A GENERALIZATION OF A JACOBSON’S COMMUTATIVITY THEOREM

A. H. Yamini and A. Zaeembashi*

Department of Mathematics, AmirKabir University, Shahid Rajaee University, Tehran, Islamic Republic of Iran

Abstract

In this paper we study the structure and the commutativity of a ring $R$, in which for each $x, y \in R$, there exist two integers $k \geq o, n > 1$ depending on $x, y$ such that $[x, y]_k$ equals $x^n$ or $y^n$.

Throughout, $R$ will represent an associative ring with center $C$, and $N$ the set of all nilpotent elements of $R$.

A ring $R$ is said to be periodic if for every $x \in R$, $x^n = x^m$ for some distinct positive integers $m, n$. Any $x \in R$ is called a potent element if $x = x^n$ for some integer $n > 1$. For $x, y \in R$, $[x, y] = [x, y]_1 = xy - yx$ is the usual commutator, and for every positive integer $k > 1$, we define inductively $[x, y]_k = [[x, y]_{k-1}, y]$. Also we define $[x, y]_k = x$.

A ring $R$ is called left (resp. right) s-unital [5] if for each $x \in R$ we have $x \in Rx$ (resp. $x \in xR$). A ring $R$ is called s-unital if for each $x \in R$, $x \in Rx \cap xR$. If $R$ is an s-unital ring, then for any finite subset $F$ of $R$ there exists an element e in $R$ such that $ex = xe = x$ for all $x$ in $F$ (see [5]).

By a remarkable result of Jacobson [3], we know that if for each $x \in R$, $x = x^n$ for some integer $n > 1$, then $R$ is commutative. Here we study the commutativity behavior of a ring $R$, which satisfies the following property:

(p) for each $x, y \in R$, there exist two integers $k \geq o, n > 1$ depending on $x, y$ such that $[x, y]_k$ equals $x^n$ or $y^n$.

Note that if $R$ satisfies (p), with $k = o$ for all $x \in R$ (and $y = x$), then by the above mentioned Jacobson’s theorem, $R$ must be commutative. However the ring

$$R = \left\{ \begin{array}{ccc}
o & a & b \\
o & o & c \\
o & o & o \\
\end{array} \right. \quad (a, b, c \text{ are real numbers})$$

is a non commutative ring satisfying (p), in fact in this ring $[x, y]_3 = o = x^3$ for all $x, y \in R$.

In preparation for the proof of our main theorems, first we state and prove the following lemmas.

**Lemma 1.** Let $R$ be a ring which satisfies (p), then each element of $R$ is either a potent or a nilpotent element. In particular $R$ is a periodic ring.

**Proof.** Suppose that $x \in R$, and $x \neq x^n$ for all integers $n > 1$, then $o = [x, x]_k = x^n$ for some $m \geq 1$ and $k \geq 1$.

**Lemma 2.** Let $R$ be a ring which satisfies (p). Then every idempotent element of $R$ is in the center of $R$.

**Proof.** Let $e^2 = e \in R$, $x \in R$, and put $a = xe - exe$. It is easy to see that for all $k \geq o$, $[a, e]_k = a$. Therefore by (p), for some $n > 1$, $a = a^n$ or $a = e^n$. Since $a^2 = o$,

*E-mail: z7413981@cic.aku.ac.ir*
and \( a = e^n = e \) implies that \( a = 0 \), hence in any case \( a = 0 \), i.e., \( xe = exe \). Similarly \( ex = exe \). This shows that \( e \) is a central element.

**Lemma 3.** Let \( R \) be a ring which satisfies (p), then the commutator ideal of \( R \) is nil.

**Proof.** Since \( R \) is a periodic ring, hence for each \( x \in R \) there exists an integer \( m \geq 1 \) such that \( x^{2m} = x^m \). Therefore, by Lemma (2) and [2] the commutator ideal of \( R \) is nil.

**Lemma 4.** If the commutator ideal of \( R \) is nil, then \( N \) is an ideal.

**Proof.** See [1].

**Lemma 5.** If \( N \) is a commutative ideal, then \( N^2 \subseteq C \).

**Proof.** See [5].

Now we are ready to state and prove our fundamental theorem.

**Theorem 1.** Let \( R \) be a semiprime ring with the following property:

(i) \( R \) satisfies (p).

(ii) \( N \) is commutative (i.e. for all \( a, b \in N \), \( ab = ba \)).

Then \( R \) is a commutative ring.

**Proof.** By the above Lemmas, \( R \) is a periodic ring and \( N \) is an ideal with \( N^2 \subseteq C \). We claim that \( [x, y]^2 = 0 \) for all \( x, y \in N \), since \( x, y \in N \), therefore

\[
[x, y]^2 = [x, y]xy - [x, y]yx = [x, y]xy - [x, y]yx = 0.
\]

Now, since \( N \) is a commutative ideal, \( [x, y] \in R, y \in R \). But \( R \) is a semiprime ring, hence \( [x, y] = 0 \) for all \( x, y \in N \), \( y \in R \). This shows that \( N \subseteq C \).

On the other hand if \( a^2 = 0 \) then \( N \subseteq C \) implies that \( aRa = Ra^2 = 0 \) and therefore \( a = 0 \), since \( R \) is a semiprime ring. This shows that \( N = \{0\} \) and thus \( R \) is commutative, by Lemma 3.

**Theorem 2.** Let \( R \) be a left \( s \)-unital ring which satisfies (p), and \( m > 1 \) is a fixed positive integer. The following conditions are equivalent:

(i) \( R \) is a commutative ring.

(ii) For each \( a \in N \), \( y \in R \), \( [a, y^m] = 0 \); and for any \( x, y \in R \) if \( m(x, y) = 0 \), then \( [x, y] = 0 \).

(iii) For each \( x, y \in R \) \( [x^m, y^m] = 0 \) and \( m(x, y) = 0 \) implies that \( [x, y] = 0 \).

(iv) For each \( x, y \in R \), \( (xy)^m = (yx)^m \) and \( m(x, y) = 0 \) implies that \( [x, y] = 0 \).

(v) For each \( x, y \in R \), \( (xy)^m = x^m y^m \) and \( m(m - 1)(x, y) = 0 \) implies that \( [x, y] = 0 \).

**Proof.** First of all, for any \( x \in R \), \( x = ex \) for some \( e \in R \) \((R \text{ is left } s \text{-unital})\). Also, by Lemma 1 and [4, Lemma 3], \( e^k = e^{2k} \) for some integer \( k \geq 1 \), and by Lemma 2; \( e^k \in C \), hence \( x = ex = xe^k \in xR \). Therefore \( R \) is an \( s \)-unital ring. Obviously (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (v).

We complete the proof as follows.

(ii) \( \Rightarrow \) (i) Let \( a, b \in N \), then \( b^n = 0 \) for some integer \( n > 1 \) and \( ae = ea = a \), \( be = eb = b \) for some \( e \in R \). By (ii), \( [a, (e + b^{n-1})] = 0 \) so \( m(a, b^{n-1}) = 0 \), or in fact \( [a, b^{n-1}] = 0 \). By repeating this argument we can see that \( [a, b] = 0 \). Thus \( N \) is commutative, and as the Proof of Theorem 1 shows: \( [a, y]^2 = 0 \), for all \( a \in N \), \( y \in R \).

Now, let \( a \in N \), \( y \in N \). By Lemma 3, \( [a, y] \in N \), hence \( ([a, y]_k) = [a, y]^k \) for some integer \( k \geq 0 \), \( n > 1 \), (by (p)). Since \( [a, y]^n = 0 \) (by above), hence \( [a, y]_{k+1} = 0 \). First, let \( k \geq 1 \). Considering \( a' = [a, y]_{k-1} \), we have \( [a', y] = 0 \). On the other hand, \( a' \in N \), by Lemma 3. Therefore, \( [a', y]^m = 0 \), by (ii).

Hence, by [4, Lemma 2], \( m y^m [a', y] = 0 \) and so \( [a', y] = 0 \), by [4, Lemma 2] and (ii). That is \( [a, y]_{k+1} = 0 \). Also, it is obvious that if \( k = 0 \), then \( [a, y] = 0 \). This shows that \( N \subseteq C \). Hence for each \( x \in R \); there exists an integer \( n > 1 \) such that \( x - x^n \in C \), by Lemma 1. Therefore, by a well-known result of Herstein [2], \( R \) is a commutative ring.

(iii) \( \Rightarrow \) (ii). See [5, Lemma 2].

(iv) \( \Rightarrow \) (ii). Let \( o \neq a \in N \), \( o \neq y \in R \). By Lemma 1.2, \( ae = ea = a \), \( ye = ey \) for some central idempotent element \( e \in R \). Let \( z = e - a \), \( z' = e + a + a^2 + \ldots \), then by (iv), \( (zyz')^m = (zy)w = (ey)w = y^m \).

On the other hand \( (zye')^m = zy^m z' \), hence \( zy^m z' = y^m \) and therefore \( [z, y^m] = 0 \), or in fact \( [a, y^m] = 0 \).

(v) \( \Rightarrow \) (ii). Let \( a \in N \), \( y \in R \) be two nonzero elements of \( R \) such that \( a^m = 0 \). Using the above notations when
\(a\) is replaced by \(a^{n-1}\), we have \(z^ny^m = (z^ny^m)^m = z^my^m, \) by (v). Therefore, \([z^{m-1}, y^m] = 0, \) i.e., \([e - a^{n-1}, y^m] = 0, \) or in fact \(a^{n-1}, y^m] = 0, \) by (v). By repeating this argument, we can easily see that \([a, y^m] = 0.\)

**Theorem 3.** Let \(R\) be a ring which satisfies (p) and \(N\) a commutative subset of \(R\). Then \(R\) is a subdirect product of commutative nil rings and local rings.

**Proof.** By a well-known result of Birkhoff, \(R\) is a subdirect product of subdirectly irreducible rings \(R_i, i \in I\). Obviously, each \(R_i\) satisfies (p). Thus \(R_i = N_i\) is a nil ring, or \(R_i\) contains a nonzero potent element. On the other hand, if \(\diamond \neq a_i \in R_i\) is not a nilpotent element, then by Lemma 1, \(a_i = a_i^n\) for some integer \(n > 1\).

Clearly, \(e = a_i^{n-1}\) is an idempotent element which lies in the center of \(R_i\), by Lemma 2. Since \(R_i\) is a subdirectly irreducible ring, \(e = 1\) must be the identity of \(R_i\), and \(a_i\) is a unit of \(R_i\). Since by Lemmas 3,4, \(N_i\) is an ideal of \(R_i\), hence each \(R_i\) is a nil or a local ring.

To complete the proof it suffices to observe that if \(f : R \to R'\) is a ring epimorphism, then \(f(N)\) coincides with \(N'\) the set of all nilpotent elements, of \(R'\). Let \(f(a) = a^n \in N'\) be a nonzero element of \(R'\), then by Lemma 1, \(a \in N\) (otherwise, there exists an integer \(n > 1\) such that \(a = a^n = a^{n^2} = \cdots, \) i.e., \(f(a) = a^n = (a^n^n) = (a^n)^n = \cdots = 0\). This completes the proof.

**Corollary 3.1.** Let \(R\) be non-nil subdirectly irreducible ring which satisfies (p). Then \(\text{char } R = p^n\), for some prime \(p\).

**Proof.** By Theorem 3, \(R\) is a local ring with 1. Consider \(2 = 1 + 1\). By Lemma 1, either \(2^n = 0\), or \(2^n = 2\) for some integer \(n > 1\). If \(2^n = 0\), then \(\text{char } R = 0\). But as a division ring with a positive characteristic, \(\text{char } R / N = p\), where \(p\) is a prime number. Therefore, in \(R, p^n = 0\) for some integer \(n > 1\). This completes the proof.

**Theorem 4.** Let \(R\) be a ring of characteristic zero, which satisfies (p). Then \(R = \oplus_{e \in E} Re + N\), where \(E\) is the set of all idempotent elements of \(R,\) and each \(Re\) is a local ring.

**Proof.** Let \(x \in R \setminus N\), then by Lemma 1, \(x = x^n\) for some integer \(n > 1\). Clearly \(x^{n-1} = e\) is an idempotent element, and \(x \in Re\). Now, suppose that \(e, e' \in E\). Since according to Lemma 2, \(E \subseteq C,\) we can easily see that

\[(e + e')^n = e + e' + (2^n - 2)ee'\]

for all integers \(n \geq 1\). By Lemma 1, either \((e + e')^n = 0\) or \((e + e')^n = e + e'\) for some integer \(n \geq 1\). If \((e + e')^n = 0\) then by (v), \(e + e' = (2 - 2^n)ee'\). Multiplying the last equation by \(e - e'\), yields \(e = e'\).

If \((e + e')^n = e + e'\), then in view of (v) we have \((2^n - 2)ee' = 0\). But \(\text{char } R = 0\), hence \(ee' = 0\). These observations show that if \(e \neq e'\), then \(Re \cap Re' = 0\). So far, we have seen that \(R = \oplus_{e \in E} Re + N\). To complete the proof we need to show that for each \(e \in E, Re\) is a local ring. Obviously, \(Ne = Re \cap N\) is an ideal of \(R\) (by Lemma 3, 4). Let \(x \in Re, x \in N\) then by Lemma 1, \(x^m = x\) for some integer \(m > 1\). Clearly \(x^{m-1} = e\) is a central idempotent, by Lemma 2, and it is easy to see that \(e = e'\) (otherwise \(e' = 0\)). Therefore \(x\) is a unit of \(Re\) and the proof is completed.

In Theorem 4, if \(N = 0\) then by Lemma 1, \(R\) satisfies the Jacobson’s condition, i.e. for any \(x \in R, x = x^m,\) for some integer \(m > 1\). Therefore, in view of Theorem 4, we have:

**Corollary 4.1.** If for each \(x \in R,\) there exists an integer \(n = n(x) > 1\) and \(\text{char } R = 0,\) such that \(x = x^m,\) then \(R\) is a direct sum of fields.

Note that \(\text{char } R > 0,\) if \(R\) is a ring with 1, which satisfies the Jacobson’s conditions.

**Remarks.** Each one of the conditions in Theorem 1 is essential, because:

**Example 1.** Let \(R = Z_2[x, y]\) with \(xy = yx + 1\). Clearly \(R\) is a semiprime ring and \(N = \{0\}\). But \(R\) does not satisfy (p).

**Example 2.** Consider the non-commutative ring

\[R = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} | a, b, c \in GF(2) \right\}\]

which satisfies (p).

In this ring, \(N\) is commutative but \(R\) is not semiprime.

**Example 3.** Let \(R = M_2(Z_2),\) this ring is a non-commutative semiprime ring which satisfies (p), and \(N\) is not commutative, because
Example 4. In Theorem 2, the torsion-freeness restrictions on commutators can not be deleted. For, in the following non-commutative ring

\[
R = \begin{pmatrix}
 a & b & c \\
 b & a & d \\
 c & d & a
\end{pmatrix} \quad |a, b, c, d \in GF(2)
\]

\[x^4 = o \quad \text{or} \quad x^4 = 1, \text{ for all } x \in R.
\]

References