathrm{M}}(\bigcup_{i=1}^{n} A_i) \subseteq \min(\{\bigcup_{i=1}^{n} K_i \mid \forall i \in \{1,\dots,n\}$ $K_i \in \overline{M}(A_i)\}, \subseteq)$.

On the other hand let $L_1 \in \overline{\mathrm{M}}(A_1),\dots,L_n \overline{\mathrm{M}}(A_n)$ be such that $\bigcup_{i=1}^{n} L_i \in \min(\{\bigcup_{i=1}^{n} K_i \mid \forall i \in \{1,\dots,n\} K_i \in$ $\overline{M}(A_i)\}, \subseteq)$, then:

$\forall i \in \{1,\dots,n\} \quad L_i \in \overline{\mathrm{M}}(A_i)$

$\Rightarrow \forall a \in \bigcup_{i=1}^{n} A_i \quad a\bigcup_{i=1}^{n} L_i = a\mathrm{E}(X)$

$\Rightarrow \exists K \in \overline{\mathrm{M}}(\bigcup_{i=1}^{n} A_i \quad K \subseteq \bigcup_{i=1}^{n} L_i$

$\exists K \in \overline{\mathrm{M}}(\bigcup_{i=1}^{n} A_i) \quad K = \bigcup_{i=1}^{n} L_i$

(since $\overline{\mathrm{M}}(\bigcup_{i=1}^{n} A_i) \subseteq \min(\{\bigcup_{i=1}^{n} K_i \mid \forall i \in \{1,\dots,n\} K_i \in \overline{M}(A_i)\}, \subseteq))$.

Therefore $\bigcup_{i=1}^{n} L_i \in \overline{\mathrm{M}}(\bigcup_{i=1}^{n} A_i)$ and:

$\overline{\mathrm{M}}(\bigcup_{i=1}^{n} A_i) = \min(\{\bigcup_{i=1}^{n} K_i \mid \forall i \in \{1,\dots,n\}$
$K_i \in \overline{M}(A_i)\}, \subseteq)$.

2. We have:

$$\begin{aligned}
&\mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(\bigcup_{i=1}^{n} A_i) \\
&= \sup\{\mathrm{h}(K) \mid K \in \overline{\mathrm{M}}(\bigcup_{i=1}^{n} A_i)\} \\
&= \sup\{\mathrm{h}(K) \mid K \in \min(\{\bigcup_{i=1}^{n} K_i \mid \forall i \in \{1,\dots,n\} \\
&\qquad K_i \in \overline{M}(A_i)\}, \subseteq)\} \\
&\le \sup\{\mathrm{h}(\bigcup_{i=1}^{n} K_i) \mid \forall i \in \{1,\dots,n\} \quad K_i \in \overline{M}(A_i)\} \\
&\le \sup\{\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{h}(K_i) + (n-1) \mid \forall i \in \{1,\dots,n\} K_i \in \overline{M}(A_i)\} \\
&\qquad \text{(by Lemma 8)} \\
&= \sum_{\mathrm{i}=1}^{\mathrm{n}} \sup\{\mathrm{h}(K_i) \mid K_i \in \overline{M}(A_i)\} + (n-1) \\
&= \sum_{i=1}^{n} \mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(A_i) + (n-1)
\end{aligned}$$

3. Let $K \in \overline{\mathrm{M}}(\bigcup_{i=1}^{n} A_i)$ and $p \in \mathrm{S}(K)$, for each $i \in \{1,\dots,n\}$, choose $K_i \in \overline{\mathrm{M}}(A_i)$ such that $K = \bigcup_{i=1}^{n} K_i$. Choose $i_0 \in \{1,\dots,n\}$ such that $p \in K_{i_0}$. By $K_{i_0} \subseteq K \quad = pK \subseteq K_{i_0} K \subseteq K_{i_0}$, we have $K = K_{i_0} \in \overline{\mathrm{M}}(A_i)$.

4. $\bigcup_{i=1}^{n} A_i \in \overline{\mathcal{M}}(X,S)$, then for all $K \in \overline{\mathrm{M}}(\bigcup_{i=1}^{n} A_i)$, $\emptyset \ne \mathrm{F}(\bigcup_{i=1}^{n} A_i, K) \subseteq \mathrm{S}(K)$, and for each $j \in \{1,\dots,n\}$, $\mathrm{F}(\bigcup_{i=1}^{n} A_i, K) \subseteq \mathrm{F}(A_j, K)$, now use (3).

5. Use Corollary 3 and a similar method described for (3) and (4).

6. Use the above items and (3) in Note 4.

Corollary 13. In the transformation semigroup (X, S), let A be a nonempty subset of X, we have:

1. For each $a \in A$, if $\mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(A) = \mathrm{Dim}^{\mathrm{T}}(a) < +\infty$, then $\overline{\mathrm{M}}(A) \subseteq \mathrm{M}(a)$.

2. "$\mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(A) = 1$" if and only if "$\max\{\mathrm{Dim}^{\mathrm{T}}(a) \mid a \in A\} = 1$ and $\overline{\mathrm{M}}(A) = \bigcap\{M(a) \mid a \in A,\ \mathrm{Dim}^{\mathrm{T}}(a) = 1\}$.

Proof.

1. Let $a \in A$, $\mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(A) = \mathrm{Dim}^{\mathrm{T}}(a) < +\infty$, and $K \in \overline{\mathrm{M}}(A)$. We have:

$$\begin{aligned}
&K \in \overline{\mathrm{M}}(A) \\
&\Rightarrow aK = a\mathrm{E}(X,S) \\
&\Rightarrow \exists L \in \mathrm{M}(a) \quad L \subseteq K \quad \text{([5, Corollary 3])} \\
&\Rightarrow \exists L \in \mathrm{M}(a) \quad (L \subseteq K \wedge \mathrm{Dim}^{\mathrm{T}}(a) = \mathrm{h}(L) \le \mathrm{h}(K) \\
&\qquad \le \mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(A)) \\
&\Rightarrow \exists L \in \mathrm{M}(a) \quad (L \subseteq K \wedge \mathrm{h}(L) \le \mathrm{h}(K) < +\infty) \\
&\Rightarrow \exists L \in \mathrm{M}(a) \quad L = K \quad \text{(Note 4)} \\
&\Rightarrow K \in \mathrm{M}(a)
\end{aligned}$$

2. Let $\mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(A) = 1$. For each $a \in A$, $\mathrm{Dim}^{\mathrm{T}}(a) \le$

$\mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(A)=1$, thus by (2) in Note 4 we have $\max\{\mathrm{Dim}^{\mathrm{T}}(a)\,|\,a\in A\}\leq 1$. If $\max\{\mathrm{Dim}^{\mathrm{T}}(a)\,|\,a\in A\}<1$, then $\mathrm{Dim}^{\mathrm{T}(-)}(A)=0$ and by (3) in Note 4 we have $\mathrm{Dim}^{\mathrm{T}(\overline{\mathrm{M}})}(A)=0$ which is a contraction, so $\max\{\mathrm{Dim}^{\mathrm{T}}(a)\,|\,a\in A\}=1$. By (1), $\overline{\mathrm{M}}(A)\subseteq\bigcap\{M(a)\,|\,a\in A\,,\ \mathrm{Dim}^{\mathrm{T}}(a)=1\}$, more-over if $K\in\bigcap\{M(a)\,|\,a\in A\,,\ \mathrm{Dim}^{\mathrm{T}}(a)=1\}$, then for each $a\in A$ such that $\mathrm{Dim}^{\mathrm{T}}(a)=1$, $aK=a\mathrm{E}(X)$ and for each $b\in A$ such that $\mathrm{Dim}^{\mathrm{T}}(b)=0$, $bK=b\mathrm{E}(X)$ (since b is almost periodic by (3) in Note 4), so there exists $L\in\overline{\mathrm{M}}(A)(\subseteq\bigcap\{M(a)\,|\,a\in A\,,\mathrm{Dim}^{\mathrm{T}}(a)=1\})$ such that $L\subseteq K$ [5, Corollary 3]. Suppose $a_0\in A$ be such that $\mathrm{Dim}^{\mathrm{T}}(a)=1$, by $L\subseteq K$ and $K,L\in\mathrm{M}(a_0)$ we have $K=L$ and $K\in\overline{\mathrm{M}}(A)$, so $\overline{\mathrm{M}}(A)=\bigcap\{M(a)\,|\,a\in A\,,\ \mathrm{Dim}^{\mathrm{T}}(a)=1\}$.

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