A GENERALIZATION OF A JACOBSON'S COMMUTATIVITY THEOREM

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Abstract

In this paper we study the structure and the commutativity of a ring R, in which for each $x,y \in R$, there exist two integers $k \geq \circ, n > 1$ depending on x,y such that $[x,y]_k$ equals x^n or y^n.

Throughout, R will represent an associative ring with center C, and N the set of all nilpotent elements of R.

A ring R is said to be periodic if for every $x \in R$, $x^m = x^n$ for some distinct positive integers m, n. Any $x \in R$ is called a potent element if $x = x^n$ for some integer $n > 1$. For $x, y \in R$, $[x,y]_1 = [x,y] = xy - yx$ is the usual commutator, and for every positive integer $k > 1$, we define inductively $[x,y]_k = [[x,y]_{k-1}, y]$. Also we define $[x,y]_\circ = x$.

A ring R is called left (resp. right) s-unital [5] if for each $x \in R$ we have $x \in Rx$ (resp. $x \in xR$). A ring R is called s-unital if for each $x \in R$, $x \in Rx \cap xR$. If R is an s-unital ring, then for any finite subset F of R there exists an element e in R such that $ex = xe = x$ for all x in F (see [5]).

By a remarkable result of Jacobson [3], we know that if for each $x \in R$ $x = x^n$ for some integer $n > 1$, then R is commutative. Here we study the commutativity behavior of a ring R, which satisfies the following property:

(p) for each $x, y \in R$, there exist two integers $k \geq \circ$, $n > 1$ depending on x, y such that $[x,y]_k$ equals x^n or y^n.

Note that if R satisfies (p), with $k = \circ$ for all $x \in R$ (and $y = x$), then by the above mentioned Jacobson's theorem, R must be commutative. However the ring

$$R = \left\{ \begin{bmatrix} \circ & a & b \\ \circ & \circ & c \\ \circ & \circ & \circ \end{bmatrix} \middle| a,b,c \quad are \quad real \quad numbers \right\}$$

is a non commutative ring satisfying (p), in fact in this ring $[x,y]_3 = \circ = x^3$ for all $x, y \in R$.

In preparation for the proof of our main theorems, first we state and prove the following lemmas.

Lemma 1. Let R be a ring which satisfies (p), then each element of R is either a potent or a nilpotent element. In particular R is a periodic ring.

Proof. Suppose that $x \in R$, and $x \neq x^n$ for all integers $n > 1$, then $\circ = [x,x]_k = x^m$ for some $m \geq 1$ and $k \geq 1$.

Lemma 2. Let R be a ring which satisfies (p). Then every idempotent element of R is in the center of R.

Proof. Let $e^2 = e \in R$, $x \in R$, and put $a = xe - exe$. It is easy to see that for all $k \geq \circ$, $[a,e]k = a$. Therefore by (p), for some $n > 1$; $a = a^n$ or $a = e^n$. Since $a^2 = \circ$,

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and $a = e^n = e$ implies that $a = \circ$, hence in any case $a = \circ$, *i.e.*, $xe = exe$. Similarly $ex = exe$. This shows that e is a central element.

Lemma 3. Let *R* be a ring which satisfies (p), then the commutator ideal of *R* is nil.

Proof. Since *R* is a periodic ring, hence for each $x \in R$ there exists an integer $m \geq 1$ such that $x^{2m} = x^m$. Therefore, by Lemma (2) and [2] the commutator ideal of *R* is nil.

Lemma 4. If the commutator ideal of *R* is nil, then *N* is an ideal.

Proof. See [1].

Lemma 5. If *N* is a commutative ideal, then $N^2 \subseteq C$.

Proof. See [5].

Now we are ready to state and prove our fundamental theorem.

Theorem 1. Let *R* be a semiprime ring with the following property:

(i) *R* satisfies (p).

(ii) *N* is commutative (i.e. for all $a, b \in N$, $ab = ba$).

Then *R* is a commutative ring.

Proof. By the above Lemmas, *R* is a periodic ring and *N* is an ideal with $N^2 \subseteq C$. We claim that $[x, y]^2 = \circ$ for all $x \in N$, $y \in R$. Since $[x, y] \in N$, therefore

$$\begin{aligned}[x,y]^2 &= [x,y]xy - [x,y]yx \\ &= [x,y]xy - x[x,y]y \\ &= [x,y]xy - [x,y]xy \\ &= \circ.\end{aligned}$$

Now, since *N* is a commutative ideal, $[x, y]R[x, y] = \circ$ for all $x \in N$, $y \in R$. But *R* is a semiprime ring, hence $[x,y]=\circ$ for all $x \in N$, $y \in R$. This shows that $N \subseteq C$. On the other hand if $a^2 = \circ$ then $N \subseteq C$ implies that $aRa = Ra^2 = \circ$ and therefore $a = \circ$, since *R* is a semiprime ring. This shows that $N = \{\circ\}$ and thus *R* is commutative, by Lemma 3.

Theorem 2. Let *R* be a left s-unital ring which satisfies (p), and $m > 1$ is a fixed positive integer. The following conditions are equivalent:

(i) *R* is a commutative ring.

(ii) For each $a \in N$, $y \in R$, $[a, y^m] = \circ$; and for any $x, y \in R$ if $m[x, y] = \circ$, then $[x, y] = \circ$.

(iii) For each $x, y \in R$ $[x^m, y^m] = \circ$ and $m[x, y] = \circ$ implies that $[x, y] = \circ$.

(iv) For each $x, y \in R$ $(xy)^m = (yx)^m$ and $m[x, y] = \circ$ implies that $[x, y] = \circ$.

(v) For each $x, y \in R$, $(xy)^m = x^m y^m$ and $m(m-1)[x, y] = \circ$ implies that $[x, y] = \circ$.

Proof. First of all, for any $x \in R$, $x = ex$ for some $e \in R$ (*R* is left s-unital). Also, by Lemma 1 and [4, Lemma 3], $e^k = e^{2k}$ for some integer $k \geq 1$, and by Lemma 2; $e^k \in C$, hence $x = e^k x = xe^k \in xR$. Therefore *R* is an s-unital ring. Obviously (i)⇒(ii)-(v). We complete the proof as follows.

(ii)⇒(i) Let $a, b \in N$, then $b^n = \circ$ for some integer $n > 1$ and $ae = ea = a$, $be = eb = b$ for some $e \in R$. By (ii), $[a, (e + b^{n-1})^m] = \circ$ so $m[a, b^{n-1}] = \circ$, or in fact $[a, b^{n-1}] = \circ$. By repeating this argument we can see that $[a, b] = \circ$. Thus *N* is commutative, and as the Proof of Theorem 1 shows: $[a, y]^2 = \circ$, for all $a \in N$, $y \in R$.

Now, let $a \in N$, $y \notin N$. By Lemma 3, $[a, y] \in N$, hence $[[a, y], y]_k = [a, y]^n$ for some integer $k \geq \circ$, $n > 1$, by (p). Since $[a, y]^n = \circ$ (by above), hence $[a, y]_{k+1} = \circ$. First, let $k \geq 1$. Considering $a' = [a, y]_{k-1}$, we have $[[a', y], y] = \circ$. On the other hand, $a' \in N$, by Lemma 3. Therefore, $[a', y^m] = \circ$, by (ii).

Hence, by [4, Lemma 2], $my^{m-1}[a', y] = \circ$ and so $[a', y] = \circ$, by [4, Lemma 2] and (ii). That is $[a, y]_k = \circ$ inductively, $[a, y] = \circ$. Also, it is obvious that if $k = \circ$, then $[a, y] = \circ$. This shows that $N \subseteq C$. Hence for each $x \in R$; there exists an integer $n > 1$ such that $x - x^n \in C$, by Lemma 1. Therefore, by a well-known result of Herstein [2], *R* is a commutative ring.

(iii)⇒(ii). See [5, Lemma 2].

(iv)⇒(ii). Let $\circ \neq a \in N$, $\circ \neq y \in R$. By Lemma 1.2, $ae = ea = a$, $ye = ey = y$ for some central idempotent element $e \in R$. Let $z = e - a$, $z' = e + a + a^2 + \ldots$, then by (iv), $(zyz')^m = (zz'y)^m = (ey)^m = y^m$.

On the other hand $(zyz')^m = zy^m z'$, hence $zy^m z' = y^m$ and therefore $[z, y^m] = \circ$, or in fact $[a, y^m] = \circ$.

(v)⇒(ii). Let $a \in N$, $y \in R$ be two nonzero elements of *R* such that $a^n = \circ$. Using the above notations when

a is replaced by a^{n-1}, we have $zy^m z' = (zyz')^m = z^m y^m z^m$, by (v). Therefore, $[z^{m-1}, y^m] = \circ$, *i.e.*, $[(e-a^{n-1})^{m-1}, y^m] = \circ$, or in fact $[a^{n-1}, y^m] = \circ$, by (v). By repeating this argument, we can easily see that $[a, y^m] = \circ$.

Theorem 3. Let R be a ring which satisfies (p) and N a commutative subset of R. Then R is a subdirect product of commutative nil rings and local rings.

Proof. By a well-known result of Birkhoff, R is a subdirect product of subdirectly irreducible rings $R_i, i \in I$. Obviously, each R_i satisfies (p). Thus $R_i = N_i$ is a nil ring, or R_i contains a nonzero potent element. On the other hand, if $\circ \neq a_i \in R_i$ is not a nilpotent element, then by Lemma 1, $a_i = a_i^n$ for some integer $n > 1$. Clearly, $e = a_i^{n-1}$ is an idempotent element which lies in the center of R_i, by Lemma 2. Since R_i is a subdirectly irreducible ring, $e = 1$ must be the identity of R_i, and a_i is a unit of R_i. Since by Lemmas 3,4, N_i is an ideal of R_i, hence each R_i is a nil or a local ring.

To complete the proof it suffices to observe that if $f: R \to R^*$ is a ring epimorphism, then $f(N)$ concides with N^* the set of all nilpotent elements, of R^*. Let $f(a) = a^* \in N^*$ be a nonzero element of R^*, then by Lemma 1, $a \in N$ (otherwise, there exists an integer $n > 1$ such that $a = a^n = a^{n^2} = \cdots$, *i.e.*, $f(a) = a^* = (a^*)^n = (a^*)^{n^2} = \cdots = \circ$. This completes the proof.

Corollary 3.1. Let R be non-nil subdirectly irreducible ring which satisfies (p). Then $\operatorname{char} R = p^n$, for some prime p.

Proof. By Theorem 3, R is a local ring with 1. Consider 2=1+1. By Lemma 1, either $2^n = \circ$, or $2^n = 2$ for some integer $n > 1$. If $2^n = \circ$, then char$R$ is a power of 2. Therefore, we may assume that $2^n - 2 = \circ$, or in fact has $\operatorname{char} R = m > \circ$. But as a division ring with a positive characteristic, $\operatorname{char} R/N = p$, where p is a prime number. Therefore, in $R, p^n = \circ$ for some integer $n > 1$. This completes the proof.

Theorem 4. Let R be a ring of characteristic zero, which satisfies (p). Then $R = \oplus_{e \in E} Re + N$, where E is the set of all idempotent elements of R, and each Re is a local ring.

Proof. Let $x \in R \setminus N$, then by Lemma 1, $x = x^n$ for some integer $n > 1$. Clearly $x^{n-1} = e$ is an idempotent element, and $x \in Re$. Now, suppose that $e, e' \in E$. Since according to Lemma 2, $E \subseteq C$, we can easily see that

$$(*) \qquad (e+e')^n = e + e' + (2^n - 2)ee'$$

for all integers $n \geq 1$. By Lemma 1, either $(e+e')^n = \circ$ or $(e+e')^n = e+e'$ for some integer $n \geq 1$. If $(e+e')^n = \circ$ then by $(*)$, $e + e' = (2 - 2^n)ee'$. Multiplying the last equation by $e - e'$, yields $e = e'$.

If $(e+e')^n = e+e'$, then in view of $(*)$ we have $(2^n - 2)ee' = \circ$. But $\operatorname{char} R = \circ$, hence $ee' = \circ$. These observations show that if $e \neq e'$, then $Re \cap Re' = \circ$. So far, we have seen that $R = \oplus_{e \in E} Re + N$. To complete the proof we need to show that for each $e \in E$, Re is a local ring. Obviously, $Ne = Re \cap N$ is an ideal of Re (by Lemma 3, 4). Let $x \in Re$, $x \notin N$ then by Lemma 1, $x^n = x$ for some integer $n > 1$. Clearly $x^{n-1} = e'$ is a central idempotent, by Lemma 2, and it is easy to see that $e = e'$ (otherwise $e'e = \circ$). Therefore x is a unit of Re and the proof is completed.

In Theorem 4, if $N = \circ$ then by Lemma 1, R satisfies the Jacobson's condition, i.e. for any $x \in R$, $x = x^{n(x)}$, for some integer $n(x) > 1$. Therefore, in view of Theorem 4, we have:

Corollary 4.1. If for each $x \in R$, there exists an integer $n = n(x) > 1$ and $\operatorname{char} R = \circ$, such that $x = x^n$, then R is a direct sum of fields.

Note that $charR > \circ$, if R is a ring with 1, which satisfies the Jacobson's conditions.

Remarks. Each one of the conditions in Theorem 1 is essential, because:

Example 1. Let $R = Z_2[x, y]$ with $xy = yx + 1$. Clearly R is a semiprime ring and $N = \{\circ\}$. But R does not satisfy (p).

Example 2. Consider the non-commutative ring

$$R = \left\{ \begin{bmatrix} a & b \\ \circ & c \end{bmatrix} \middle| a, b, c \in GF(2) \right\}$$

which satisfies (p).

In this ring, N is commutative but R is not semiprime.

Example 3. Let $R = M_2(Z_2)$, this ring is a non-commutative semiprime ring which satisfies (p), and N is not commutative, because

$$x = \begin{bmatrix} \circ & 1 \\ \circ & \circ \end{bmatrix}, \qquad y = \begin{bmatrix} \circ & \circ \\ 1 & \circ \end{bmatrix},$$

$x^2 = \circ = y^2$ *But* $xy \neq yx$.

Example 4. In Theorem 2, the torsion-freeness restrictions on commutators can not be deleted. For, in the following non-commutative ring

$$R = \left\{ \begin{bmatrix} a & b & c \\ \circ & a & d \\ \circ & \circ & a \end{bmatrix} | a, b, c, d \in GF(2) \right\}$$

$x^4 = \circ$ or $x^4 = 1$, for all $x \in R$.

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