SOME BOUNDARY VALUE PROBLEMS FOR A NON-LINEAR THIRD ORDER O.D.E.

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Abstract

Existence of periodic solutions for non-linear third order autonomous differential equation (O.D.E.) has not been investigated to as large an extent as non-linear second order. The popular Poincare-Bendixon theorem applicable to second order equation is not valid for third order equation (see [3]). This conclusion opens a way for further investigation.

Let us consider the following third order non-linear differential equation

\[ x''' + ax' + f(x) = 0, \quad a > 0 \]  

(1)

where \( f(x) \) is a continuous real-valued function with \( xf(x) > 0 \). Under this assumption, we shall establish the following theorem.

Theorem. Let us assume that there exist constants \( D > 0, \ c_2 > 0, \ \alpha \) and \( \beta \) such that

(i) \( a\alpha + f(x) \leq 0 \leq a\beta + f(x) \) for all \( |x| \leq D \)

(ii) \( c_2 > d \) where \( d = \max(|\alpha|, |\beta|) \)

(iii) \( 3M \leq a^{3/2}(c_2 - d) \) with \( M = \|f\|_{\infty} \)

\[ \|f\|_{\infty} = \max \{|f(x)| : |x| \leq D\} \]

(iv) \( 3m \leq D \) where \( m = \max(M, c_2 + d + \frac{6M}{a^{3/2}}) \)

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Then there exists at least one \( \omega \in \left[ \frac{\pi}{2a}, \frac{3\pi}{2a} \right] \) such that Equation (1) has a non-trivial solution satisfying the following boundary conditions:

\[ x'(0) = x' (\omega), \quad \int_{0}^{\omega} f(x(s)) \, ds = 0 \]  

(2)

Proof. For each \( c_1 \in [\alpha, \beta] \), let us define the function \( x(t) = x(t, c_1) \) as the solution of the following integral equation:

\[ x(t) = x(t) + c_1 \sin \sqrt{a} t + c_2 \cos \sqrt{a} t + F(t, x(t)), \]

\[ F(t, x(t)) = -a^{-1} \int_{0}^{\omega} \left[ 1 - \cos \sqrt{a} (t-s) \right] f(x(s)) \, ds \]  

(3)

Obviously,

\[ x(t) = -a^{1/2} \sqrt{a} c_1 \cos \sqrt{a} t c_2 \sin \sqrt{a} t - a^{-1} \int_{0}^{\omega} f(x(s)) \sin \sqrt{a} (t-s) \, ds \]

One can easily verify that \( x(t) \) satisfies Equation (1). By (iii) for \( |x| \leq D \), we obtain

\[ \left| a^{-1} \int_{0}^{\omega} f(x(s)) \sin \sqrt{a} (t-s) \, ds \right| \leq \frac{3M}{a^{3/2}}. \]

Let \( H(t) = x'(0) - x'(t) \), then clearly we have

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\[ H(\frac{3\pi}{2\sqrt{a}}) = \sqrt{a}\left[c_1 + c_2 - a^{-1}\int_{0}^{\frac{3\pi}{2}} f(x(s)) \cos \sqrt{a} \, ds \right], \]

and

\[ H(\frac{\pi}{2\sqrt{a}}) = \sqrt{a}\left[c_1 - c_2 + a^{-1}\int_{0}^{\frac{\pi}{2}} f(x(s)) \cos \sqrt{a} \, ds \right]. \]

For \( |t| \leq D \), we obtain the inequalities:

\[ \int_{0}^{\frac{3\pi}{2}} f(x(s)) \cos \sqrt{a} \, ds \leq \frac{3M}{\sqrt{a}} \]

and

\[ \int_{0}^{\frac{\pi}{2}} f(x(s)) \cos \sqrt{a} \, ds \leq \frac{M}{\sqrt{a}}. \]

Hence,

\[ H(\frac{3\pi}{2\sqrt{a}}) \leq \frac{3M}{a} + \frac{\sqrt{a}(c_1 - c_2)}{2} \leq \frac{3M}{a} + \sqrt{a}(d - c_2) \leq 0 \]

and

\[ H(\frac{\pi}{2\sqrt{a}}) \geq \frac{-M}{a} + \frac{\sqrt{a}(c_1 + c_2)}{2} = \frac{-M + a^{3/2}(c_1 + c_2)}{2} \geq 0 \]

Therefore,

\[ H(\frac{3\pi}{2\sqrt{a}})H(\frac{\pi}{2\sqrt{a}}) \leq 0. \]

If \( H(\frac{3\pi}{2\sqrt{a}}) = 0 \) or \( H(\frac{\pi}{2\sqrt{a}}) = 0 \) then \( x'(0) = x'(\omega) \) with \( \omega = \frac{3\pi}{2\sqrt{a}} \) or \( \frac{\pi}{2\sqrt{a}} \).

Hence, w.l.o.g., we may assume \( H(\frac{3\pi}{2\sqrt{a}})H(\frac{\pi}{2\sqrt{a}}) < 0 \) and the obvious continuity of \( H(t) \) implies the existence of one \( \omega \in (\frac{3\pi}{2\sqrt{a}}, \frac{\pi}{2\sqrt{a}}) \) such that \( H(\omega) = 0 \) i.e. \( x'(0) = x'(\omega) \). To complete the proof of the theorem, we introduce the Banach space \( B = C[0, \omega] \times \mathbb{R} \) with norm \( \| (\delta, l) \| = \| \delta \|_{C} + \| l \| \), \( \| \delta \|_{C} := \max_{0 \leq t \leq \omega} |\delta(t)| \). Now we define the map \( T : B \to B \) by the rule: \( T(\delta, l) = (\delta', l') \) where

\[ \delta' = l + c_1 \sin \sqrt{a} t + c_2 \cos \sqrt{a} t + F(t, \delta(t)) \]

and

\[ l' = l - \frac{1}{\omega} \int_{0}^{\omega} f(\delta'(s)) \, ds. \]

Then \( T \) is a continuous map from \( B \) into \( B \). Next, we are going to prove that the closed subset \( K' \) of \( B \) defined by

\[ K' := \{ (\delta, l) \in B : \| \delta \| \leq D, \| l \| \leq 2m \} \]

is \( T \)-invariant. Since

\[ \| \delta' \| \leq l + d + c_2 + \frac{6M}{a^{3/2}} \leq 2m + m = 3m \leq D, \]

it only remains to prove \( \| l' \| \leq 2m \). For the proof of \( \| l' \| \leq 2m \), three distinct cases will be considered.

**Case I.** \( \| l \| = m \): Since \( \| \delta' \| \leq D \), it follows that

\[ \frac{1}{\omega} \int_{0}^{\omega} f(\delta'(s)) \, ds \leq M \leq m \]

and hence,

\[ -2m \leq l - \frac{1}{\omega} \int_{0}^{\omega} f(\delta'(s)) \, ds \leq 2m \text{ i.e. } \| l' \| \leq 2m. \]

**Case II.** \( m < l < 2m \): Obviously

\[ \| \delta' - l \| \leq d + c_2 + \frac{6M}{a^{3/2}} \leq m, \]

which implies that \( 3m \geq \delta' \geq l - m > 0 \) and hence, \( f(\delta') > 0 \), since by our assumption \( \delta' \cdot f(\delta') > 0 \). Therefore, we obtain

\[ 0 < \frac{1}{\omega} \int_{0}^{\omega} f(\delta'(s)) \, ds \leq M \leq m, \]

which clearly means:

\[ 0 < l - m \leq l' \leq 2m \text{ or } \| l' \| \leq 2m \] (5)

**Case III.** \( -2m \leq l < -m \): With similar arguments we obtain \(-3m < \delta' < 0 \) implying

\[ 0 < -\frac{1}{\omega} \int_{0}^{\omega} f(\delta'(s)) \, ds < m \]

and hence,

\[ -2m \leq l' = l - \frac{1}{\omega} \int_{0}^{\omega} f(\delta'(s)) \, ds < l + m < 0 \text{ i.e. } \| l' \| \leq 2m \] (6)
Thus, $K^*$ is a $T$-invariant closed subset of the Banach space $B$. Using Schauder’s fixed point Theorem (for more discussion on the subject see [1,4]), there exists at least one element $(\delta, \gamma) \in K^*$ such that $T(\delta, \gamma) = (\delta, \gamma)$ i.e.

\[ \delta(t) = \delta(t; c_1) = \gamma + c_1 \sin \sqrt{\alpha t} + c_2 \cos \sqrt{\alpha t} + F(t, \delta(t; c_1)) \]

and

\[ \int_0^\alpha (\delta(s; c_1)) ds = 0, \]

which completes the proof of the Theorem.

**Example.** Let us consider the equation $x'' + x' + x = 0$. With taking $\alpha = -1, \beta = -1, d = 1, c_2 = 4, D = 34, m = 11, M = 1$, all assumptions of the Theorem are fulfilled. Hence, there exists $\omega \in \left[ \frac{\pi}{2}, \frac{3\pi}{2} \right]$ such that

\[ x'(0) = x'(\omega) \quad \text{and} \quad \int_0^\alpha f(x(s)) ds = 0. \]

**References**