THE DUALITY OF THE L^∞-REPRESENTATION ALGEBRA $\mathfrak{R}(S)$ OF A FOUNDATION SEMIGROUP S AND FUNCTION ALGEBRAS

M. Lashkarizadeh Bami*

Department of Mathematics, University of Isfahan, Isfahan, Islamic Republic of Iran

Abstract

In the present paper for a large family of topological semigroups, namely foundation semigroups, for which topological groups and discrete semigroups are elementary examples, it is shown that $\mathfrak{R}(S)$ is the dual of a function algebra.

Introduction

The notation of the L^∞-representation Banach algebra of a commutative topological semigroup S was introduced and extensively studied by Dunkl and Ramirez in [3]. Recall that an L^∞-representation of S is a triple (Ω, μ, T) where μ is a complete probability measure on the set Ω, and $s \mapsto T_s$ is a homomorphism of S into the unit ball of $L^\infty(\Omega, \mu)$ (where $L^\infty(\Omega, \mu)$ has the pointwise multiplication) and is weak-* (i.e., $\sigma(L^\infty(\Omega, \mu), L^1(\Omega, \mu))$ continuous (see [3]). The *representation algebra* $\mathfrak{R}(S)$ is defined to be the set of all functions

$$s \mapsto \int_\Omega (T_s g) d\mu$$

of S into $\mathbb{C}$, where (Ω, μ, T) is an L^∞-representation of S and $g \in L^1(\Omega, \mu)$. It is shown in [4] that $\mathfrak{R}(S)$ is a Banach algebra of bounded continuous complex-valued functions on S, with pointwise multiplication and the norm

$$\|f\|_{\mathfrak{R}} = \inf\left\{\|g\|_1 : f(s) \equiv \int_\Omega Tg d\mu\right\}.$$

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We denote by $\mathfrak{R}_+(S)$ the set of all $f \in \mathfrak{R}(S)$ such that $f(s) = \int T_s g d\lambda (s \in S)$ for some L^∞-representation (Ω, λ, T) and some $0 \leq g \in L^1(\Omega, \lambda)$.

It is well-known that in general $\mathfrak{R}(S)$ is not the dual of any Banach space (see [3], Example 2.2.1). In the present paper we prove, for a large class of topological semigroups, the so-called foundation semigroups, that $\mathfrak{R}(S)$ is the dual of a function algebra. As an application of this result we give a version of the Bochner-Eberlein theorem on $\mathfrak{R}(S)$. We also prove an analogue of one of our earlier results in [5] on the *- semisimplicity of the Banach *-algebras $M(S)$ and $M_a(S)$ of a foundation *-semigroup S (not necessarily commutative) in terms of *-representations, by proving that the commutative measure algebras $M(S)$ and $M_a(S)$ of a commutative foundation semigroup S are semisimple if and only if $\mathfrak{R}(S)$ separates the points of S. It should be noted that in the case where S in a topological semigroup (not necessarily commutative) with an involution, a representation algebra $F(S)$ is defined by Lau in [8] which satisfies the inclusion $F(S) \subseteq \mathfrak{R}(S)$ whenever S is commutative with an involution. In the Example 4.2 of [8] Lau has shown that for the additive group Z of integer numbers with the involution $n^* = n (n \in Z)$ this inclusion in proper. Note that a mapping * on a topological semigroup S is called an involution if $x^{**} = x$ and $(xy)^* = y^* x^*$ for every $x, y \in S$.

*E-mail: lashkari@math.ui.ac.ir
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Preliminaries

Throughout this article, S will denote a locally compact Hausdorff topological semigroup. Let $M(S)$ denote the space of all bounded complex regular measures on S and δ_x be the Dirac measure at x. We denote by $M_a(S)$ the space of all measures $\mu \in M(S)$ such that the mappings: $x \to \delta_x * |\mu|$ and $x \to |\mu| * \delta_x$ of S into $M(S)$ are weakly continuous. A topological semigroup S is called a *foundation semigroup* if $\bigcup \{\mathrm{supp}(\mu)\colon \mu \in M_a(S)\}$ is dense in S. A nonzero complex-valued function χ on S is called a *semicharacter* if $\chi(xy) = \chi(x)\chi(y)$ for every $x, y \in S$. If S is a foundation semigroup then by Theorem 4.4 of [1] $\hat{S}$ is homeomorphic to $\widehat{M_a(S)}$ (the maximal ideal space of $M_a(S)$) whenever $\hat{S}$ has the compact open topology and $\widehat{M_a(S)}$ has the Gelfand topology. We denote by $\hat{S}$ the set of all bounded continuous semicharacters on S. In particular, $\hat{S}$ with the pointwise multiplication and the compact open topology is a locally compact Hausdorff topological semigroup. Moreover the Gelfand transform $\hat{\mu}$ of $\mu \in M_a(S)$ is given by $\hat{\mu}(\chi) = \int_S \chi(x) d\mu(x) \quad (\chi \in \hat{S})$.

$\mathfrak{R}(S)$ as the Dual of a Function Algebra

We commence with the following theorem in which we assume familiarity with the notion of function algebras.

Theorem 1. *Let S be a commutative foundation semigroup. Then $(\mathfrak{R}(S), \|\cdot\|_{\mathfrak{R}})$ is the dual of the function algebra the L^∞-representation Banach algebra of the completion of $\widehat{M_a(S)}$ in the Banach algebra $C_0(\hat{S})$.*

Proof. For simplicity we denote the completion of $\widehat{M_a(S)}$ in $C_0(\hat{S})$ by A. So by the Gelfand representation theorem, A defines a function algebra on $\hat{S}$. For every $f \in \mathfrak{R}(S)$ we define the linear functional τ_f on $\widehat{M_a(S)}$ by

$$\tau_f(\hat{\mu}) = \int_S f(x) d\mu(x) \quad (\mu \in M_a(S)).$$

We claim that $\|\tau_f\| \le \|f\|_{\mathfrak{R}}$ $(f \in \mathfrak{R}(S))$. To see this, let $f \in \mathfrak{R}(S)$ and suppose $\varepsilon > 0$ is given. Then there exists an L^∞-representation (Ω, T, λ) and $g \in L^1(\Omega, \lambda)$ such that $\|g\|_1 < \|f\|_{\mathfrak{R}} + \varepsilon$ and

$$f(x) = \int_\Omega T_x g d\lambda \quad (x \in S).$$

Define $\tilde{T}\colon M_a(S) \to L^\infty(\Omega, \lambda)$ by

$$\left\langle \tilde{T}\mu, h \right\rangle = \int_S \left(\int_\Omega T_x h d\lambda \right) d\mu(x) \quad (\mu \in M_a(S), h \in L^1(\Omega, \lambda))$$

,

where $\langle , \rangle$ denotes the pairing between $L^\infty(\Omega, \mu)$ and $L^1(\Omega, \mu)$. As in the proof of Theorem 3 of [7] we have $\|\tilde{T}_\mu\|_\infty \le \|\hat{\mu}\|_u$ $(\mu \in M_a(S))$, where $\|\cdot\|_u$ denotes the norm of $C_0(\hat{S})$. Thus for every $(\mu \in M_a(S)$

$$\left|\tau_f(\hat{\mu})\right| = \left|\left\langle \tilde{T}\mu, g \right\rangle\right| \le \|\tilde{T}_\mu\|_\infty \|g\|_1 \le \|\hat{\mu}\|_u (\|f\|_{\mathfrak{R}} + \varepsilon).$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $\|\tau_f\| \le \|f\|_{\mathfrak{R}}$ and hence our claim is established. Since $M_a(S)$ is dense in A, we can extend τ_f uniquely to a bounded linear functional $\tilde{\tau}_f$ on A such that

$$\|\tilde{\tau}_f\| \le \|f\|_{\mathfrak{R}} \tag{1}$$

Now suppose that $\Phi \in A^*$, the dual of A. So by the Hahn-Banach Theorem and the Riesz representation theorem there exists $\lambda \in M(\hat{S})$ such that $\|\lambda\| = \|\Phi\|$ and

$$\Phi(\hat{\mu}) = \int_S \hat{\mu}(\chi) d\lambda(\chi) \quad (\mu \in M_a(S)).$$

Putting $d\lambda = g d\nu$ for some probability measure ν on $\hat{S}$ and $g \in L^1(\hat{S}, \nu)$ with $\|g\|_1 = \|\lambda\|$ and using Fubini's theorem we obtain

$$\begin{aligned} \Phi(\hat{\mu}) &= \int_{\hat{S}} \left[\int_S \chi(x) d\mu(x) \right] g(\chi) d\nu(\chi) \\ &= \int_S \left[\int_{\hat{S}} \chi(x) g(\chi) d\nu(\chi) \right] d\mu(x) \end{aligned} \tag{2}$$

Now we define f_0 on S by

$$f_0(x) = \int_{\hat{S}} \chi(x) g(\chi) d\nu(\chi) \quad (x \in S).$$

Since as in the proof of Lemma 2.2 of [7] the triple $(\hat{S}, \nu, \hat{x})$ (where $\hat{x}\colon \hat{S} \to \mathbb{C}$ is given by $\hat{x}(\chi) = \chi(x)$ $(\chi \in \hat{S})$ defines an L^∞-representation on S, it flows that $f_0 \in \mathfrak{R}(S)$ and

$$\|f_0\|_{\mathfrak{R}} \le \|g\|_1 = \|\lambda\| = \|\Phi\| \tag{3}$$

Since by (2) for every $\hat{\mu} \in \widehat{M_a(S)}$, $\tau_f(\hat{\mu}) = \Phi(\hat{\mu})$, and $\widehat{M_a(S)}$ is dense in A, we deduce that

$$\|\Phi\| = \|\tilde{\tau}_{f_0}\| \tag{4}$$

A combination of (3) and (4) with the aid of (1) yields $\|\Phi\| = \|\tilde{\tau}_{f_0}\| = \|f_0\|_{\mathfrak{R}}$. This completes the proof. ■

As a consequence of the above theorem we obtain the following version of the Bochner-Eberlein theorem.

Corollary 2. *Let S be a commutative foundation semigroup. Let (f_α) be a net in $\Re(S)$ such that $\|f_\alpha\|_\Re \leq M$ for all α, where M is a fixed positive number. Suppose that there is a bounded continuous complex-valued function f on S such that for every $\mu \in M_a(S)$, $\int_S f_\alpha d\mu \to \int_S f d\mu$. Then $f \in \Re(S)$ and $\|f_\alpha\|_\Re \leq M$.*

Proof. From Theorem 1 and the Banach-Alaoglu theorem (by passing to a subnet if necessary) it follows that there exists $g \in \Re(S)$ such that $\|g\|_\Re \leq M$ and

$$\int_S f_\alpha d\mu \to \int_S g d\mu \quad (\mu \in M_a(S)).$$

Hence $\int_S f_\alpha d\mu = \int_S g d\mu$ $(\mu \in M_a(S))$. So $f=g$, by Lemma 2.2 of [5]. ■

The following result is a counterpart of Theorem 2.5 of [6] for the case that S is commutative.

Theorem 3. *Let S be a commutative foundation semigroup. Then the following are equivalent:*

(i) The Banach algebra $M(S)$ is semisimple.

(ii) The Banach algebra $M_a(S)$ is semisimple.

(iii) $\hat{S}$ separates the points of S.

(iv) The L^∞-representation algebra $\Re(S)$ separates the points of S.

Proof. By Theorem 3.6 of [1] we only need to prove the equivalence of *(iii)* and *(iv)*.

(iii)⇒(iv). This is clear, since by Proposition 1.1.6 of [3] $\hat{S} \subset \Re(S)$.

(iv)⇒(iii). To see this, let $x, y \in S$ with $x \neq y$. Since $\Re(S)$ separates the points of S, we can find $f \in \Re(S)$ such that $f(x) \neq f(y)$. By Theorem 3 of [7] there exists $\lambda \in M(\hat{S})$ such that $f(x) = \int_{\hat{S}} \chi(x) d\lambda(\chi)$ $(x \in S)$. So there is $\chi \in \hat{S}$ such that $\chi(x) \neq \chi(y)$. ■

The following example shows that the result of Theorem 3 is not valid in general for non-foundation semigroups.

Example 4. Let $S=[0,1]$. Then with the multiplication $xy=\max(x,y)(x,y \in S)$ and the usual topology S is a non-foundation semigroup.

By Theorem 5 of [2] $\Re(S)=BV(S)$ (the space of continuous functions of bounded variation on S) and since $\hat{S}=\{1\}$ (where $1(x)=1$ for every $x \in S$), then it is clear that $\Re(S)$ separates the points of S, but this is not the case for $\hat{S}$.

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