# A MIXED PARABOLIC WITH A NON-LOCAL AND GLOBAL LINEAR CONDITIONS

S. M. Hosseini* and N. Aliev

*Department of Mathematics, Tarbiat Modarres University, Tehran, Islamic Republic of Iran*

## Abstract

Krein [1] mentioned that for each PD equation we have two extreme operators, one is the minimal in which solution and its derivatives on the boundary are zero, the other one is the maximal operator in which there is no prescribed boundary conditions. They claim it is not possible to have a related boundary value problem for an arbitrarily chosen operator in between. They have only considered local conditions and so their claim is justified, particularly, for partial differential boundary value problems of odd orders. By considering more general (non-local and global) conditions, we showed this is not necessarily true. With similarly general conditions as considered in this paper one can define a boundary value problem for PDEs of odd order and also problems with local conditions will be a particular case of this general form. In this paper, a mixed problem for a parabolic equation with general conditions is analytically investigated and, in a closed from, its unique solution is shown.

## 1. Introduction

We consider $u_t = \Delta u + f(x,t)$, where $t > 0$, $x \in D \subset \mathbf{R}^2$ and $D$ is bounded, $\Gamma$ or $\partial D$ the boundary of $D$ is a Lyapunov curve. This problem with different types of (mainly local) boundary conditions has been resolved by many authors. But we consider the same problem with a general type of conditions (non-local and global).

## 2. The Mixed Problem for the Parabolic Equation

$$\frac{\partial u(x,t)}{\partial t} = \Delta u(x,t) + f(x,t), \quad x \in D \subset \mathbf{R}^2, t > 0 \qquad (1)$$

**Keywords:** Non-local conditions; Global conditions; Parabolic equations; AMS classification: 35A05

$$\begin{aligned} l_j u \equiv & \sum_{k=1}^{2}\Big[\sum_{s=1}^{2} \alpha_{js}^{(k)}(x_1)\frac{\partial u}{\partial x_s}\Big|_{x_2=\gamma_k(x_1)} \\ & + \alpha_j^{(k)}(x_1) u(x_1, \gamma_k(x_1), t)\Big] \\ & + \int_{a_1}^{b_1} \sum_{k=1}^{2}\Big[\sum_{s=1}^{2} K_{js}^{(k)}(x_1,\eta_1)\frac{\partial u(\eta,t)}{\partial \eta_s}\Big|_{\eta_2=\gamma_k(\eta_1)} \\ & + K_j^{(k)}(x_1,\eta_1) u(\eta_1, \gamma_k(\eta_1), t)\Big] d\eta_1 = f_j(x_1,t), \\ & t \ge 0, \quad j = 1,2, x_1 \in [a_1, b_1] \end{aligned} \qquad (2)$$

$$u(x,0) = \psi(x) \quad x \in \overline{D}. \qquad (3)$$

All the coefficients in (2), and the right hand side of (1), (2) and (3) are real-valued continuous functions. $\lambda \in \mathbf{C}$, the set of complex numbers, $\overline{D}$ is $D \cup \Gamma$. To denote the real part of a complex quantity, we prefix it

* *E-mail: hossei_m@net1cs.modares.ac.ir* (also known as M. Hosseini Aliabadi)

with $R_e$. The interval $[a_1, b_1]$ is the projection of $\overline{D}$ on the $x_1$ axis. We also assume that the following conditions are satisfied:

$1^0 := D$ is a bounded domain in $\mathbf{R}^2$ where each line parallel to $x_2$-axis cross it at most at two points. $\Gamma$, the boundary of $D$ is divided in two pieces, the equation of the lower part is denoted by $x_2 = \gamma_1(x_1)$ and the upper one $x_2 = \gamma_2(x_1)$, $x_1 \in [a_1, b_1]$, $\gamma_1(x_1) < \gamma_2(x_1)$ for $x_1 \in (a_1, b_1)$.

Using the Laplace transformation [2] or the formal scheme of Rasulov [3], the mixed problem could be transformed to the following boundary value problem:

$$\Delta\tilde{u}(x,\lambda) - \lambda\tilde{u}(x,\lambda) = F(x,\lambda), \quad x \in D \subset \mathbf{R}^2 \tag{4}$$

$$\begin{aligned} l_j\tilde{u} \equiv &\sum_{k=1}^{2}[\sum_{s=1}^{2}\alpha_{js}^{(k)}(x_1)\frac{\partial\tilde{u}}{\partial x_s}\Big|_{x_2=\gamma_k(x_1)} \\ &+\alpha_j^{(k)}(x_1)\tilde{u}(x_1,\gamma_k(x_1),\lambda)] \\ &+\int_{a_1}^{b_1}\sum_{k=1}^{2}[\sum_{s=1}^{2}K_{js}^{(k)}(x_1,\eta_1)\frac{\partial\tilde{u}(\eta,t)}{\partial\eta_s}\Big|_{\eta_2=\gamma_k(\eta_1)} \\ &+K_j^{(k)}(x_1,\eta_1)\tilde{u}(\eta_1,\gamma_k(\eta_1),\lambda)]d\eta_1 = \tilde{f}_j(x_1,\lambda), \\ &j=1,2, \quad x_1 \in [a_1,b_1] \end{aligned} \tag{5}$$

where, $F = -\tilde{f}(x,\lambda) - \psi(x)$, the complex value $\lambda$ is the Laplace transform parameter.

**Theorem 1.** (The essential conditions) If $\Gamma$ is a Lyapunov curve, then the solution of (4) should satisfy the following conditions:

$$\begin{aligned} \frac{1}{2}\tilde{u}(\xi,\lambda) = &\int_\Gamma \tilde{u}(x,\lambda)\frac{\partial U(x-\xi,\lambda)}{\partial n}dx \\ &-\int_\Gamma U(x-\xi,\lambda)\frac{\partial\tilde{u}(x,\lambda)}{\partial n}dx \\ &+\int_D F(x,\lambda)U(x-\xi,\lambda)dx, \quad \xi \in \Gamma \end{aligned} \tag{6}$$

$$\begin{aligned} \frac{1}{2}\frac{\partial\tilde{u}}{\partial\xi_1} = &\int_\Gamma\left[A(x,\xi,\lambda) - \lambda\tilde{u}(x,\lambda)U(x-\xi,\lambda)\right]\cos(n,x_1)dx \\ &+\int_\Gamma B(x,\xi,\lambda)\cos(n,x_2)dx \\ &-\int_D F(x,\lambda)\frac{\partial U(x-\xi,\lambda)}{\partial x_1}dx, \quad \xi \in \Gamma \end{aligned} \tag{7}$$

$$\begin{aligned} \frac{1}{2}\frac{\partial\tilde{u}}{\partial\xi_2} = &-\int_\Gamma\left[A(x,\xi,\lambda) + \lambda\tilde{u}(x,\lambda)U(x-\xi,\lambda)\right]\cos(n,x_2)dx \\ &+\int_\Gamma B(x,\xi,\lambda)\cos(n,x_1)dx \\ &-\int_D F(x,\lambda)\frac{\partial U(x-\xi,\lambda)}{\partial x_2}dx, \quad \xi \in \Gamma \end{aligned} \tag{8}$$

where $n$ is the outer unit normal vector on $\Gamma$ and

$$A(x,\xi,\lambda) = \frac{\partial\tilde{u}(x,\lambda)}{\partial x_1}\frac{\partial U(x-\xi,\lambda)}{\partial x_1} - \frac{\partial\tilde{u}(x,\lambda)}{\partial x_2}\frac{\partial U(x-\xi,\lambda)}{\partial x_2} \tag{9}$$

$$B(x,\xi,\lambda) = \frac{\partial\tilde{u}(x,\lambda)}{\partial x_2}\frac{\partial U(x-\xi,\lambda)}{\partial x_1} + \frac{\partial\tilde{u}(x,\lambda)}{\partial x_1}\frac{\partial U(x-\xi,\lambda)}{\partial x_2} \tag{10}$$

$U$ is the generalized or fundamental solution of (4), (see [2] for fundamental solution), which is as follows:

$$U = -\frac{i}{4}H_0^{(1)}\left(i\sqrt{\lambda}|x-\xi|\right)$$

where $H_0^{(1)}$ is the Hankel function.

**Proof.** In this proof, $\tilde{u}$ and $U$ are to represent $\tilde{u}(x,\lambda)$ and $U(x-\xi,\lambda)$, respectively. Using the second Green's formula, (6) is obtained. (7) and (8) are obtained similarly. In the following we show how (7) is obtained.

On multiplying (4) by $\dfrac{\partial U}{\partial x_1}$ we obtain:

$$\int_D \frac{\partial^2\tilde{u}}{\partial x_1^2}\frac{\partial U}{\partial x_1}dx + \int_D \frac{\partial^2\tilde{u}}{\partial x_2^2}\frac{\partial U}{\partial x_1}dx - \lambda\int_D \tilde{u}\frac{\partial U}{\partial x_1}dx = \int_D F(x,\lambda)\frac{\partial U}{\partial x_1}dx.$$

The first integral, after integrating by parts, yields

$$\int_D \frac{\partial^2\tilde{u}}{\partial x_1^2}\frac{\partial U}{\partial x_1}dx = \int_\Gamma \frac{\partial\tilde{u}}{\partial x_1}\frac{\partial U}{\partial x_1}\cos(n,x_1)dx - \int_D \frac{\partial\tilde{u}}{\partial x_1}\frac{\partial^2 U}{\partial x_1^2}dx.$$

Integrating by parts in such a way that the highest derivative in each integral over $D$ and $\Gamma$ is 2 and 1, respectively, the second integral reduces to the following:

$$\begin{aligned} \int_D \frac{\partial^2\tilde{u}}{\partial x_2^2}\frac{\partial U}{\partial x_1}dx = &\int_\Gamma \frac{\partial\tilde{u}}{\partial x_2}\frac{\partial U}{\partial x_1}\cos(n,x_2)dx \\ &-\int_\Gamma \frac{\partial\tilde{u}}{\partial x_2}\frac{\partial U}{\partial x_2}\cos(n,x_1)dx + \int_\Gamma \frac{\partial\tilde{u}}{\partial x_1}\frac{\partial U}{\partial x_2}\cos(n,x_2)dx \\ &-\int_D \frac{\partial\tilde{u}}{\partial x_1}\frac{\partial^2 U}{\partial x_2^2}dx. \end{aligned}$$

Similarly, for the third term we obtain

$$-\lambda\int_D \tilde{u}\frac{\partial U}{\partial x_1}dx = -\lambda\int_\Gamma \tilde{u}U\cos(n,x_1)dx + \lambda\int_D \frac{\partial\tilde{u}}{\partial x_1}Udx.$$

On substituting these three partial results in the first formula, we obtain

$$\int_\Gamma \frac{\partial\tilde{u}}{\partial x_1}\frac{\partial U}{\partial x_1}\cos(n,x_1)dx+\int_\Gamma \frac{\partial\tilde{u}}{\partial x_2}\frac{\partial U}{\partial x_1}\cos(n,x_2)dx$$
$$-\int_\Gamma \frac{\partial\tilde{u}}{\partial x_2}\frac{\partial U}{\partial x_2}\cos(n,x_1)dx+\int_\Gamma \frac{\partial\tilde{u}}{\partial x_1}\frac{\partial U}{\partial x_2}\cos(n,x_2)dx$$
$$-\lambda\int_\Gamma \tilde{u}U\cos(n,x_1)dx-\int_D F(x,\lambda)\frac{\partial U}{\partial x_1}dx=$$
$$\int_D \frac{\partial\tilde{u}}{\partial x_1}\left[\Delta U-\lambda U\right]dx=\begin{cases}\dfrac{\partial\tilde{u}}{\partial\xi_1}, & \text{if } \xi\in D\\ \dfrac{1}{2}\dfrac{\partial\tilde{u}}{\partial\xi_1}, & \text{if } \xi\in\Gamma\end{cases}$$

Now, from this result, for $\xi\in\Gamma$, we obtain

$$\frac{1}{2}\frac{\partial\tilde{u}}{\partial\xi_1}=\int_\Gamma\left[\frac{\partial\tilde{u}}{\partial x_1}\frac{\partial U}{\partial x_1}-\frac{\partial\tilde{u}}{\partial x_2}\frac{\partial U}{\partial x_2}-\lambda\tilde{u}U\right]\cos(n,x_1)dx+$$
$$\int_\Gamma\left[\frac{\partial\tilde{u}}{\partial x_1}\frac{\partial U}{\partial x_2}+\frac{\partial\tilde{u}}{\partial x_2}\frac{\partial U}{\partial x_1}\right]\cos(n,x_2)dx-\int_D F(x,\lambda)\frac{\partial U}{\partial x_1}dx,$$

which verifies (7).●

**Theorem 2.** (Regularization) Assuming $1^0$ holds and $\Gamma$ is a Lyapunov curve, $\alpha_{jk}^{(s)}(x_1)$ the coefficients of the boundary conditions (5) belong to the Holder's class and also continuous in $[a_1, b_1]$ and $\tilde{f}_j(x_1,\lambda)$ differentiable and satisfies $\tilde{f}_j(a_1,\lambda)=\tilde{f}_j(b_1,\lambda)=0$ for $\lambda$ complex, and if the kernels of the integrals in (5) are continuous or at most weakly singular, $F(x,\lambda)$ continuous, then with a suitable linear combination of the conditions (7), (8) and using the boundary conditions (5) we obtain a regularized condition.

**Proof.** It is enough to consider the following linear combination

$$-\alpha_{j2}^{(1)}(\xi_1)\frac{\partial\tilde{u}(\xi,\lambda)}{\partial\xi_1}\Big|_{\xi_2=\gamma_1(\xi_1)}+\alpha_{j1}^{(1)}(\xi_1)\frac{\partial\tilde{u}(\xi,\lambda)}{\partial\xi_2}\Big|_{\xi_2=\gamma_1(\xi_1)}$$
$$+\alpha_{j2}^{(2)}(\xi_1)\frac{\partial\tilde{u}(\xi,\lambda)}{\partial\xi_1}\Big|_{\xi_2=\gamma_2(\xi_1)}-\alpha_{j1}^{(2)}(\xi_1)\frac{\partial\tilde{u}(\xi,\lambda)}{\partial\xi_2}\Big|_{\xi_2=\gamma_2(\xi_1)}$$

For details see [4].●

**Theorem 3.** (Fredholm) With the same conditions as in Theorem 2 and if $\det(A(x_1))\neq 0$ for each $x_1\in[a_1,b_1]$ then the boundary value problem (4)-(5) is Fredholm, where

$$\det(\mathbf{A}(x_1))=\det\begin{bmatrix}\alpha_{11}^{(1)} & \alpha_{11}^{(2)} & \alpha_{12}^{(1)} & \alpha_{12}^{(2)}\\ \alpha_{21}^{(1)} & \alpha_{21}^{(2)} & \alpha_{22}^{(1)} & \alpha_{22}^{(2)}\\ -\alpha_{12}^{(1)} & \alpha_{12}^{(2)} & \alpha_{11}^{(1)} & -\alpha_{11}^{(2)}\\ -\alpha_{22}^{(1)} & \alpha_{22}^{(2)} & \alpha_{21}^{(1)} & -\alpha_{21}^{(2)}\end{bmatrix}$$

**Proof.** For details see [4].●

## 3. The Adjoint Problem

Multiplying the homogeneous part of (4), $l\tilde{u}=\Delta\tilde{u}-\lambda\tilde{u}$, by $\overline{\varphi}\in C^{(2)}(D)\cap C^{(1)}(\overline{D})$ and integrating over $D$, using integration by parts, we obtain

$$\int_D l\tilde{u}\,\overline{\varphi}dx=\int_\Gamma\left\{\left[\frac{\partial\tilde{u}}{\partial x_1}\overline{\varphi}-\tilde{u}\frac{\partial\overline{\varphi}}{\partial x_1}\right]\cos(\nu,x_1)\right.$$
$$\left.+\left[\frac{\partial\tilde{u}}{\partial x_2}\overline{\varphi}-\tilde{u}\frac{\partial\overline{\varphi}}{\partial x_2}\right]\cos(\nu,x_2)\right\}dx \tag{11}$$
$$+\int_D\tilde{u}\left[\Delta\overline{\varphi}-\lambda\overline{\varphi}\right]dx=0$$

where $\nu$ is the outer unit normal on $\Gamma=\partial D$.

From (11) we have the adjoint equation as

$$l^*\varphi\equiv\Delta\varphi-\overline{\lambda}\varphi \tag{12}$$

To obtain the adjoint boundary conditions we force the first term in (11) to be equal to zero. Hence, we obtain boundary conditions similar to the following formula in which no values of the derivatives of unknown function $\varphi$ on the boundary can appear in the integral sign

$$l_j^*\varphi=\sum_{k=1}^{2}\left[\sum_{s=1}^{2}\beta_{js}^{(k)}(x_1)\frac{\partial\varphi(x,\lambda)}{\partial x_s}\Big|_{x_2=\gamma_k(x_1)}\right.$$
$$\left.+\beta_j^{(k)}(x_1)\varphi(x_1,\gamma_k(x_1),\lambda)\right] \tag{13}$$
$$+\int_{a_1}^{b_1}\sum_{k=1}^{2}M_j^{(k)}(x_1,\eta_1)\varphi(\eta_1,\gamma_k(\eta_1),\lambda)d\eta_1,$$
$$j=1,2,\ x_1\in[a_1,b_1],$$

where $\beta_{js}^{(k)}(x_1)$, $\beta_j^{(k)}(x_1)$, $M_j^{(k)}(x_1,\eta_1)$ depend on the coefficient in (5).

**Theorem 4.** Assuming all the conditions given for the problem (1)-(3) are satisfied, we obtain some sufficient conditions for which the problem (12)-(13) is the adjoint of the problem (4)-(5).

**Sketch of proof.** The first condition is

$$\Delta(x_1)=\begin{vmatrix}\alpha_{12}^{(1)}-\alpha_{11}^{(1)}\gamma_1' & \alpha_{12}^{(2)}-\alpha_{11}^{(2)}\gamma_2'\\ \alpha_{22}^{(1)}-\alpha_{21}^{(1)}\gamma_1' & \alpha_{22}^{(2)}-\alpha_{21}^{(2)}\gamma_2'\end{vmatrix}\neq 0$$

for each $x_1\in[a_1,b_1]$. Then, under this condition, from the two linearly independent boundary conditions (5) a second kind Fredholm system is obtained in which the unknowns are $\frac{\partial \tilde{u}}{\partial x_2}\Big|_{x_2=\gamma_k(x_1)}$, $k = 1, 2$. Assuming this system is solvable enables us to obtain $\frac{\partial \tilde{u}}{\partial x_2}\Big|_{x_2=\gamma_k(x_1)}$, in terms of $\tilde{u}(x_1,\gamma_k(x_1),\lambda)$ and $\tilde{u}'(x_1,\gamma_k(x_1),\lambda)$. On substituting $\frac{\partial \tilde{u}}{\partial x_2}\Big|_{x_2=\gamma_k(x_1)}$, $k = 1, 2$, in the first term in (11), integrating by parts, taking factor of all the coefficients of $\tilde{u}$ in the integral sign, the adjoint boundary conditions (13) are obtained. For details see [5].●

**Remark 1.** Comparing problems (4)-(5) and (12)-(13) the sufficient conditions obtained for the boundary value problem (4)-(5) to be self-adjoint.

## 4. Existence and Uniqueness for the Solution of the Boundary Value Problem

From some results in operator theory (see [1] and [6]) it is clear that if we show the uniqueness of the solution for the adjoint problem it implies the existence of the solution for the main problem. Hence, the uniqueness of the solution of the main boundary value problem (BVP) is obtained and then the uniqueness of the solution of the problem is shown.

### 4.1. Uniqueness Criteria for the Main BVP

Assuming that the problem (4)-(5) has two different solutions $\tilde{u}_1$ and $\tilde{u}_2$ then we have

$$\Delta(\tilde{u}_1-\tilde{u}_2)-\lambda(\tilde{u}_1-\tilde{u}_2)=0\,.$$

As $\tilde{u}_1$, $\tilde{u}_2$ satisfy the boundary conditions (2) then $\tilde{u}=\tilde{u}_1-\tilde{u}_2$ satisfies the following homogeneous BVP:

$$\begin{cases}\Delta\tilde{u}-\lambda\tilde{u}=0\\ l_j(\tilde{u})=0, \qquad j=1,2\end{cases} \tag{14}$$

Now we show that under some certain sufficient conditions the problem (6) has the only solution $\tilde{u}\equiv 0$. To this end, we multiply it by $\overline{\tilde{u}}$ and integrate over $D$, we have

$$\int_D(\Delta\tilde{u}-\lambda\tilde{u})\overline{\tilde{u}}\,dx=0\Rightarrow$$

$$\int_\Gamma\left[\frac{\partial\tilde{u}}{\partial x_1}\overline{\tilde{u}}\cos(\nu,x_1)+\frac{\partial\tilde{u}}{\partial x_2}\overline{\tilde{u}}\cos(\nu,x_2)\right]dx \tag{15}$$

$$-\int_D\left[\left|\frac{\partial}{\partial x_1}\tilde{u}\right|^2+\left|\frac{\partial}{\partial x_2}\tilde{u}\right|^2\right]dx-\lambda\int_D|\tilde{u}|^2dx=0$$

Then using (5) and (15) we obtain some certain sufficient conditions for the problem (14) to have the only solution $\tilde{u}\equiv 0$ in $D$ and hence $\tilde{u}_1\equiv\tilde{u}_2$. For details see [5].

### 4.2. Uniqueness for the Adjoint Problem

Similar to 4.1 some sufficient conditions are obtained for the adjoint problem to have unique solution.

## 5. Existence of the Solution for the Mixed Problem

From the Green's formula, we obtain the solution of the BVP (4)-(5) as follows:

$$\tilde{u}(\xi,\lambda)=-\sum_{k=1}^{2}\int_\Gamma\left[\frac{\partial\tilde{u}(x,\lambda)}{\partial x_k}U(x-\xi,\lambda)\right.$$
$$\left.-\tilde{u}(x,\lambda)\frac{\partial U(x-\xi,\lambda)}{\partial x_k}\right]\cos(\nu,x_k)dx$$
$$+\int_D F(x,\lambda)U(x-\xi,\lambda)dx, \qquad \xi\in D$$

where

$$U=-\frac{i}{4}H_0^{(1)}(i\sqrt{\lambda}|x-\xi|)$$

$$\frac{\partial U}{\partial x_j}=-\frac{\sqrt{\lambda}}{4}H_1^{(1)}(i\sqrt{\lambda}|x-\xi|)\frac{x_j-\xi_j}{\sqrt{(x_1-\xi_1)^2+(x_2-\xi_2)^2}}$$

$$j=1,2$$

From [2], we see that the asymptotical Hankel functions are as follows: as $|z|\to\infty$

$$H_\nu^{(1)}(z)=\sqrt{\frac{2}{\pi z}}e^{i(z-\frac{\pi}{2}\nu-\frac{\pi}{4})}+O\left(|z|^{-\frac{3}{2}}\right)$$

So, as $|\lambda|\to\infty$ we have

$$U=-\frac{ie^{-i\pi/4}}{4}\sqrt{\frac{2}{\pi|x-\xi|i\sqrt{\lambda}}}e^{-|x-\xi|\sqrt{\lambda}}$$

Hence, as long as $\lambda$ is on the $c-i\infty$ to $c+i\infty$ where $c>0$, and we choose $\sqrt{\lambda}$ such that to have positive real part then $U(x-\xi,\lambda)\to 0$ as $|\lambda|\to\infty$. The same happens for $\frac{\partial U}{\partial x_k}$.

Therefore, $\tilde{u}(x,\lambda)\to 0$ as $|\lambda|\to\infty$, $x\in D$ for $\lambda\in[c-i\infty,c+i\infty]$ with positive real part for $\sqrt{\lambda}$.

By the inverse Laplace transform, we have

$$u(x,t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{\lambda t}\tilde{u}(x,\lambda)d\lambda \tag{16}$$

where $[c-i\infty,c+i\infty]$ used to represent the straight Laplace line.

Clearly, for $t = 0$ we have no difficulty but for $t > 0$, it may be the case that the integral part of the solution is undefined. So, to treat this difficulty we continue as follows:

As our main problem is parabolic, it is clear that its spectral region only contains $\lambda$ inside a parabola with vertex $(h, 0)$, for some $h \in \mathbf{R}$, which is symmetric about the real axis and opens to the left like $\overline{S}$ where $\overline{S}$ denotes the curve which is symmetric to $S$ about the Laplace line and $S$ is an infinite curve lying in a region $R_\delta$ of the complex $\lambda$-plane and that it approaches the straight lines

$$\cos \arg \lambda = \delta$$

asymptotically [3]. Hence, we can let the straight Laplace line to be replaced with $\overline{S}$.

From the methods discussed in [3] by Rasulov, the operators of the mixed problem, i.e., of the main equation and its boundary conditions, can also be written under the integral sign expressed in (16).

Hence, by the Rasulov methods, the existence of the solution of the mixed problem is resolved. The uniqueness of $u(x,t)$ is clearly obtained from (16) and also the uniqueness of $\widetilde{u}(x,t)$.

**Theorem 5.** From Theorem 4.3 in [5], if $\psi(x) \in C^4(D)$ and $\psi(x)$ and its first and second derivatives on the boundary $\Gamma$ are zero, then the solution of the mixed problem (1)-(3) exists as given in (16) and is also unique.

**Proof.** On substituting (16) in (1), we obtain

$$\frac{\partial u(x,t)}{\partial t} - \Delta u(x,t) =$$
$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\lambda t}\left[\lambda \widetilde{u}(x,\lambda) - \Delta \widetilde{u}(x,\lambda)\right] d\lambda$$
$$= -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\lambda t} F(x,\lambda)\, d\lambda$$
$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\lambda t} \widetilde{f}(x,\lambda)\, d\lambda + \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\lambda t} \psi(x)\, d\lambda$$
$$= f(x,t) + \psi(x)\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\lambda t} d\lambda, \quad \text{with } \lambda = c + i\varepsilon$$
$$= f(x,t) + \psi \frac{e^{ct}}{2\pi}\int_{-\infty}^{\infty} e^{i\varepsilon t} d\varepsilon$$
$$= f(x,t) + \psi(x) e^{ct}\delta(t) = f(x,t), \qquad t > 0.$$

So, (16) satisfies (1).

On substituting (16) in the boundary conditions (2), we obtain

$$l_j u = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\lambda t} l_j \widetilde{u}\, d\lambda$$
$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\lambda t} \widetilde{f}_j(x_1,\lambda)\, d\lambda, \qquad \text{by (5)},$$
$$= f_j(x_1,t), \qquad t > 0.$$

So, the conditions (2) are also satisfied by (16).

For the initial condition, it is enough to use the conditions stated for $\psi(x)$ in this theorem and asymptotically apply on $\widetilde{u}(x,\lambda)$ in the same lines of Rasulov [3] to see that the initial condition (3) is also satisfied by (16).●

## Conclusion

Using the property of the fundamental solution of the equation (4), and the asymptotic property of $\widetilde{u}$ and the method of Rasulov, we showed the existence and uniqueness of the solution of a mixed problem for parabolic equation. The essential conditions were used, after regularization, to show that the BVP (4)-(5) is Fredholm. The conditions (2) are linear and so general that the limitations mentioned in the abstract are resolved. It must be emphasized that with this method, the solution of the mixed problem is obtained in a closed form and hence it is useful for any analytical investigation such as continuity of the solution to the related data. We think the solution in the form of (16) can also be useful for obtaining good approximations.

## References

1. Krein, S. G. *Functional analysis.* 1st ed., Nauka, Moscow, p. 135-136, (1964).
2. Vladimirov, V. S. *Equations of mathematical physics.* Mir-Publication, Moscow, 1st ed., p. 160-162, (1984).
3. Rasulov, M. L. *Methods of countor integration,* North-Holland Publication, 1st ed., p. 20-155, (1967).
4. Aliev, N. and Hosseini, S. M. An analysis of a parabolic problem with a general (non-local and global) supplementary linear conditions I, accepted for publication in *Italian Journal of Pure and Applied Mathematics*.
5. Aliev, N. and Hosseini, S. M. An analysis of a parabolic problem with a general (non-local and global) supplementary linear conditions II, (submitted for publication).
6. Bitsadze, A. V. *Boundary-value problems for second-order elliptic equations,* North-Holland Publication, 1st ed., p. 12-15, (1968).