SOLUTION OF AN INVERSE PARABOLIC PROBLEM WITH UNKNOWN SOURCE-FUNCTION AND NONCONSTANT DIFFUSIVITY VIA THE INTEGRAL EQUATION METHODS

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Abstract

In this paper, a nonlinear inverse problem of parabolic type, is considered. By reducing this inverse problem to a system of Volterra integral equations the existence, uniqueness, and stability of the solution will be shown.

1. Introduction

The problem of determining unknown parameters in a parabolic partial differential equation has been treated previously by some authors [2,4]. In some applications the boundary conditions are not all linear [3,6-9], and the diffusivity is not constant [1]. Moreover, it may happen that the source-function is also unknown and one has to determine it from some suitable overspecified boundary conditions [5].

A very powerful method for solving partial differential equations is reducing them to a system of integral equations [1].

In this paper, we consider the following inverse problem:

Suppose T is a given positive constant, $D_T = \{(x, t)|0 < x < 1, 0 < t < T\}$, a is a given positive continuous function of time defined on the interval $[0,T]$ with a positive minimum A, g and h are given piecewise-continuous functions, s is a given continuously differentiable function, and l is a given two times continuously differentiable function, each defined on appropriate domains. We are seeking the functions u, f, G and H such that satisfy the following conditions:

$$u_t(x,t) = a(t)(u_{xx}(x,t) + f(t)), \quad 0 < x < 1, \quad 0 < t < T \tag{1.1}$$

$$u(x,0) = l(x), \qquad 0 \le x \le 1 \tag{1.2}$$

$$u(0,t) = G(u_x(0,t)), \qquad 0 \le t \le T \tag{1.3}$$

$$u(1,t) = H(u_x(1,t)), \qquad 0 \le t \le T \tag{1.4}$$

and the overspecified conditions:

$$u_x(0,t) = g(t), \qquad 0 \le t \le T \tag{1.5}$$

$$u_x(1,t) = h(t), \qquad 0 \le t \le T \tag{1.6}$$

$$u\left(\frac{1}{2},t\right) = s(t), \qquad 0 \le t \le T \tag{1.7}$$

It is clear that for any given functions f, a, g, h, s, G and H there may be no function $u(x,t)$ satisfying all of the conditions (1.1)-(1.7). However, for any given piecewise-continuous functions f, g and h the problem

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(1.1), (1.2), (1.5) and (1.6) has a unique solution $u(x,t)$.

Now we give the following definition:

If for some choice of the functions *f*, *G* and *H* the solution $u(x,t)$ to the problem (1.1), (1.2), (1.5) and (1.6) also satisfies (1.3), (1.4) and (1.7), then we call the quadruple (*u*, *f*, *G*, *H*) a solution to the inverse problem (1.1)-(1.7).

A physical interpretation for such a problem arises from the one dimensional conduction of heat in a homogeneous bar of unit length, one end located at the origin, when the diffusivity is not necessarily constant, and the source-function is an unknown function of time. For given functions *l*, *g*, *h*, and *s* there exist suitable source-function *f* and suitable radiation terms *G* and *H* such that the overspecified problem (1.1)-(1.7) is satisfied.

In section 2 we construct a system of Volterra integral equations equivalent to the direct problem and prove the existence and uniqueness of the solution to this system, then we prove the existence of solution to the original problem. The uniqueness of the solution will be considered in section 3. In section 4 we prove the stability of the solution.

2. Existence

In this section, we consider the problem of determining a solution (*u*, *f*, *G*, *H*) of problem (1.1)-(1.7). For this purpose, we introduce the following transformation

$$\tau = z(t) = \int_0^t a(y)dy\,. \tag{2.1}$$

Since

$$z'(t) = a(t) > 0 \tag{2.2}$$

z is an increasing function of *t*, and has an inverse *w*. By inverse function theorem *w* is differentiable and

$$w'(\tau) = [z'(w(\tau))]^{-1} = (a(w(\tau)))^{-1} = \frac{1}{a(t)}, \quad 0 < \tau < z(T) \tag{2.3}$$

Now, putting

$$U(x,\tau) = u(x, w(\tau)) \tag{2.4}$$

we have

$$\begin{aligned} U_\tau(x,\tau) &= u_t(x, w(\tau))(a(w(\tau)))^{-1} = \frac{1}{a(t)} u_t(x,t) \\ &= u_{xx}(x,t) + f(t) = U_{xx}(x,\tau) + f(w(\tau)). \end{aligned} \tag{2.5}$$

By putting $T_1 = z(T)$ the problem (1.1)-(1.7) in terms of $U(x,\tau)$ becomes

$$U_\tau(x,\tau) = U_{xx}(x,\tau) + f(w(\tau)), \quad 0 < x < 1, \quad 0 < \tau < T_1 \tag{2.6}$$

$$U(x,0) = l(x), \qquad 0 \le x \le 1 \tag{2.7}$$

$$U(0,\tau) = G(u_x(0,\tau)), \qquad 0 \le \tau \le T_1 \tag{2.8}$$

$$U(1,\tau) = H(U_x(1,\tau)), \qquad 0 \le \tau \le T_1 \tag{2.9}$$

and the overspecified conditions

$$U_x(0,\tau) = g(w(\tau)), \qquad 0 \le \tau \le T_1 \tag{2.10}$$

$$U_x(1,\tau) = h(w(\tau)), \qquad 0 \le \tau \le T_1 \tag{2.11}$$

$$U\left(\frac{1}{2}, \tau\right) = s(\tau), \qquad 0 \le \tau \le T_1\,. \tag{2.12}$$

Now putting

$$F(\tau) = f(w(\tau)), \qquad 0 \le \tau \le T_1 \tag{2.13}$$

and

$$U(x,\tau) = v(x,\tau) + \int_0^\tau F(\eta)d\eta, \; 0 \le x \le 1, \; 0 \le \tau \le T_1 \tag{2.14}$$

in terms of *v* we have

$$v_\tau(x,\tau) = v_{xx}(x,\tau), \qquad 0 < x < 1, \quad 0 < \tau < T_1 \tag{2.15}$$

$$v(x,0) = l(x), \qquad 0 \le x \le 1 \tag{2.16}$$

$$v(0,\tau) = G(v_x(0,\tau)), \qquad 0 \le \tau \le T_1 \tag{2.17}$$

$$v(1,\tau) = H(v_x(1,\tau)), \qquad 0 \le \tau \le T_1 \tag{2.18}$$

$$v_x(0,\tau) = g(w(\tau)), \qquad 0 \le \tau \le T_1 \tag{2.19}$$

$$v_x(1,\tau) = h(w(\tau)), \qquad 0 \le \tau \le T_1 \tag{2.20}$$

$$v\left(\frac{1}{2}, \tau\right) = s(w(\tau)), \qquad 0 \le \tau \le T_1 \tag{2.21}$$

so *v* specially satisfies

$$v_\tau(x,\tau) = v_{xx}(x,\tau), \qquad 0 < x < 1, \quad 0 < \tau < T_1 \tag{2.22}$$

$$v(x,0) = l(x), \qquad 0 \le x \le 1 \tag{2.23}$$

$$v_x(0,\tau) = g(w(\tau)), \qquad 0 \le \tau \le T_1 \tag{2.24}$$

$$v_x(1,\tau) = h(w(\tau)), \qquad 0 \le \tau \le T_1 \tag{2.25}$$

Now we extend the function *l* to a bounded and continuous function with the same name which has compact support, and put

$$p(x,\tau) = \int_{-\infty}^{\infty} k(x-\mu, \tau) l(\mu) d\mu \tag{2.26}$$

where $k(x,t)=\frac{1}{\sqrt{4\pi t}}\exp\left\{\frac{-x^2}{4t}\right\}$ is the fundamental solution of the heat equation. Since $k_x(-x,t)=-k_x(x,t)$, the problem (2.22)-(2.25) has a solution of the form

$$v(x,\tau)=p(x,\tau)-2\int_0^\tau k(x,\tau-\eta)\phi(\eta)d\eta+2 \\ +\int_0^\tau k(x-1,\tau-\eta)\psi(\eta)d\eta \tag{2.27}$$

if and only if [1]

$$\phi(\tau)=g(w(\tau))-p_x(0,\tau)+2\int_0^\tau k_x(1,\tau-\eta)\psi(\eta)d\eta \tag{2.28}$$

and

$$\psi(\tau)=h(w(\tau))-p_x(1,\tau)+2\int_0^\tau k_x(1,\tau-\eta)\phi(\eta)d\eta. \tag{2.29}$$

This is a system of Volterra integral equations of the second kind. Putting

$$\begin{aligned}&H_1(\tau,\eta,\phi(\eta),\psi(\eta))=\\&H_2(\tau,\eta,\psi(\eta),\phi(\eta))=\\&2k_x(1,\tau-\eta)\psi(\eta)=\\&\frac{-1}{2\pi^{\frac{1}{2}}(\tau-\eta)^{\frac{3}{2}}}\exp\left\{\frac{-1}{4(\tau-\eta)}\right\}\psi(\eta)\end{aligned} \tag{2.30}$$

and noting that the relation $\exp\left(\frac{-1}{\alpha}\right)<\alpha$ is satisfied for any positive number α, we have

$$\begin{aligned}&|H_1(\tau,\eta,\phi_1(\eta),\psi_1(\eta))-H_2(\tau,\eta,\phi_2(\eta),\psi_2(\eta))|\\&\le L(\tau,\eta)\{\|\phi_1-\phi_2\|_T+\|\psi_1-\psi_2\|_T\},\end{aligned} \tag{2.31}$$

where

$$L(\tau,\eta)=\frac{2}{\pi^{\frac{1}{2}}}(\tau-\eta)^{\frac{-1}{2}}, \qquad 0<\eta<\tau \tag{2.32}$$

Now, since

$$\int_{\tau_1}^{\tau_2}L(\tau_2,\eta)d\eta=\frac{4}{\pi^{\frac{1}{2}}}(\tau_2-\tau_1)^{\frac{1}{2}}=\alpha(\tau_2-\tau_1) \tag{2.33}$$

α is an increasing function and

$$\lim_{\tau\to 0}\alpha(\tau)=0 \tag{2.34}$$

the system (2.28), (2.29) has a unique and stable solution (ϕ,ψ) [1]. Having found ϕ and ψ we obtain

$$U(x,\tau)=p(x,\tau)-2\int_0^\tau k(x,\tau-\eta)\phi(\eta)d\eta \\ +2\int_0^\tau k(x-1,\tau-\eta)\psi(\eta)d\eta+\int_0^\tau F(\eta)d\eta \tag{2.35}$$

and noting that $k(-x,\tau-\eta)=k(x,\tau-\eta)$, from (2.12) and (2.35) and (2.35) we find

$$\int_0^\tau F(\eta)d\eta=s(w(\tau))-p\left(\frac{1}{2},\tau\right) \\ +2\int_0^\tau k\left(\frac{1}{2},\tau-\eta\right)\{\phi(\eta)-\psi(\eta)\}d\eta \tag{2.36}$$

and since

$$\frac{\partial}{\partial\tau}\int_0^\tau k(x,\tau-\eta)z(\eta)d\eta=-\int_0^\tau k_\eta(x,\tau-\eta)z(\eta)d\eta \tag{2.37}$$

differentiating (2.36) with respect to τ, using (2.26) and integrating by parts we find

$$\begin{aligned}F(\tau)&=s'(w(\tau))w'(\tau)\\&-\int_{-\infty}^{\infty}k\left(\frac{1}{2}-\mu,\tau\right)l''(\mu)d\mu\\&-2\int_0^\tau k_\eta\left(\frac{1}{2}-\mu,\tau\right)\{\phi(\eta)-\psi(\eta)\}d\eta\end{aligned} \tag{2.38}$$

and so

$$\begin{aligned}f(t)&=\frac{1}{a(t)}s'(t)\\&-\int_{-\infty}^{\infty}k\left(\frac{1}{2}-\mu,z(t)\right)l''(\mu)d\mu\\&-\int_0^{z(t)}k_\eta\left(\frac{1}{2},z(t)-\eta\right)\{\phi(\eta)-\psi(\eta)\}d\eta.\end{aligned} \tag{2.39}$$

Moreover from (2.36) we find

$$\begin{aligned}U(x,\tau)&=2\int_0^\tau\left\{k(x-1,\tau-\eta)-k\left(\frac{1}{2},\tau-\eta\right)\right\}\psi(\eta)d\eta\\&-p\left(\frac{1}{2},\tau\right)+s(w(\tau))\\&-2\int_0^\tau\left\{k(x,\tau-\eta)-k\left(\frac{1}{2},\tau-\eta\right)\right\}\phi(\eta)d\eta\\&+p(x,\tau)\end{aligned} \tag{2.40}$$

thus

$$\begin{aligned}u(x,t)&=2\int_0^{z(t)}\left\{k(x-1,z(t)-\eta)-k\left(\frac{1}{2},z(t)-\eta\right)\right\}\psi(\eta)d\eta\\&-p\left(\frac{1}{2},z(t)\right)\\&-2\int_0^{z(t)}\left\{k(x,z(t)-\eta)-k\left(\frac{1}{2},z(t)-\eta\right)\right\}\phi(\eta)d\eta\\&+p(x,z(t))+s(t)\end{aligned} \tag{2.41}$$

finally from (1.5), (1.6) and (2.41) we obtain

$$G(g(t))=2\int_0^{z(t)}\left\{k(1,z(t)-\eta)-k\left(\frac{1}{2},z(t)-\eta\right)\right\}\psi(\eta)d\eta$$
$$-p\left(\frac{1}{2},z(t)\right)$$
$$-2\int_0^{z(t)}\left\{k(0,z(t)-\eta)-k\left(\frac{1}{2},z(t)-\eta\right)\right\}\phi(\eta)d\eta$$
$$+p(0,z(t))+s(t) \tag{2.42}$$

$$H(h(t))=2\int_0^{z(t)}\left\{k(0,z(t)-\eta)-k\left(\frac{1}{2},z(t)-\eta\right)\right\}\psi(\eta)d\eta$$
$$-p\left(\frac{1}{2},z(t)\right)$$
$$-2\int_0^{z(t)}\left\{k(1,z(t)-\eta)-k\left(\frac{1}{2},z(t)-\eta\right)\right\}\phi(\eta)d\eta$$
$$+p(1,z(t))+s(t). \tag{2.43}$$

If we assume that both of functions g and h are bounded and invertible on $0<t<T$, then putting $m=g(t)$ and $n=h(t)$, equations (2.42) and (2.43) yield, respectively

$$G(m)=2\int_0^{z(g^{-1}(m))}\{k(1,z(g^{-1}(m))-\eta)$$
$$-k(1/2,z(g^{-1}(m))-\eta)\}\psi(\eta)d\eta$$
$$-2\int_0^{z(g^{-1}(m))}\{k(0,z(g^{-1}(m))-\eta)$$
$$-k(1/2,z(g^{-1}(m))-\eta)\}\phi(\eta)d\eta$$
$$+s(g^{-1}(m))+p(0,z(g^{-1}(m)))-p(1/2,z(g^{-1}(m))) \tag{2.44}$$

and

$$H(n)=2\int_0^{z(h^{-1}(n))}\{k(0,z(h^{-1}(n))-\eta)$$
$$-k(1/2,z(h^{-1}(n))-\eta)\}\psi(\eta)d\eta$$
$$-2\int_0^{z(h^{-1}(n))}\{k(0,z(h^{-1}(n))-\eta)$$
$$-k(1/2,z(h^{-1}(n))-\eta)\}\phi(\eta)d\eta$$
$$+s(h^{-1}(n))+p(1,z(h^{-1}(n)))-p(1/2,z(h^{-1}(n))) \tag{2.45}$$

The above discussion shows the existence of the solution to problem (1.1)-(1.7) under the assumption that g and h are invertible.

3. Uniqueness

To prove the uniqueness, we consider again, the problem (1.1)-(1.7), and suppose this problem has two sets of solutions (u_1, f_1, G_1, H_1) and (u_2, f_2, G_2, H_2). Then the equivalence of the problems (2.22)-(2.25) and the system of integral equations (2.28), (2.29), and the uniqueness of the solution to this system of integral equations proves that $u_1-u_2=0$ and $f_1-f_2=0$ [1]. So we must only prove the uniqueness of G and H. For any $0\le t\le T$, we have

$$G_1(g(t))-G_2(g(t))=G_1\left(\frac{\partial u_1}{\partial x}(0,t)\right)-G_2\left(\frac{\partial u_2}{\partial x}(0,t)\right)$$
$$-u_1(0,t)-u_2(0,t)=0 \tag{3.1}$$

and

$$H_1(h(t))-H_2(h(t))=H_1\left(\frac{\partial u_2}{\partial x}(1,t)\right)-H_2\left(\frac{\partial u_2}{\partial x}(1,t)\right)$$
$$-u_1(1,t)-u_2(1,t)=0 \tag{3.2}$$

So G_1 and G_2 are equal on the range of g. Similarly H_1 and H_2 are equal on the range of h, and the uniqueness of the solution to (1.1)-(1.7) is proved.

4. Stability

In this section we prove the stability of the solution to (1.1)-(1.7). Suppose that (u_1, f_1, G_1, H_1) and (u_2, f_2, G_2, H_2) are solutions to (1.1)-(1.7) for the given data (l_1, g_1, h_1, s_1) and (l_2, g_2, h_2, s_2), respectively. Then (2.41) yields

$$|u_1(x,t)-u_2(x,t)|\le\frac{4}{\sqrt{\pi}}(z(T))^{\frac{1}{2}}$$
$$\left\{\|\phi_1-\phi_2\|_{z(T)}^{(0)}+\|\psi_1-\psi_2\|_{z(T)}^{(0)}\right\} \tag{4.1}$$
$$+2\|l_1-l_2\|_\infty^{(0)}+\|s_1-s_2\|_T^{(0)}$$

Similarly we have

$$|G_1(g_1(t))-G_2(g_2(t))|\le\frac{4}{\sqrt{\pi}}(z(T))^{\frac{1}{2}}$$
$$\left\{\|\phi_1-\phi_2\|_{z(T)}^{(0)}+\|\psi_1-\psi_2\|_{z(T)}^{(0)}\right\} \tag{4.2}$$
$$+2\|l_1-l_2\|_\infty^{(0)}+\|s_1-s_2\|_T^{(0)}$$

and also

$$|H_1(h_1(t))-H_2(h_2(t))|\le\frac{4}{\sqrt{\pi}}(z(T))^{\frac{1}{2}}$$
$$\left\{\|\phi_1-\phi_2\|_{z(T)}^{(0)}+\|\psi_1-\psi_2\|_{z(T)}^{(0)}\right\} \tag{4.3}$$
$$+2\|l_1-l_2\|_\infty^{(0)}+\|s_1-s_2\|_T^{(0)}$$

where for any bounded function h belonging to $C^{(n)}[0,\alpha]$ we define the functional norm

$$\|h\|_\alpha^{(n)}=\sum_{k=0}^{n}\sup_{0\le z\le\alpha}\left|h^{(k)}(z)\right| \tag{4.4}$$

and also for any bounded C^n function l defined on $(-\infty,\infty)$ we define the functional norm

$$\|l\|_\infty^{(n)} = \sum_{k=0}^{n} \sup_{-\infty<z<\infty} \left|l^{(k)}(z)\right| . \tag{4.5}$$

So from the stability of the solution to (2.28) and (2.29) we deduce the stability of the solutions for u, G, H.

Finally we note that for any nonzero x and any positive t the relation

$$|k_t(x,t)| < \frac{70}{|x|^3} \tag{4.6}$$

is satisfied [9]. Specifically for $x = \frac{1}{2}$ we obtain

$$\left|k_t\left(\frac{1}{2},t\right)\right| < 560 \tag{4.7}$$

So if a(t) has a positive minimum A on $0 \le t \le T$, from (2.39) we have

$$\begin{aligned} |f_1(t) - f_2(t)| &\le 1120 z(T) \\ &\left\{\|\phi_1 - \phi_2\|_{z(T)}^{(0)} + \|\psi_1 - \psi_2\|_{z(T)}^{(0)}\right\} \\ &+ \frac{1}{A}\|s_1 - s_2\|_T^{(1)} + \|l_1 - l_2\|_\infty^{(2)} \end{aligned} \tag{4.8}$$

and the stability of the solution to (1.1)-(1.7) is proved.

We summarise the above results in the following theorem:

Suppose l is a two times continuously differentiable function which is defined on $0 \le x \le 1$, a, s, g and h are given functions on $0 \le t \le T$, g and h are piecewise-continuous, s is continuously differentiable, and a is a continuous function which has a positive minimum A. Then there exist unique functions u defined on D_T, f defined on $0 \le t \le T$ and unique functions G and H, where the domains of G and H include the ranges of g and h, respectively, and satisfy the problem (1.1)-(1.7). Moreover this solution is stable.

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