GENERALIZED POSITIVE DEFINITE FUNCTIONS AND COMPLETELY MONOTONE FUNCTIONS ON FOUNDATION SEMIGROUPS

M. Lashkarizadeh Bami

Department of Mathematics, Institute for Studies in Theoretical physics and Mathematics, Tehran, P. O. Box 19395-1795, Islamic Republic of Iran

Abstract

A general notion of completely monotone functionals on an ordered Banach algebra $B$ into a proper $H^*$-algebra $A$ with an integral representation for such functionals is given. As an application of this result we have obtained a characterization for the generalized completely continuous monotone functions on weighted foundation semigroups. A generalized version of Bochner’s theorem on foundation semigroups is also obtained.

Introduction

In the present paper we shall introduce the concept of a completely monotone functional on an ordered Banach algebra $B$ into a proper $H^*$-algebra $A$ and we shall give an integral representation for such functionals with respect to $A$-valued measures on $\Delta_+(B)$, the space of all positive multiplicative linear functionals on $B$. As an application of the theory we shall obtain an integral representation for the generalized $w$-bounded continuous completely monotone $A$-valued functions with respect to positive $A$-valued measures on $\Gamma^-_w$, the space of all $w$-bounded continuous nonnegative semicharacters on a foundation semigroup $S$ with a Borel measurable weight function $w$. We will also give a generalization of our earlier version of Bochner’s theorem [4; Theorem 4.2].

Keywords: Locally compact semigroups; Positive definite functions; $H^*$-algebras; Spectral measures

1. Preliminaries

Recall that (see, [11], [12], [13], [17]) a proper $H^*$-algebra is a Banach algebra $A$ whose norm is a Hilbert space norm and which has an involution: $x \mapsto x^*$ on $A$ such that $(y, x^*z) = (xy, z) = (z, xy^*)$ for all $x, y, z \in A$. Let $\tau(A) = \{xy : x, y \in A\}$ be the trace class of $A$. It is a Banach algebra with respect to a norm $\tau(.)$ which is related to the norm $\|\cdot\|$ of $A$ by $\tau(a^*a) = \|a\|^2$ for all $a \in A$. There is a trace $tr$ defined on $\tau(A)$ such that $tr(ab) = tr(ba) = (a, b^*)$ for all $a, b \in A$, where $(., .)$ denotes the scalar product on $A$. if $a = b^*b$ for some $b \in A$ then $a$ is called positive and we write $a \geq 0$. It is obvious that $a \geq 0$ if and only if $(ax, x) \geq 0$ for all $x \in A$. A right module $H$ over $A$ is called a Hilbert module if there is a $\tau$-valued function $(\cdot, \cdot)$ on $H \times H$ with the following properties

1. $(\xi + \eta, \phi) = (\xi, \phi) + (\eta, \phi)$ for all $\xi, \eta, \phi \in H$.
2. $(\xi, \eta)^* = (\eta, \xi^*)$ for all $\xi, \eta \in H$.
3. $(\xi, \eta) = (\xi, \eta) = (\xi^*, \eta^*)a$ for all $\xi, \eta \in H$ and each $a \in A$.
4. $(\xi, \xi) \geq 0$ for all $\xi \in H$ and $(\xi, \xi) = 0$ if and only

1991 Mathematics Subject Classification. Primary 43 A35, 43 A10; This research was in part supported by grant from IPM.

E-mail: lashkari@math.ui.ac.ir
5. \( |\text{tr}(\xi, \eta)|^2 \leq \tau(\xi, \xi) \tau(\eta, \eta) \) for all \( \xi, \eta \in H \).

6. \( H \) is complete in the norm \( \|\cdot\| = |\tau(\xi, \xi)|^{1/2} \).

The function \( (\cdot, \cdot) \) is called a generalized scalar product. There is a linear structure on \( H \) such that \( H \) is an ordinary Hilbert space with respect to the scalar product \( (\xi, \eta) = \text{tr}(\eta, \xi) \). An \( A \)-linear operator on \( H \) is an additive linear mapping \( T : H \to H \) such that \( T(\xi \omega) = (T\xi)\omega \) for all \( \xi \in H, a \in A, \omega \) is bounded in the sense that \( \|T\xi\| \leq M\|\xi\| \) for some \( M \geq 0 \) and every \( \xi \in H \). For each bounded \( A \)-linear operator \( T \) its adjoint \( T^* \) is \( A \)-linear and has the property that \( (T\xi, \eta) = (\xi, T^*\eta) \) for all \( \xi, \eta \in H \).

By a real ordered Banach algebra we shall mean a real Banach algebra \( \mathcal{B} \) for every \( A \)-positive elements \( a, b, \ldots, b_n \) in \( \mathcal{B} \) if for every \( x \in \mathcal{B} \) and for every \( n \geq 2 \)

\[
\Delta_n f(b, b_1, \ldots, b_n) \geq 0 \\
\Delta_{n-1} f(b_1, b_2, \ldots, b_{n-1}) - \Delta_{n-2} f(b_1, b_2, \ldots, b_{n-2})
\]

for all \( \xi \in H \) and for every \( n \geq 2 \), where \( f \in B^* \) is called completely monotone if

\[
\Delta_n f(b, h_1, \ldots, h_n) \geq 0 \\
\Delta_{n-1} f(b, h_1, \ldots, h_{n-1}) - \Delta_{n-2} f(b, h_2, \ldots, h_{n-2})
\]

for all \( n \in \mathbb{N}_+ \) and \( b, b_1, \ldots, b_n \in \mathcal{B} (B) \).

An operator-valued transformation \( U : B \to \mathcal{L}(H) \) (the space of all bounded linear operators on a Hilbert space \( H \)) is called completely monotone if for every \( \xi \in H \), the mapping \( \varphi_N : b \mapsto \langle U_N \xi, \xi \rangle(b \in B) \) defines a completely monotone functional on \( B \).

We now recall some definitions concerning topological semigroups.

Throughout this paper \( S \) will denote a locally compact, Hausdorff topological semigroup.

**Definition 1.2.** On a commutative topological semigroup \( S \) with \( C_b(S) \) (the space of bounded continuous complex-valued functions on \( S \)) inductive identity, for each \( n \in \mathbb{N}_+ \) we define the operator \( \Delta_n \) on by

\[
\Delta_n f(x) = f(x),
\]

\[
\Delta_{n-1} f(x, h_1) = \Delta_{n-1} f(x) - \Delta_{n-2} f(x, h_1) = f(x) - f(x, h_1)
\]

for every \( n \geq 2 \)

\[
\Delta_n f(x, h_1, \ldots, h_n) = \Delta_{n-1} f(x, h_1, \ldots, h_{n-1}) - \Delta_{n-2} f(x, h_2, \ldots, h_{n-1})
\]

for \( f \in C_b(S), x, h_1, \ldots, h_n \in S, n = 1, 2, \ldots \). A function \( f \in C_b(S) \) is called completely monotone if \( \Delta_n f \geq 0 \) for every \( f \in C_b(S) \) (cf. [5; p. 43]).

**Definition 1.3.** An operator-valued transformation \( T : S \to \mathcal{L}(H) \) is called completely monotone if for every \( \xi \in H \) the mapping

\[
x \mapsto \langle T_x \xi, \xi \rangle(x \in S)
\]

is completely monotone on \( S \).

**Definition 1.4.** Let \( B \) be an ordered commutative Banach algebra and \( H \) be a Hilbert module over a proper \( H \)-algebra \( A \). A linear mapping \( f : B \to A \) is called a completely monotone \( A \)-functional if for every \( n \in \mathbb{N}_+ \)

\[
\Delta_n f(b, b_1, \ldots, b_n) \geq 0
\]

for every \( (n + 1) \)-positive elements \( b, b_1, \ldots, b_n \) of \( B \) where

\[
\Delta_0 f(b) = f(b)
\]

\[
\Delta_1 f(b, h) = \Delta_{0,1} f(b) = \Delta_{0} f(bh) = f(b) - f(bh)
\]

and for every \( n \geq 2 \)

\[
\Delta_n f(b, b_1, \ldots, b_n) = \Delta_{n-1} f(b, b_1, \ldots, b_{n-1}) - \Delta_{n-2} f(b_1, b_2, \ldots, b_{n-2})
\]

for \( f \in B^*, b, b_1, \ldots, b_n \in B; n = 1, 2, \ldots \). A linear functional
\[ \Delta_n f(b; h_1, \ldots, h_n) = \Delta_{n-1} f(b; h_1, \ldots, h_{n-1}) - \Delta_{n-1} f(bb; h_1, \ldots, h_{n-1}). \]

**Definition 1.5.** Let \( S \) be a commutative topological semigroup with an identity. A mapping \( f : S \to A \) is called **completely monotone** if \( \Delta_n f(x; h_1, \ldots, h_n) \geq 0 \) for all nonnegative integers \( n \) and all \( x, h_1, \ldots, h_n \in S \) where

\[ \Delta_0 f(x) = f(x) \]

\[ \Delta_1 f(x; h) = \Delta_0 f(x) - \Delta_0 f(xh) = f(x) - f(xh) \]

and for every \( n \geq 2 \)

\[ \Delta_n f(x; h_1, \ldots, h_n) = \Delta_{n-1} f(x; h_1, \ldots, h_{n-1}) - \Delta_{n-1} f(xh; h_1, \ldots, h_{n-1}). \]

**Definition 1.6.** Let \( B \) be a Banach \(*\)-algebra and \( A \) be a proper \( H^\prime \)-algebra. A linear mapping \( f : B \to A \) is called a **positive \( A \)-functional** if

\[ \sum_{i=1}^{n} \sum_{j=1}^{n} a_i^* f(b_i^* b_j) a_j \geq 0 \]

for all \( b_1, \ldots, b_n \in B \) and \( a_1, \ldots, a_n \in A \).

**Definition 1.7.** Let \( S \) be a \(*\)-semigroup. Then a mapping \( \varphi : S \to A \) is called **positive definite** if

\[ \sum_{i=1}^{n} \sum_{j=1}^{n} a_i^* \varphi(x_i^* x_j) a_j \geq 0 \]

for all \( x_1, \ldots, x_n \in S \) and \( a_1, \ldots, a_n \in A \).

Recall that a Borel measurable mapping \( w : S \to \mathbb{R}_+ \) (the set of nonnegative real numbers) with \( w(xy) \leq w(x)w(y) \) \((x, y \in S)\) and such that \( w \) and \( \frac{1}{w} \) are locally bounded (i.e., bounded on compact subsets of \( S \)) is called a **weight function** on \( S \). A function \( f : S \to C \) is called \( w \)-bounded if there is a \( k > 0 \) such that \( |f(x)| \leq kw(x) \) for all \( x \in S \).

Recall also that \( M(S, w) \) denotes the set of all complex, regular, signed measures \( \mu \) (not necessarily bounded) of the form \( \mu = \mu_1 - \mu_2 + i(\mu_3 - \mu_4) \) where \( \mu_i \) is a positive regular measure on \( S \) with \( w \in L^1(S, \mu_i) \) \(i = 1, 2, 3, 4\) (see, for example [2], [7], [9]). Note that for an element \( \mu \in M(S, w) \) and a Borel set \( B \), \( \mu(B) \) is well-defined whenever \( B \) is relatively compact. For every \( \mu \in M(S, w) \), the equation

\[ \int_S f d(w, \mu) = \int_S f \text{wd}\mu \quad (f \in C_0(S)), \]

defines a measure \( w, \mu \in M(S) \), the space of all bounded regular complex measures on \( S \). With the norm

\[ \|w, \mu\| = \|w\| \quad (\mu \in M(S, w)), \]

where \( \|w, \mu\| \) denotes the total variation of \( w, \mu \), the space \( M(S, w) \) defines a Banach lattice, and with the convolution product

\[ (\mu * \nu)(f) = \int_S \int_S f(xy)d\mu(x)d\nu(y) \]

(1)

\[ (\mu, \nu \in M(S, w), f \in C_0(S)), \]

where \( C_0(S) \) denotes the set of all functions in \( C(S) \) with compact support, defines a Banach algebra. From part (iii) of Theorem 4.6 of [7], we conclude that (1) also holds for every \( w \)-bounded Borel measurable function \( f \) on \( S \).

We also recall (see, for example, [1], [6], [18]) that \( M_s(S) \) (or \( \hat{L}(S) \)) denotes the set of all measures \( \mu \in M(S) \) for which the mappings \( x \mapsto \delta_x * \mu \) and \( x \mapsto |\mu| * \delta_x \) (where \( \delta_x \) denotes the Dirac measure at \( x \)) from \( S \) into \( M(S) \) are weakly continuous. As in [7], we can define \( M_s(S, w) \) (or \( \hat{L}(S, w) \)) as the set of measures \( \mu \in M(S, w) \) for which \( w, \mu \in M_s(S) \). Then, \( M_s(S, w) \) is a closed, two-sided \( L \)-ideal of \( M(S, w) \). Finally, we call \( S \) a foundation semigroup if \( \cup \{\mu \in M(S) : \mu \in M_s(S)\} \) is dense in \( S \). A mapping \( \chi : S \to \mathbb{C} \) is called a **semicharacter** if \( \chi(xy) = \chi(x)\chi(y) \) for all \( x, y \in S \). We denote by \( \Gamma_w \) the set of all \( w \)-bounded continuous semicharacters on \( S \), and by \( \Gamma_w^0 \) the set of nonnegative semicharacters in \( \Gamma_w \). If \( S \) is commutative and foundation, then \( \Gamma_w \) is homomorphic to \( \Delta(M_s(S, w)) \) whenever \( \Gamma_w \) has the compact open topology and \( \Delta(M_s(S, w)) \) has the Gelfand topology. In particular, \( \Gamma_w \) is a locally compact Hausdorff space (see, Theorem 2.10 of [8]).

An operator-valued transformation \( U : S \to \mathcal{L}(H) \) is called \( w \)-bounded (continuous, respectively) if for every \( \xi, \eta \in H \) the map \( x \mapsto \langle U_x \xi, \eta \rangle \) is \( w \)-bounded (continuous, respectively). Finally if \( U : S \to \mathcal{L}(H) \) is such that \( U_x y = U_x U_y (x, y \in S) \), then \( U \) is called a **representation**. For further information on the representation theory of topological semigroups and \(*\)-algebras the reader is referred to [7].
2. Generalized Representations and Positive-Definite Functions on Weighted Foundation Semigroups

We start this section with the following result which is indeed a generalization of our earlier result (Theorem 4.4 of [7]).

**Theorem 2.1.** Let $S$ be a foundation $*$-semigroup with identity and with a Borel measurable weight function $w$ such that $w(x^*) = w(x)$ $(x \in S)$. Let $T$ be a $*$-representation of $M_a(S, w)$ by bounded $A$-linear operators on a Hilbert module $H$ over a proper $H^*$-algebra $A$ such that for every $0 \in \xi \in H$ there exists a measure $\mu \in M_a(S, w, \xi, \eta)$ such that $T_\mu \xi \neq 0$. Then there exists a unique $w$-bounded continuous $*$-representation $V$ of $S$ by $A$-linear operators on $H$ such that

$$\langle \eta, T_\mu \xi \rangle = \int_S \langle \eta, V_\xi \eta \rangle d\mu(x) \quad (\xi, \eta \in H, \mu \in M_a(S, w)).$$

(2)

**Proof.** Recall that by Theorem 1 of [11] $H$ with the inner product $\langle , \rangle$ where $\langle \xi, \eta \rangle = tr(\eta^* \xi)$ defines a Hilbert space and by Theorem 4 of [11], the adjoint operator $T^*$ of $T$ defines a bounded $A$-linear operator on $H$. So by Theorem 5.4 of [7] there exists a bounded continuous $*$-representation $V$ of $S$ by bounded operators on the Hilbert space $(H, \langle , \rangle)$ such that

$$\langle T_\mu \xi, \eta \rangle = \int_S \langle V_\xi \eta \rangle d\mu(x) \quad (\xi, \eta \in H, \mu \in M_a(S, w)).$$

(3)

Now let $R(A)$ denote the space of the right centralizers of $A$. From Lemma 2 of [14] and Theorem 1 of [11] for every $U \in R(A)$ we have

$$trU(\eta, T_\mu \xi) = tr(U^* \eta, T_\mu \xi) = \langle T_\mu \xi, U^* \eta \rangle = \int_S \langle V_\xi \eta \rangle d\mu(x) = \int_S trU(\eta, V_\xi \eta) d\mu(x).$$

So by Theorem 2 of [16]

$$\langle \eta, T_\mu \xi \rangle = \int_S \langle \eta, V_\xi \eta \rangle d\mu(x) \quad (\xi, \eta \in H).$$

This proves formula (2).

We shall now use formula (3) and prove that if $T_\mu$ is $A$-linear for every $\mu \in M_a(S, w)$, then $V_\mu$ is $A$-linear for every $x \in S$. To see this from (3) for every $\xi, \eta \in H$, and $\alpha \in A$ we have

$$\int_S \langle \eta, V_\xi (\alpha \eta) \rangle d\mu(x) = \int_S \langle \eta, T_\mu (\alpha \eta) \rangle = tr(T_\mu (\alpha \eta), \eta)$$

Since both the mappings: $x \rightarrow \langle \eta, V_\xi (\alpha \eta) \rangle$ and $x \rightarrow \langle \eta, (V_\xi \alpha) \rangle$ are $w$-bounded and continuous and $S$ is a foundation semigroup, from Lemma 4.8 of [7] we conclude that $V_\xi (\alpha \eta) = (V_\xi \alpha) \eta$ $(x \in S, \alpha \in A)$.

The following result is indeed a generalization of our earlier version of Bochner’s theorem [4; Theorem 4.2].

**Theorem 2.2.** (Generalized Bochner’s theorem on foundation semigroups). Let $S$ be a commutative foundation topological $*$-semigroup with identity and with a Borel measurable weight function $w$. Let $A$ be a proper $H^*$-algebra over a Hilbert module $H$. Then a mapping $\varphi : S \rightarrow \tau(A)$ is $w$-bounded and continuous and positive definite if and only if there exists a unique positive $A$-valued measure $\lambda_\varphi$ on $\Gamma_s$ such that

$$\varphi(x) = \int_{\Gamma_s} \chi(x) d\lambda_\varphi(\chi) \quad (x \in S).$$

**Proof.** Since $\varphi$ is $w$-bounded and continuous, by Theorem 1 of [16] there exists a $w$-bounded weakly continuous $*$-representation $V$ of $S$ by bounded $A$-linear operators on a Hilbert $A$-module $K$ with some $\xi_0 \in K$ such that $\varphi(x) = \langle \xi_0, (V_\xi) \rangle$ and $\|V_\xi\| \leq w(x)$ for every $x \in S$.

Using the integration theory on page 120 of [13] and Lemma 2 of the same reference, we conclude that the mapping $\Phi : M_a(S, w) \rightarrow \tau(A)$ given by

$$\Phi(\mu) = \int_{\Gamma_s} \chi(x) d\mu(x) = \int_{\Gamma_s} \langle \xi_0, V_\xi \rangle d\mu(x) \quad (\mu \in M_a(S, w))$$

is well-defined. It is also easy to see the $\Phi$ defines a positive $A$-functional on the Banach $*$-algebra $M_a(S, w)$. Therefore, by Theorem 3 of [15] there exists a positive $\tau(A)$-valued measure $\lambda$ on $\Delta(M_a(S, w))$ such that

$$\Phi(\mu) = \int_{\Delta(M_a(S, w))} \lambda(\sigma) d\lambda(\sigma).$$

Using Theorem 2.10 of [7], we conclude that

$$\Phi(\mu) = \int_{\Gamma_s} \chi(x) d\mu(x) = \int_{\Gamma_s} \langle \xi_0, V_\xi \rangle d\mu(x) \quad (\mu \in M_a(S, w)).$$

By Fubini’s theorem

$$\int_S \varphi(x) d\mu(x) = \int_S \int_{\Gamma_s} \chi(x) d\lambda(\chi) d\mu(x) \quad (\mu \in M_a(S, w)).$$

Since both functions $\varphi$ and $x \rightarrow \int_{\Gamma_s} \chi(x) d\lambda(\chi)$ are
w-bounded and weakly continuous and $S$ is a foundation semigroup, we infer that

$$\phi(x) = \int_x^\infty f(x) d\lambda(x) \quad (x \in S).$$

The uniqueness of $\lambda$ follows in the same lines as those of Theorem 4.2 of [4].

3. Completely Monotone Functionals on Ordered Banach Algebras

Our starting point of this section is the following:

**Theorem 3.1.** Let $B$ be a commutative ordered Banach algebra with a bounded approximate identity $(e_n)$ in $P_1(B)$. Let $k$ be the set of all completely monotone functionals $f$ in $B'$ such that $\|f\| \leq 1$. Then $K$ is a convex and weak*−compact subset of $A'$. If $f$ is an extreme point of $K$, then $f(a) \geq 0$ for all $a \in P(B)$ and $f(ab) = f(a)f(b)$ for all $a, b \in B$.

**Proof.** It is clear that $K$ is a convex and weak*−closed subset of the unit ball of $B'$ and so by the Banach Alaoglu theorem is weak*−compact. Let $f$ be an extreme point of $K$. Then it is clear that $f(a) \geq 0$ for all $a \in P(B)$. Since $P(B)$ spans $B$, to prove that $f(ab) = f(a)f(b)$ for all $a, b \in B$, it suffices to show that $f(ab) = f(a)f(b)$ for all $a, b \in P(B)$. For every $a \in B$ we define $f_a \in B^*$ by $f_a(b) = f(ab)$ $(b \in B)$. It is easy to see that

$$\Delta_n(f - f_a)(b; b_1, \ldots, b_n) = \Delta_{n+1}f(b; b_1, \ldots, b_n, a),$$

for all $n \in \mathbb{Z}_{+}$, and $a, b, b_1, \ldots, b_n \in P(B)$. Thus, $f - f_a$ is also completely monotone. So

$$(f - f_a)(e_n) = \Delta_n(f - f_a)(e_n) \geq 0,$$

and

$$(f - f_a)(e_n) = (f - f_a)(e_n) \geq 0,$$

for all $a, b \in P_1(B)$. From these two inequalities it follows that

$$0 \leq (f - f_a)(e_\alpha) \leq (f - f_a)(e_\alpha) = f(e_\alpha) - f(ae_\alpha) \leq 1 - f(ae_\alpha)$$

for all $\alpha$ and all $a, b \in P_1(B)$. Since $(e_\alpha)$ is a bounded approximate identity for $B$, it follows that

$$0 \leq (f - f_a)(b) \leq 1 - f(a) \quad (a, b \in P_1(B)). \quad (4)$$

Using the fact that $f$ is completely monotone, we conclude that

$$0 \leq \Delta_n f(ab) = f(ab) - f(a) = f(ab) \quad (a, b \in P_1(B)).$$

and

$$0 \leq \Delta_n f(ab) = f(ab) - f(a) = f(ab) \quad (a, b \in P_1(B)).$$

Thus

$$0 \leq f(ab) \leq f(a) \quad (a, b \in P_1(B)). \quad (5)$$

We shall now consider three cases.

**Case 1.** $f(a) = 0$. So by (5), $f(ab) = 0$. Hence $f(ab) = 0 = f(a)f(b)$ $(a, b \in P_1(B))$.

**Case 2.** $f(a) = 1$. Then by (4),

$$f(b) = f(ab)(a, b \in P_1(B))$$

and so

$$f(ab) = f(b) = f(a)f(b)(a, b \in P_1(B)).$$

**Case 3.** $0 < f(a) < 1$. In this case we write

$$f = (1 - f(a))f + f(a)f_a.$$  

From (4) it follows that $(f - f_a)(1 - f(a)) \in K$, and (5) implies that $f_a/f(a)$ also belongs to $K$. Since $f$ is an extreme point of $K$, it follows that $f_a/f(a) = f$. So

$$f(ab) = f(a)f(b) \quad (a, b \in P_1(B)).$$

This completes the proof.

**Theorem 3.2.** Let $B$ be a commutative ordered Banach algebra with a bounded approximate identity $(e_\alpha)$ in $P_1(B)$. Then a linear transformation $U : B \to \mathcal{L}(H)$ ($H$ is a Hilbert space) is completely monotone if and only if there is a positive operator-valued measure $E$ on $\Delta_1(B)$ such that

$$\langle U_b \xi, \eta \rangle = \int_{\Delta_1(B)} \sigma(b) d\langle E_\sigma(\xi), \eta \rangle \quad (\xi, \eta \in H, b \in B). \quad (6)$$

Moreover, $U$ is a representation if and only if $E$ is a spectral measure.

**Proof.** Let $U : B \to \mathcal{L}(H)$ be completely monotone. Without loss of generality, we may assume that $\|U_b\| = \|b\| (b \in B)$. For every $\xi \in H$ with $\|\xi\| = 1$ we define the linear functional $L_{\xi}$ on $B$ by

$$L_{\xi}(b) = \langle U_b \xi, \xi \rangle \quad (b \in B).$$

It is clear that $L_{\xi}$ defines a completely monotone functional on $B$ with $\|L_{\xi}\| \leq 1$. By the integral form of the Krein-Milman theorem [10; p. 6] and Theorem 3.1, there exists a unique regular probability measure $\mu_{\xi, z}$ on
\[
\Delta_+(B) \text{ such that }
L_\xi(b) = \int_{\Delta_+(B)} \sigma(b) d\mu_{\xi\xi}(\sigma) \quad (b \in B).
\]

So if \(0 \neq \xi \in H\) is arbitrary, then there exists a unique positive regular measure \(\mu_{\xi\xi}\) with \(\mu_{\xi\xi} \leq \|\xi\|^2\) and
\[
\langle U_b \xi, \xi \rangle = \int_{\Delta_+(B)} \sigma(b) d\mu_{\xi\xi}(\sigma) \quad (b \in B).
\]

By the polarization identity for every \(\xi, \eta \in H\) and \(B \in \mathcal{B}(H)\) we have
\[
\langle U_b \xi, \eta \rangle = \frac{1}{4}\left(\langle U_b (\xi + \eta), \xi + \eta \rangle - \langle U_b (\xi - \eta), \xi - \eta \rangle + i\langle U_b (\xi + i\eta), \xi + i\eta \rangle - i\langle U_b (\xi - i\eta), \xi - i\eta \rangle\right).
\]

Thus
\[
\langle U_b \xi, \eta \rangle = \int_{\Delta_+(B)} \sigma(b) d\mu_{\xi\eta}(\sigma) \quad (b \in B, \xi, \eta \in H),
\]

where
\[
\mu_{\xi\eta} = \frac{1}{4}\left(\mu_{\xi+\eta, \xi+\eta} - \mu_{\xi-\eta, \xi-\eta} + i\mu_{\xi+i\eta, \xi+i\eta} - i\mu_{\xi-i\eta, \xi-i\eta}\right).
\]

Now let \(\mathcal{B}(\Delta_+(B))\) denote the \(\sigma\)-algebra of all Borel subsets of \(\Delta_+(B)\). Define the operator-valued measure \(E\) on \(\mathcal{B}(\Delta_+(B))\) by
\[
\langle E(M) \xi, \eta \rangle = \mu_{\xi\eta}(M) \quad (\xi, \eta \in H, M \in \mathcal{B}(\Delta_+(B))).
\]

It is easy to see that \(E\) is positive, in the sense that \(\langle E(M) \xi, \xi \rangle \geq 0\) for all \(\xi \in H\) and \(M \in \mathcal{B}(\Delta_+(B))\). Moreover,
\[
\langle U_b \xi, \eta \rangle = \int_{\Delta_+(B)} \sigma(b) d\langle E_{\sigma}(\xi), \eta \rangle \quad (b \in B, \xi, \eta \in H).
\]

For simplicity, we abbreviate this equality as
\[
U_b = \int_{\Delta_+(B)} \sigma(b) dE_{\sigma} \quad (b \in B).
\]

Now for every \(b \in B\) we denote by \(\hat{b}\) the restriction of the Gelfand transform of \(b\) to \(\Delta_+(B)\), that is
\[
\hat{b}(\sigma) = \sigma(b) \quad \text{for all } \sigma \in \Delta_+(B).
\]

Since by the Gelfand representation theorem \(\mathcal{P} = \{\hat{b} : b \in B\}\) separates the points of \(\Delta_+(B)\), from the Stone-Weierstrass theorem it follows that it is dense in \(C_0(\Delta_+(B))\), the space of all continuous complex-valued functions on \(\Delta_+(B)\) vanishing at infinity. Now if \(U\) is multiplicative, then for every \(a, b \in B\) we have
\[
\int_{\Delta_+(B)} \hat{a}(\sigma) \hat{b}(\sigma) dE_{\sigma} = \int_{\Delta_+(B)} \hat{a}(\sigma) \hat{b}(\sigma) dE_{\sigma} = U_{ab} = U_a U_b = \int_{\Delta_+(B)} \hat{a}(\sigma) \hat{b}(\sigma) dE_{\sigma}.
\]

Since for every fixed \(b \in B\), each of the functions \(\hat{a} \mapsto \int_{\Delta_+(B)} \hat{a}(\sigma) dE_{\sigma}\) and \(\hat{a} \mapsto \int_{\Delta_+(B)} \hat{a}(\sigma) dE_{\sigma}\)

\((\Delta_+(B)), \int_{\Delta_+(B)} \hat{b}(\sigma) dE_{\sigma}(\hat{a} \in \mathcal{P})\) are bounded and linear on \(\mathcal{P}\) and \(\mathcal{P}\) is dense in \(C_0(\Delta_+(B))\), then for every Borel subset \(M\) of \(\Delta_+(B)\) we have
\[
\int_{\Delta_+(B)} f_M(\sigma) \hat{a}(\sigma) dE_{\sigma} = \int_{\Delta_+(B)} f_M(\sigma) dE_{\sigma} \int_{\Delta_+(B)} \hat{a}(\sigma) dE_{\sigma},
\]

where \(f_M\) denotes the characteristic function of the set \(M\). A similar argument shows that for every two Borel subsets \(M\) and \(N\) of \(\Delta_+(B)\)
\[
\int_{\Delta_+(B)} f_M(\sigma) f_N(\sigma) dE_{\sigma} = \int_{\Delta_+(B)} f_M(\sigma) dE_{\sigma} \int_{\Delta_+(B)} f_N(\sigma) dE_{\sigma}.
\]

That is \(E(M \cap N) = E(M) E(N)\). So \(E\) is a spectral measure on \(\Delta_+(B)\). The proof is now complete.

The following theorem gives a characterization of the completely monotone functionals on commutative ordered Banach algebras.

**Theorem 3.3.** Let \(B\) be a commutative ordered Banach algebra with a bounded approximate identity in \(P(B)\). Then a bounded linear mapping \(U\) of \(B\) into a proper \(H^*-\)algebra \(A\) is completely monotone if and only if there is a positive \(\tau(A)\)-valued measure \(E\) on \(\Delta_+(B)\) such that
\[
\langle \xi, \eta U(b) \rangle = \int_{\Delta_+(B)} \sigma(b) dE_{\sigma}(\xi, \eta, \xi U(b)) \quad (b \in B, \xi, \eta \in H).
\]

Moreover, \(U\) is a positive homomorphism if and only if \(E\) is a generalized spectral measure.

**Proof.** For every \(\xi \in H\) with \(\text{tr}(\xi, \xi) = 1\) we define
\[
L_\xi(b) = \text{tr}(\xi, \xi U(b)) = \langle \xi U(b), \xi \rangle \quad (b \in B).
\]

From
\[
\Delta_+(\xi, \xi U(b)) = \{\xi, \xi \Delta_+(U(b)) = \{\xi, \xi \Delta_+(U(b)), \xi U(b) \cdot b_1, \ldots, b_n\}
\]

\((b, b_1, \ldots, b_n) \in \mathbb{R}_+\).
and the fact that \( U \) is bounded and completely monotone we conclude that \( L_\xi \) defines a completely linear functional on \( B \). So by Theorem 3.2 there exists an operator-valued measure \( E \) bounded operators on the Hilbert space \((H, \langle \cdot, \cdot \rangle)\) such that

\[
L_\xi(b) = \langle U(b), \eta \rangle = \int_{\Delta_\xi(B)} \sigma(b) d\langle E_\sigma(\cdot), \xi, \eta \rangle \quad (b \in B).
\]

For every \( T \in \Gamma(A) \) by Lemma 2 of [14] we have

\[
trT(\eta, \xi U(b)) = tr(T\eta, \xi U(b)) = \langle \xi U(b), T\eta \rangle
\]

\[
= \int_{\Delta_\xi(B)} \sigma(b) d\langle E_\sigma(\cdot), T\eta, \xi \rangle
\]

\[
= \int_{\Delta_\xi(B)} \sigma(b) d\ trT(\eta, E_\sigma(\cdot)) \xi \}
\]

Now it is easily seen that the mapping: \( \Delta_\xi(B) \rightarrow \tau(A) \) given by: \( M \rightarrow (\eta, E(M) \xi) \) (\( M \in \mathcal{B}(\Delta_\xi(B)) \)) defines a \( \tau(A) \)-valued measure on \( \Delta_\xi(B) \). Therefore, by Lemma 2 of [14] and Theorem 2 of [16] we have

\[
(\eta, \xi U(b)) = \int_{\Delta_\xi(B)} \sigma(b) d\langle \eta, E_\sigma(\cdot), \xi \rangle \quad (b \in B).
\]

Thus, the proof is complete.

We are now in a position to state and prove the main result of this paper. Note that if \( H \) is a right Hilbert module over a proper \( H^* \)-algebra \( A \), then a mapping \( T : S \rightarrow A \) is called \( w \)-bounded and continuous if for every \( \xi, \eta \in H \) the mapping \( x \mapsto tr(\xi, \eta Tx) \) is a \( w \)-bounded continuous complex-valued function on \( S \).

**Theorem 3.4.** Let \( S \) be a commutative foundation semigroup with identity and with a Borel measurable weight function \( w \) continuous at the identity. Let \( H \) be a Hilbert module over a proper \( H^* \)-algebra \( A \). Then a mapping \( T : S \rightarrow A \) is \( w \)-bounded and completely monotone if and only if there exists a unique positive \( A \)-valued measure \( E \) on \( \Gamma(A) \) such that

\[
(\xi, \eta U(\mu)) = \int_S \chi(x) d\langle \xi, E(x) \eta \rangle \quad (\mu \in M_a(S, w), \xi, \eta \in H).
\]

Then \( T \) is a homomorphism if and only if \( E \) is a generalized \( A \)-valued spectral measure.

**Proof.** From the continuity of \( w \) at the identity of \( S \) it follows that \( M_a(S, w) \) has a bounded approximate identity in \( \Gamma(M_a(S, w)) \) (see [9]). It is also easy to see that the equation

\[
(\xi, \eta U(\mu)) = \int_S (\xi, \eta T_x) d\mu(x) \quad (\mu \in M_a(S, w), \xi, \eta \in H)
\]

defines a completely monotone \( A \)-valued bounded functional on the ordered Banach algebra \( M_a(S, w) \).

Therefore, by Theorem 3.3 there exists a unique positive \( A \)-valued measure \( E \) on \( \Delta_\xi(M_a(S, w)) \) such that

\[
(\xi, \eta U(\mu)) = \int_{\Delta_\xi(M_a(S, w))} \mu(x) d\langle \xi, E_x(\cdot), \eta \rangle
\]

\[
(\xi, \eta \in H, \mu \in M_a(S, w)).
\]

Now an application of this equality and Theorem 2.10 of [8] with the aid of Fubini’s theorem gives

\[
(\xi, \eta U(\mu)) = \int_S \int_{\Delta_\xi(M_a(S, w))} \mu(x) d\langle \xi, E_x(\cdot), \eta \rangle
\]

Now since both mappings \( x \mapsto \int_{\Delta_\xi(M_a(S, w))} \mu(x) d\langle \xi, E_x(\cdot), \eta \rangle \) and \( x \mapsto (\xi, \eta T_x) \) are \( w \)-bounded and continuous and \( S \) is also a foundation semigroup, we conclude that

\[
(\xi, \eta T_x) = \int_{\Delta_\xi(M_a(S, w))} \mu(x) d\langle \xi, E_x(\cdot), \eta \rangle \quad (\xi, \eta \in H, x \in S).
\]

**Remark.** The following example shows that the conclusion of the preceding theorem is not valid in general for non-foundation semigroups.

**Example 3.5.** Let \( S = [0,1] \). Then with the usual topology of the real line and the multiplication \( xy = \min(x,y) \) \((x, y \in S) \) \( S \) defines a non-foundation semigroup. If we choose \( w=1 \) on \( S \), then \( \Gamma^*_w = \{1\} \), where 1 denotes the function which is identically one on \( S \). It is clear that the mapping \( T : S \rightarrow L^2(S, m) \) \((m \) denotes the Lebesgue measure on \([0,1])\) given by

\[
T_x f = \hat{f} \quad (x \in S, f \in L^2(S, m)),
\]

where \( \hat{f} \) denotes the characteristics function on \([0,x]\), defines a completely monotone operator-valued transformation of \( S \) by operators on the Hilbert module \( L^2(S, m) \) (see [3]). If the formula (6) is valid for \( T \), then we arrive at the contradiction that \( T_x = I \) for every \( x \) in \( S \), where \( I \) denotes the identity operator on \( L^2(S, m) \).

**References**


