RESIDUAL SOLUBILITY OF SF-GROUPS

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Abstract

Seifert Fibre Groups (SF-Gps) have been introduced and their first derived groups have been worked out in an earlier paper by the author [2,3]. Now we aim to prove that they are residually soluble and residually finite.

1. Introduction

Our intention in earlier sections is to show that every SF-group is residually soluble. For this purpose we need some definitions as well as few lemmas which we will bring in §1 and in early parts of §2, prior to the main theorem (2.4). Residually finiteness will be worked out in §3. The use has been also taken of the following theorem, proved by Chin-Han Sah, 1969, [5]:

Theorem 1.1. *Let G be a group with signature:*

$$\{e(1), e(2), \dots, e(m); \gamma\}, \dots\dots\dots\dots\dots\dots \otimes$$

then,

1. G is residually finite;

2. G is perfect if and only if the following conditions hold:

- *i)* $\gamma = 0$,
- *ii) all 'e(i)'s,* $1 \le i \le m$, *are finite, and*
- *iii) the 'e(i)'s are pairwise co-prime;*

3. if G is not perfect then it is residually finite and soluble.

Definition 1.2. If G has the signature $\otimes$, trivial centre, and all 'e(i)'s finite for $1 \le i \le m$, m finite, i.e. there are no elements with e(i) equal to ∞, then the number:

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$$X(G) = 2 - 2\gamma - \sum_{i=1}^{m}\left(1 - \left(\frac{1}{e(i)}\right)\right)$$

is called the *curvature*[1] of G.

Groups of positive or zero curvature are very limited. For a complete list of such groups [5], App. 4, §§A7&A8. In the case of m=3, J. Milnor [4] gives the detail. We shall also discuss the groups with negative curvature which are all Fuchsian groups.

Definition 1.3. Let G be a SF-group with centre $<\zeta>$. If the quotient group $G/<\zeta>$ has negative curvature, then we call it the *Fuchsian Projection* of G. (And say that the SF-signature of G is of *Fuchsian Type*). Moreover, a subgroup S of finite index of a SF-group G, is called:

> a *broad* subgroup of G if the natural homomorphism maps it onto the Fuchsian Projection of G; and a *deep* subgroup of G if it contains the centre of G.

Let Γ denote the Fuchsian projection of a SF-group H with signature:

$$\{(m_1, q_1), (m_2, q_2), \dots, (m_r, q_r); q_0; g\} \dots\dots\dots\dots\dots \otimes$$

(Every SF-group has a signature of this type [3].

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[1] The curvature is defined for all groups with signature and in the case of groups discussed in our work, is equal to the Euler characteristic of the group.

We next let the prime mark ‘ ′ ’ on the top right-hand corner of any group-letter, represent its first derived group as usual.

Lemma 1.4. *If H is perfect and $\phi : H \to \Gamma$ is the natural surjective homomorphism from H onto Γ, then the image of H' under ϕ is just Γ'.*

Proof. H is perfect then by definition $H = H'$. So, from the presentation of H' [3] we notice that every element $h \in H$ can be written as:

$$h = \prod [\alpha_j, \beta_j] \quad for \quad 1 \le j \le k \quad and \quad \alpha_j, \beta_j \in H.$$

Since Γ is the image of the surjective homomorphism ϕ, then for every element $\gamma \in \Gamma$, $\exists h \in H$ s.t. $\phi(h) = \gamma$; whence we get $\gamma = \prod [\phi(\alpha_j), \phi(\beta_j)]$ which is an element of Γ'. Hence $\phi(H) \subset \Gamma'$. But $\phi(H) = \Gamma$ and H is perfect, then $\phi(H) = \phi(H')$, i.e. $\phi(H') \subset \Gamma'$.

On the other hand Γ' as the derived group of Γ, is obviously its subgroup, the two ways inclusions of Γ and Γ' imply that $k\phi(H')$ is equal to Γ'.

2. On Residual Solubility of SF-Groups

Definition 2.1. A group H is called *residually soluble* if for every non-identity element h in H there exist s soluble group S and a homomorphism ϕ from H onto S such that the image of h under this homomorphism is different from the identity in S.

One may notice that the definition 2.1. in the case of quotient groups is equivalent to say that:

H is residually soluble if and only if,

- a) for every non-identity element h there exists a normal subgroup N of H such that N does not contain h and H/N is soluble; or
- b) its derived series intersects in the identity only, because H/H' for any group H, is abelian.

Lemma 2.2. *Let H be a SF-group with presentation:*

$$H = <\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g; \zeta : \prod_{i=1}^{g} [\alpha_i, \beta_i] =$$
$$\zeta^l, \& \zeta \Leftrightarrow \alpha_i, \zeta \Leftrightarrow \beta_i >$$

where ζ is a central element, l an integer, and $\Leftrightarrow$ implies the commutement of two sides. If $p{:}H \to \Gamma$ is the natural projection, we assume that $p(H) = \Gamma$ is torsion-free. Then H is residually soluble.

Proof. Take any element t in H other than the identity. There are two possibilities according to the element t being central or not.

- 1) t is *not* central, i.e. $p(t) \neq 1_\Gamma$. Since Γ is residually soluble, by Sah's theorem (1.1), then there is a homomorphism ϕ from Γ onto a soluble group G such that $\phi(p(t)) \neq 1_G$. Thus we get a homomorphism ψ from H onto G by taking $\psi = \phi \circ p$ which does not kill t.
- 2) t *is* central, i.e. $p(t) = 1_\Gamma$. Then there exists a positive integer $k_t s.t.\ t = \zeta \pm kt$ where ζ is a generating element of the centre of H. Suppose q is a large number greater than kt. Then an epimorphism

$$\phi : \Gamma \to \mathbb{Z}q$$

is given by sending one element a_1, say, to 1 (mod q) and the rest to zero. Denote again $\phi \circ p$ by ψ and $ker(\psi)$ by H_1. Since H/H_1 is abelian then $H_1 \supset H'$. This is a deep subgroup of H of index q with Euler number equal to (q) times that of H, assuming that, of course, $e(H)$ is not zero ([1], Lemma 7.2, p. 24). As we mentioned before, since the factor group of a group with a broad subgroup is cyclic, then if we denote the generators of H_1 other than the powers of central element ζ, by $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$, then the factor group of H_1 which is obtained by killing all α_i, β_i, $1 \le i \le g$, is a cyclic group of order divisible by q. Hence $t = \zeta \pm kt \notin H_1'$, so $t \notin H''$. Thus the natural map:

$$p' : H \to H / H''$$

is a homomorphism of H onto a soluble group which does not kill t. If $e(H)$ was zero, then a homomorphism from H onto $\mathbb{Z}q$ could be given by taking ζ onto the generator of $\mathbb{Z}q$ and the rest of the elements to zero.

Proposition 2.3. *Let G be a group and N a normal subgroup of G with G/N soluble. Then G is residually soluble if and only if N is residually soluble.*

Proof. Solubility of G/N implies that there is a positive integer k such that k–*th* derived group of G/N is just identity element only, i.e.

$$(G / N)^{(k)} = 1$$

and this implies that

$$G^{(k)} \subset N.$$

Then for any positive integer l we get

$$G^{(k+1)} \subset N^{(l)}$$

N being residually soluble implies that

$$\bigcap_{l=1}^{\infty} N^{(l)} = \{1\},$$

then

$$\bigcap_{l=1}^{\infty} G^{(k+1)} = \{1\},$$

because

$$\bigcap_{l=1}^{\infty} G^{(k+1)} \subset \bigcap_{l=1}^{\infty} N^{(l)} = \{1\}.$$

Hence G is residually soluble.

The other way is obvious, as G being residually soluble means that the sequence of its derived subgroups intersect in the identity, hence that of derived subgroups of N intersect in the identity, thus N is residually soluble.

Theorem 2.4. *Let H be the SF-group with signature* $\otimes$ *and* Γ *its Fuchsian projection. Then*

- *a)* Γ *is perfect if H is perfect;*
- *b) H′ is perfect if H is not perfect but* Γ *is perfect; moreover, the Fuchsian projection of H′ is also* Γ, *and H/ H′ is cyclic.*
- *c) H is residually soluble if* Γ *is residually soluble.*

Proofs. a) (This is fulfilled by the proof of Lemma 1.4). b) Γ perfect, then Sah's theorem (1.1) ensures that g=0 and all $m_i, i = 1,2,\ldots r$ are relatively prime. Hence H has a presentation as follows

We abelianize H, and get H/H' isomorphic to the additive group

$$G = < g_1, g_2, \ldots, g_r; \delta : m_i g_i \\ = 1,2,\ldots, r \;\&\; \sum_{i=1}^{r} g_i = q_0 \delta >,$$

the order of which is the determinant of the following matrix

$$M = \begin{pmatrix} m_1 & 0 & 0 & \cdots & 0 & q_1 \\ 0 & m_2 & 0 & \cdots & 0 & q_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & m_r & q_r \\ 1 & 1 & 1 & \cdots & 1 & q_0 \end{pmatrix}$$

which is equal to: $m_1 m_2 \ldots m_r (q_0 - \sum_{i=1}^{r} \frac{q_i}{m_i})$, and gives k, the index of H' in H. Since the natural projection maps H' onto the Fuchsian projection of H then H' is a broad subgroup of H (by Definition 1.3), and we have the following equality:

$$e(H) = k\, e(H').$$

But as $e(H)$ is defined by:

$$e(H) = q_0 - \sum_{i=1}^{r} \frac{q_i}{m_i}$$

then we have:

$$e(H') = 1/m_1 m_2 \ldots m_r.$$

In a similar way to that argument of the above lines, A. M. Macbeath has shown[2] that if $e(H)$ is not zero then the order of the torsion subgroup H/H' is equal to

$$m_1 m_2 \ldots m_r e(H) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \odot.$$

Then the index of H'' in H', which is the same as the order of H'/H'', must be $m_1 m_2 \ldots m_r e(H')$, and by the value of $e(H')$, is equal to 1. Hence H' is perfect.

It is very easy and straight forward to see that H' has Γ as its Fuchsian projection. And one can see easily that if a group H_1 has been taken as a broad subgroup of H of index k, then H/H_1 is cyclic. In conjunction with that we have H/H' cyclic and its order the same as $\odot$.

c) Γ is not perfect then there are two possibilities:

Either (c.1) *its genus is zero and at least two of its periods are not co-prime;*

Or (c.2) *its genus is not zero.*

In the case of (c.1) let q be the prime common factor of the non-co-prime periods m_1 and m_2, say. Then Γ has the signature

$$(qr_1, qr_2, m_3, \ldots, m_r; 0)$$

where r_1 and r_2 are the numbers m_1/q and mz/q respectively. Define a homomorphism:

$$\phi_1 : \Gamma \to \mathbb{Z}q$$

by putting,

$$\phi_1(x_1) = 1, \;\; \phi_1(x_2) = -1, \text{ and } \phi_1(x_i) = 0 \text{ for } 3 \le i \le r.$$

Let Γ_1 denote the $\ker(\phi_1)$, then by using Singerman's theorem and the Riemann-Hurwitz formula ([6] & [3]) we get the following signature for Γ_1:

$$(r_1, r_2, m_3^{[q]}, m_4^{[q]}, \ldots, m_r^{[q]}; 0)$$

where $m_i^{[q]}$ means q periods m_i.

Next we define a homomorphism ϕ_2 from Γ_1 into the

[2] A. M. Macbeath, The Fundamental Groups of the 3-Dimensional Briskorn Manifolds, University of Birmingham, (1976), U.K., Unpublished.

direct sum of cyclic groups as follows:

Let

$$y_1, y_2, y_{ij}\begin{cases} i = 3,4,\ldots,r, \\ j = 1,2,\ldots,q, \end{cases}$$

be the generators of Γ_1, then define

$$\phi_2 :\to \Gamma_1 \to \mathbb{Z}_{m_3}^{q-1} \oplus \mathbb{Z}_{m_4}^{q-1} \oplus \ldots \oplus \mathbb{Z}_{m_r}^{q-1}$$

by putting

$$\phi_2(y_1) = \phi_2(y_2) = 0 \text{ and } \phi_2(y_{ij})\begin{cases} i = 3,4,\ldots,r, \\ j = 1,2,\ldots,q, \end{cases}$$

as shown in Table 1.

Table 1. Defining the homomorphism in section c of Theorem 2.4

ϕ_2	$\mathbb{Z}_{m_3}^{q-1}$	$\mathbb{Z}_{m_4}^{q-1}$	$\ldots$	$\mathbb{Z}_{m_r}^{q-1}$
y_{31}	(1,0,…,0),	(0,0,…,0),	⋯,	(0,0,…,0)
y_{41}	(0,0,…,0),	(1,0,…,0),	⋯,	(0,0,…,0)
⋮	⋮	⋮	⋮	⋮
y_{r1}	(0,0,…,0),	(0,0,…,0),	⋯,	(1,0,…,0)
y_{32}	(0,1,…,0),	(0,0,…,0),	⋯,	(0,0,…,0)
y_{42}	(0,0,…,0),	(0,1,…,0),	⋯,	(0,0,…,0)
⋮	⋮	⋮	⋮	⋮
y_{r2}	(0,0,…,0),	(0,0,…,0),	⋯,	(0,1,…,0)
⋮⋮	⋮	⋮	⋮	⋮
$y_{3(q-1)}$	(0,0,…,1)	(0,0,…,0),	⋯,	(0,0,…,0)
$y_{4(q-1)}$	(0,0,…,0),	(0,0,…,1)	⋯,	(0,0,…,0)
⋮	⋮	⋮	⋮	⋮
$y_{r(q-1)}$	(0,0,…,0),	(0,0,…,0),	⋯,	(0,0,…,1)
y_{3q}	(-1,-1,…,-1),	(0,0,…,0),	⋯,	(0,0,…,0)
y_{4q}	(0,0,…,0),	(-1,-1,…,-1),	⋯,	(0,0,…,0)
⋮	⋮	⋮	⋮	⋮
y_{rq}	(0,0,…,0),	(0,0,…,0),	⋯,	(-1,-1,…,-1)

Since, if we denote the $\phi_2(y_{ij})$ by t_{ij}, then the relation $\sum t_{ij} = 0$ is satisfied, and every element t_{ij} has the correct order, hence this is a homomorphism from Γ_1 onto the finite group on the right-hand side.

Let Γ_2 denote the $\ker(\phi_2)$. The index of Γ_2 in Γ_1 is N, say, and equals to $(m_3 m_4 \ldots m_r)^{q-1}$. By using the Singerman's theorem and Riemann-Hurwitz formula we get the following signature for Γ_2:

$$\left(r_1^{[N]}, r_2^{[N]}; g_2\right)$$

where

$$g_2 = 1 - N\left[1 - \frac{q}{2}\sum_{i=3}^{r}\left(1 - \frac{1}{m_i}\right)\right].$$

Finally, let

$$z_{ij}\begin{cases} i = 1,2\,, \\ j = 1,2,\ldots,N \quad \textit{and} \end{cases} \quad \alpha_\nu \beta_\nu, \quad \nu = 1,2,\ldots,g_2\,,$$

be the generators of Γ_2; and let ϕ_3 be a homomorphism from Γ_2 into the group

$$\mathbb{Z}_{r_1}^{N-1} \oplus \mathbb{Z}_{r_2}^{N-1}$$

defined by:

$$\phi_3(\alpha_\nu) = \phi_3(\beta_\nu) = 0,$$
$$\phi_3(z_{1j}) = (t_{1j}, 0),$$
$$\textit{and} \qquad \phi_3(z_{2j}) = (0, t_{2j});$$

such that

$$(t_{ij}, m_i) = 1 \quad \textit{and} \quad \sum_{j=1}^{N} t_{ij} \equiv 0 (\bmod m_i).$$

If Γ_3 denotes the $\ker(\phi_3)$, then the index of Γ_3 in Γ_2 is $(r_1 r_2)^{N-1}$. Once more we must use Singerman's theorem as well as the Riemann-Hurwitz formula to get the following signature for Γ_3:

$(-\ ;g_3)$ <u>*noperiods*</u> ,

where

$$g_3 = N/2(r_1 r_2)^{N-1}(-q)[\chi(\Gamma)].$$

Thus Γ_3 is a surface group.

Now let $H_i = p^{-1}(\Gamma_i)$ for $i = 1,2,3$. H_3 is residually soluble by Lemma 2.2, and H_2/H_3 is abelian hence it is soluble, then H_2 is residually soluble by Proposition 2.3. So by the same argument H_1 and H both are residually soluble.

In the case of (c.2) we have $g \geq 1$.

Let

$$x_i, i = 1,2,\ldots,r; \text{ and } a_\nu, b_\nu, \quad \nu = 1,2,\ldots,g$$

be the generators of Γ and define a map:

$$\psi_1 : \Gamma \to \mathbb{Z}_2$$

by

$$\psi_1(x_i)=0,\quad \textit{for all } i,$$
$$\psi_1(a_1)=1,$$
$$\psi_1(a_\mu)=0,\quad \textit{for } \mu>1, \textit{ and}$$
$$\psi_1(b_\nu)=0,\quad \textit{for all } \nu.$$

Denote $\ker(\psi_1)$ by Γ_1. By using Singerman's theorem and the Riemann-Hurwitz formula (or otherwise) we get the following signature for Γ_1:

$$(m_1,m_1,m_2,m_2,\ldots,\ldots,m_r,m_r;g_1)$$

where $g_1=2g-1$.
Let

$$v_{ij}\begin{cases} i=1,2,\ldots,r & ,\\ j=1,2 & ;\end{cases} \text{ and } \quad \alpha_k,\beta_k,\quad k=1,2,\ldots,g_1$$

be the generators of Γ_1.
Define a homomorphism:

$$\psi_2:\Gamma_1\to\mathbb{Z}_{m_1}\oplus\mathbb{Z}_{m_2}\oplus\ldots\oplus\mathbb{Z}_{m_r}$$

by

$$\psi_2(v_{ij})=(-1)^j t_i,\begin{cases} i=1,2,\ldots,r & ,\\ j=1,2 & ;\end{cases} \textit{where } (t_i,m_i)=1,$$

and

$$\psi_2(\alpha_k)=\psi_2(\beta_k)=0,\quad \textit{for all } k.$$

Denote $\ker(\psi_2)$ by Γ_2. Then Γ_2 has the following signature:

$(-\ ;g_2)$ <u>*noperiods*</u>,

where, $g_2=1-m\chi(\Gamma)$. Thus Γ_2 is torsion-free.

Let H_2 be $p^{-1}(\Gamma_2)$. Then H_2 is residually soluble by Lemma 2.2.

By a similar argument to (c.1) we get the results that H_1 (the inverse image of Γ_1 under the map p), and finally H are both residually soluble.

3. Residual Finiteness of SF-Groups

Definition 3.1. A group G will be *residually finite* if its subgroups of finite index intersect just in identity, i.e. 1.

Theorem 3.2. *Every SF-group H is residually finite.*

Proof. Regarding the definition 3.1, for the SF-group H to be residually finite it suffices to prove that for any element $h\in H$ other than the identity, there exists a normal subgroup K_h of finite index in H such that h does not belong to K_h.

There are two possibilities according to the image of any non-identity element $h\in H$ under the projection map p, is different from or equal to the identity in Γ.

If $p(h)\neq 1_\Gamma$, since Γ is residually finite [5], then there exists a finite group G and a homomorphism $\phi:\Gamma\to G$ such that $\phi(p(h))\neq 1_G$ for any non-identity element h of H. Hence the composition of the maps ϕ and p, denoted by ψ, is a map from H onto G which does not kill h. Thus $\ker(\psi)=K_h$ is a subgroup of finite index in H which does not contain h.

If $p(h)=1_\Gamma$, then we get $h=\zeta^m$ for some integer m. Since every plane discontinuous group has a surface group as a subgroup of finite index [3], then Γ has a subgroup Γ_1, say, of finite index l in Γ which is a surface group. This gives $p^{-1}(\Gamma_1)$ a presentation as follows:

$$\begin{cases}\text{generators}:\alpha_1,\beta_1,\ldots,\alpha_g,\beta_g;\zeta,\\ \text{relations}:\prod_{\nu=1}^{g}[\alpha_\nu,\beta_\nu]=\zeta^l.\end{cases}$$

This group admits a homomorphism ρ onto cyclic group of order l defined by:

$$\rho(\zeta)\equiv 1(\text{mod}\, l),$$
$$\rho(\alpha_\nu)=\rho(\beta_\nu)\equiv 0(\text{mod}\, l).$$

If m is not a multiple of l, then ρ does not kill ζ^m.

If m is a multiple of l, then we proceed as follows:

Let k be an integer greater than m but not a multiple of it. Define a map τ from $p^{-1}(\Gamma_1)$ onto Z_k by:

$$\tau(\alpha_1)\equiv 1(\text{mod}\, k),$$
$$\tau(\beta_1)=\tau(\alpha_\nu)=\tau(\beta_\nu)\equiv 0(\text{mod}\, k),\nu\geq 2,$$
$$\tau(\zeta)\equiv 0(\text{mod}\, k).$$

Denote $p^{-1}(\Gamma_1)$ by H_1 and $\ker(\tau)$ by H_2. Then we get:

$H_1:H_2=k$, and

H_2 as a deep subgroup of H_1.

By Riemann-Hurwitz formula $p(H_2)$ is a surface group of genus:

$$g'=k(g-1)+1,$$

and by Bailey-Neumann theorem, the Euler number of H_2 is k times that of H_1, i.e.

$$e(H_2)=k\ e(H_1).$$

([1] for details.) Then H_2 has the presentation:

$$\begin{cases} generators: \gamma_1, \delta_1, \ldots, \gamma_{g'}, \delta_{g'}; \zeta \\ relations: \prod_{\mu=1}^{g'} [\gamma_\mu, \delta_\mu] = \zeta^{kl} \ \& \ \gamma_\mu \zeta \gamma_\mu^{-1} = \zeta, \delta_\mu \zeta \delta_\mu^{-1} = \xi. \end{cases}$$

Since m is not a multiple of kl, so H_2 has a normal subgroup of finite index which does not include ζ^m, by above argument.

References

1. Bailey, G. O. Uncharacteristically Euler. Ph.D. Thesis, University of Birmingham, UK, (1977).
2. Mamagani, A. B. Fuchsian Groups & Seifert Fiber Groups. Ph.D. Thesis, University of Birmingham, UK, (1979).
3. Mamagani, A. B. The Derived Group of a Seifert Fiber Group. *Journal of Sciences, I. R. Iran,* **6**(2), (1995).
4. Milnor, J. Knots, Groups and 3-Manifolds. *Annals of Mathematics Studies.* Princeton University Press, USA, (1977).
5. Sah Ch.-H. Groups Related to Compact Riemann Surfaces. *Acta Mathematica,* 123, UPPSALA, USA, (1969).
6. Singerman, D. Subgroups of Fuchsian Groups and Finite Permutation Groups. *Bulletin of London Math. Soc.,* **2**, UK, (1970).
7. Zieschang, H., Vogt, E. and Coldewey, H. D. Lecture Notes in Mathematics. Springer-Verlag, **122**.